The Shammas Telescoping Trigonometric Functions

By
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Introduction

This study presents a new family of nested trigonometric functions that uses sine and cosine functions. I am using my last name in naming these functions, because I did find on the internet a pre-existing definition for the *Telescoping Trigonometric Series,* as sin(x) sec(3x) + ⋯ + sin(3^nx) sec(3^{n+1}x). The initial purpose of creating my new family of telescoping trigonometric functions is to generate curves with multiple minima and maxima to test various empirical curve–fitting models. These curves would not be smoothly fitted by regular third or fourth order polynomials.

The Shammas Telescoping Trigonometric Functions

The general form of the Shammas Telescoping Trigonometric Functions is:

F(x,s,t) = sin(x + s(1)*cos(t(1)*x + s(2)*sin(t(2)*x…)…)   (1)

Where variable s is an array of values in the positive or negative factors (either integer or floating point). In this study I use the values of 1 and –1 to mainly represent sign values. The variable t is an array of scaling values and can be greater, equal, or less than 1. In this study I use two sequences of values for array t—multiples of ½ and multiples of 2. You can choose your own sequences, like multiples of ½, 1/3, ¼, and so on. Thus, the values in arrays s and t can follow a sequence or can be arbitrary. Using sequences for arrays s and t makes it much easier to build a family of successive telescoping trigonometric functions. Equation (1) is the general form for a truly vast set of combinations of sine functions, cosine functions, signs, and factors. Using the arrays s and t gives the reader maximum flexibility in defining the details of function F to match the reader's needs.

Equation (1) employs sine, and cosine functions whose values range between –1 and 1. It is very easy to re-scale these values into any range desired. Equation (1) excludes the tangent function because its values can reach high value and even infinity! Placing the tangent function as an expression in a sine or cosine function may cause the values of these two functions to fluctuate wildly. Curves with wildly fluctuating values are virtually impossible to fit with any reasonable approximating model

First Sample Telescoping Functions

This section presents a relatively small set of telescoping functions that I used in studying candidate non–smooth curves.

The first two functions are $P_1(x)$ and $P_2(x)$ and are defined as:

$P_1(x) = \sin(x + \cos(x/2))$ (2)
$P_2(x) = \sin(x + \cos(x/2) + \sin(x/4)))$ (3)

Figure 1 shows the curve for equation (2). Function $P_1(x)$ is equivalent to F(x, [1], [1/2]).

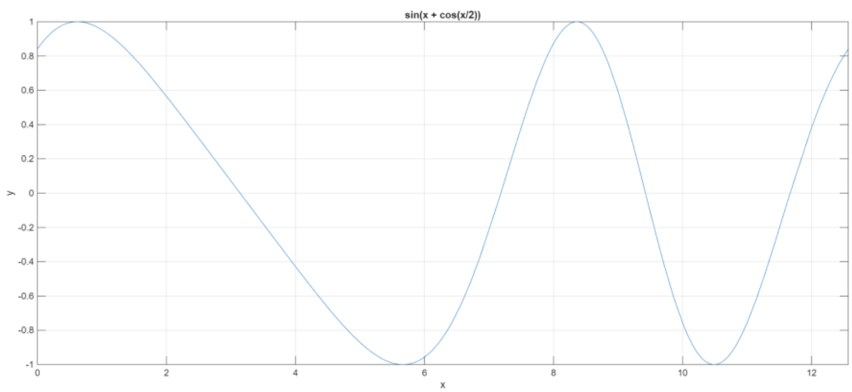

Figure 1. The curve for sin(x + cos(x/2)).

Figure 2 shows the curve for equation (3). Function $P_2(x)$ is equivalent to F(x, [1,1], [1/2,1/4])

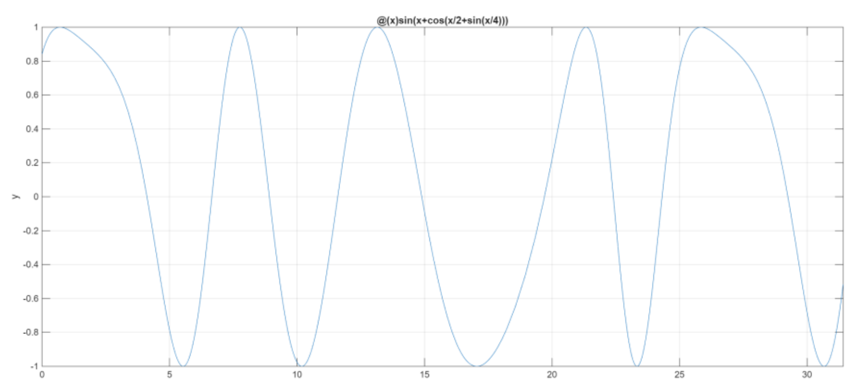

Figure 2. The curve for sin(x + cos(x/2+ sin(x/4))).

The third and fourth functions are $A_1(x)$ and $A_2(x)$ and are defined as:

$A_1(x) = \sin(x - \cos(x/2))$ (4)
$A_2(x) = \sin(x - \cos(x/2 + \sin(x/4)))$ (5)

Figure 3 shows the curve for equation (4). Function $A_1(x)$ is equivalent to F(x, [–1], [1/2]).

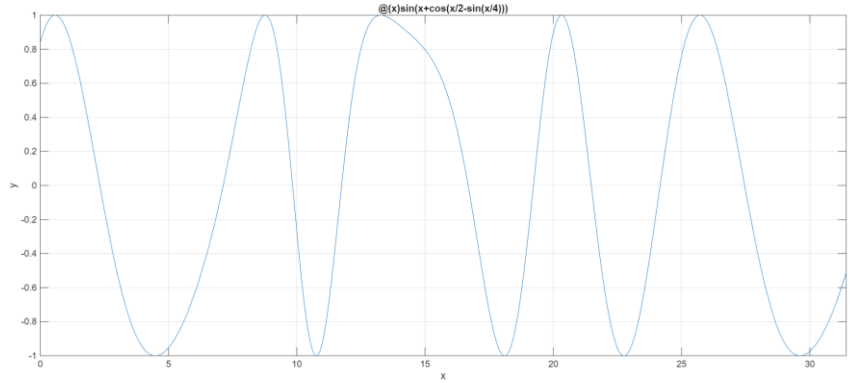

Figure 3. The curve for sin(x - cos(x/2)).

Figure 4 shows the curve for equation (5). Function $A_2(x)$ is equivalent to F(x,[–1,1],[1/2,1/4])

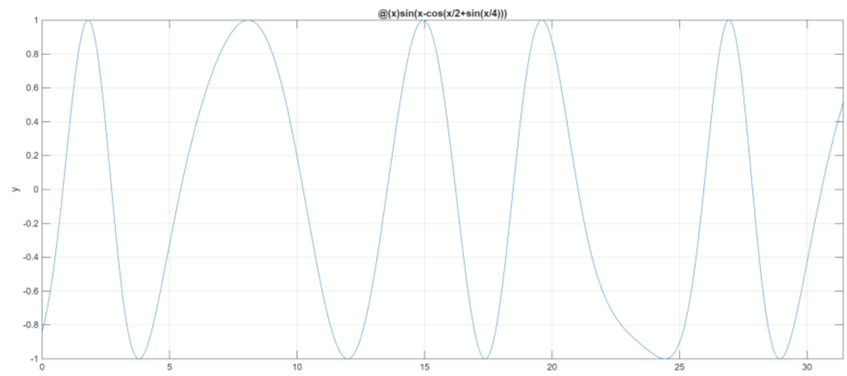

Figure 4. The curve for sin(x - cos(x/2 + sin(x/4))).

The fifth and sixth functions are $N_1(x)$ and $N_2(x)$ and are defined as:

$$N_1(x) = \sin(x - \cos(x/2)) \quad (6)$$
$$N_2(x) = \sin(x - \cos(x/2 - \sin(x/4))) \quad (7)$$

Function $N_1(x)$ is equivalent to $A_1(x)$ and F(x, [–1], [1/2]). Function $N_2(x)$ is equivalent to F(x, [–1, –1], [1/2,1/4]).

Figure 5 shows the curve for equation (7).

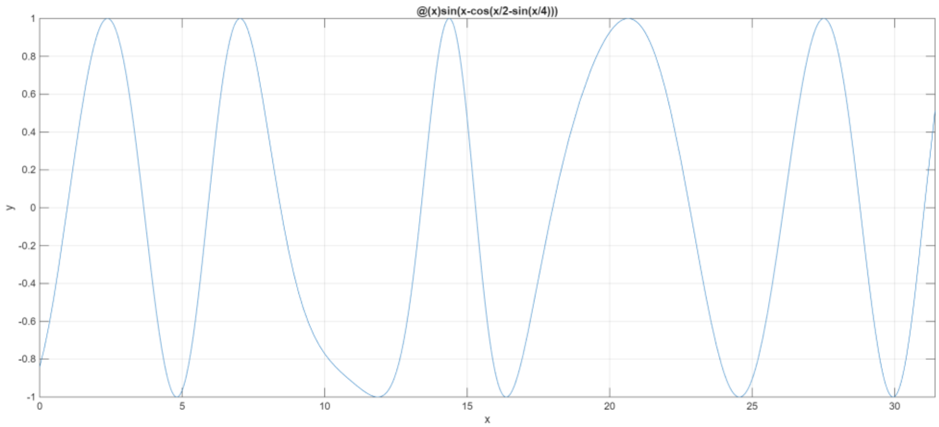

Figure 5. The curve for sin(x - cos(x/2 - sin(x/4))).

The $P_n(x)$ functions use the + operators for the sin and cos terms. The $A_n(x)$ functions alternate signs for the sin and cos terms. The $N_n(x)$ functions use the – operators for the sin and cos terms.

Second Sample Telescoping Functions

Another set of functions $B_n(x)$, $Q_n(x)$, $M_n(x)$ are like the functions $P_n(x)$, $A_n(x)$, and $N_n(x)$, except they multiply x by factors of 2, 4, 8, and so on.

$Q_1(x) = \sin(x + \cos(2*x))$ (8)
$Q_2(x) = \sin(x + \cos(2*x + \sin(4x)))$ (9)
$B_1(x) = \sin(x - \cos(2*x))$ (10)
$B_2(x) = \sin(x - \cos(2*x + \sin(4*x)))$ (11)
$M_1(x) = \sin(x - \cos(2*x))$ (12)
$M_2(x) = \sin(x - \cos(2*x - \sin(4*x)))$ (13)

Figures 6 and 7 show the graphs for functions $Q_1(x)$ and $Q_2(x)$ equations (8) and (9), respectively. This set of telescoping functions shows multiple minima and maxima for $-1 < y < 1$.

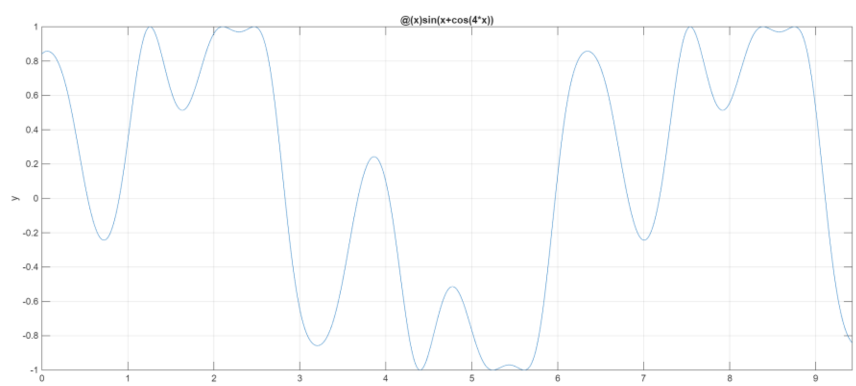

*Figure 6. The curve for sin(x + cos(2*x)).*

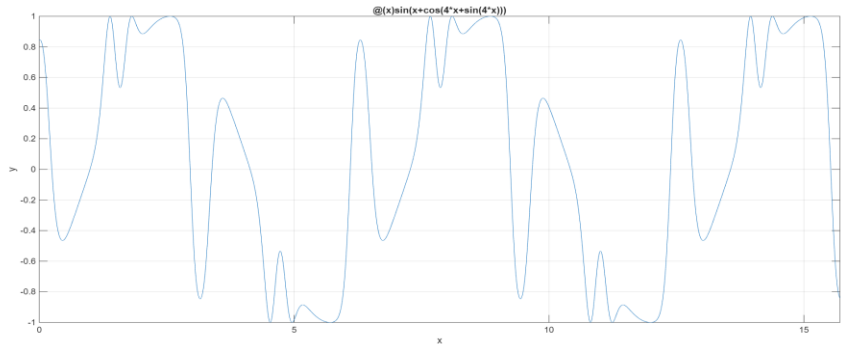

*Figure 7. The curve for sin(x + cos(2*x + sin(4*x))).*

Figures 6 and 7 show the graphs for equations (8) and (9). This set of telescoping functions shows multiple minima and maxima for -1 < y < 1.

Figures 8 and 9 show the graphs for functions $B_1(x)$ and $B_2(x)$ equations (10) and (11), respectively. This set of telescoping functions shows multiple minima and maxima for –1 < y < 1.

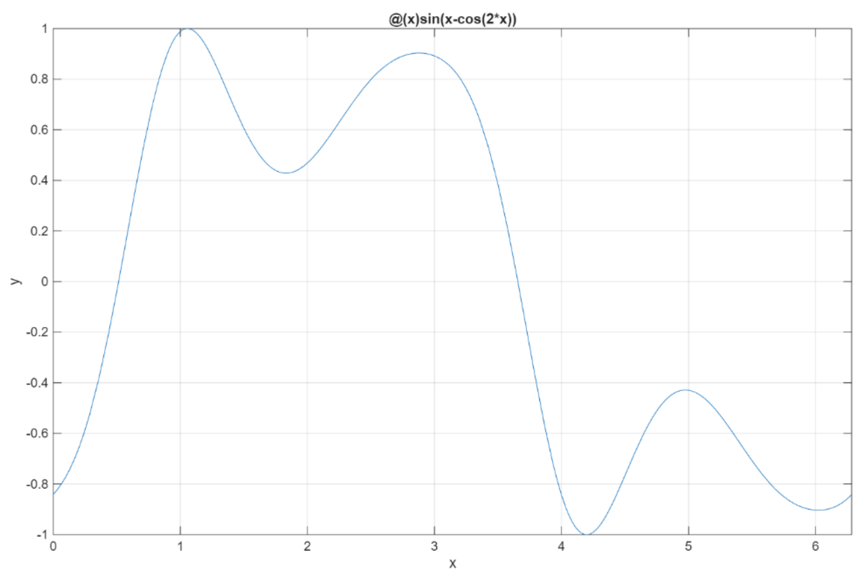

Figure 8. The curve for sin(x – cos(2*x)).

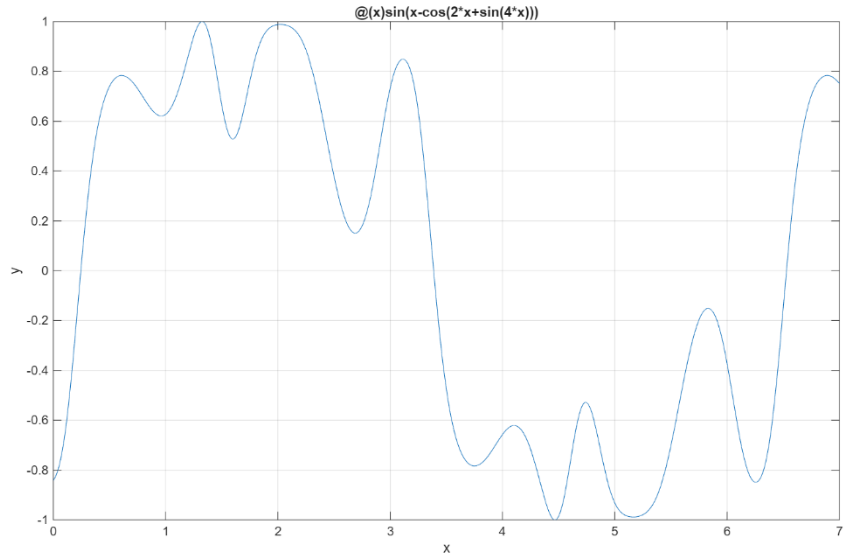

*Figure 9. The curve for sin(x – cos(2*x + sin(4*x))).*

Figures 10 and 11 show the graphs for functions $M_1(x)$ and $M_2(x)$ equations (12) and (11), respectively. This set of telescoping functions shows multiple minima and maxima for $-1 < y < 1$.

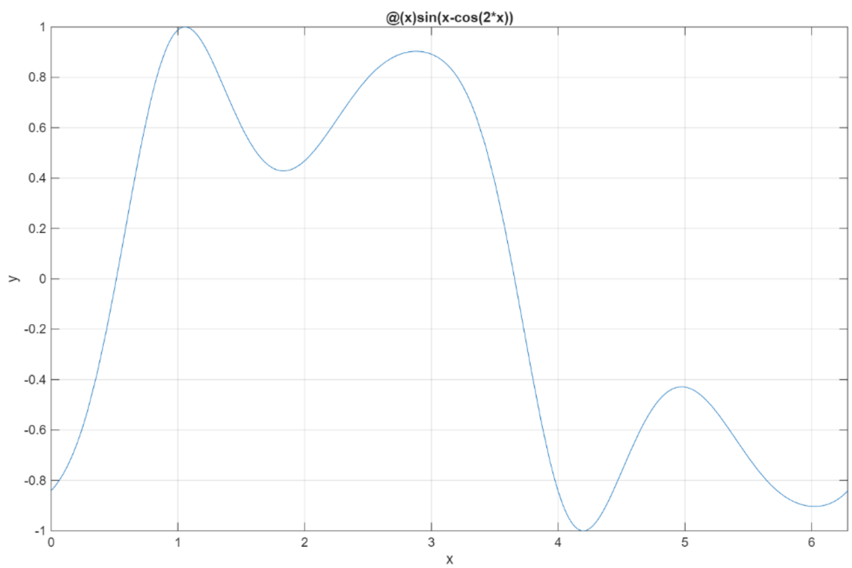

Figure 10. The curve for sin(x – cos(2*x)).

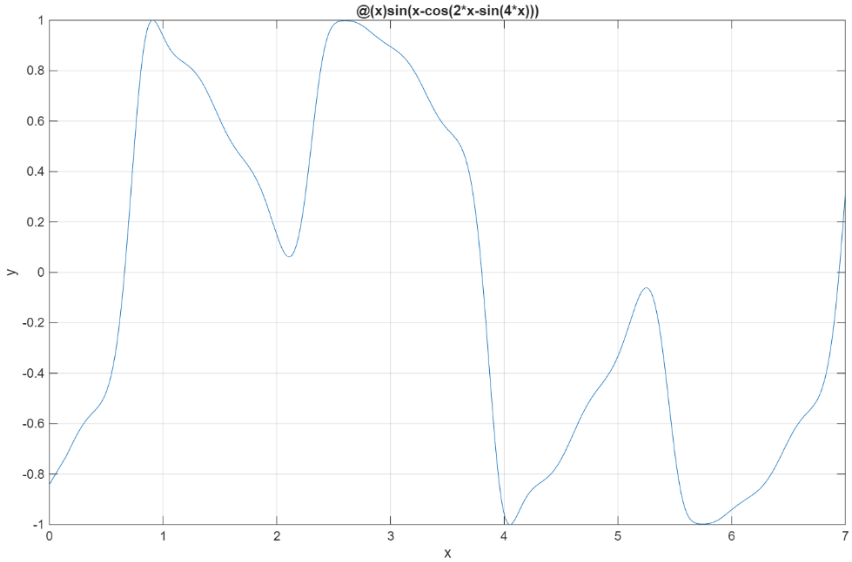

*Figure 11. The curve for sin(x – cos(2*x – sin(4*x))).*

The functions that I presented above calculate values of sine functions with expressions containing nested cosine and sine functions. You are welcome to reverse the order of using the trigonometric functions and make your custom functions calculate the cosine of expressions containing nested sine and cosine functions.

Third Sample Telescoping Functions

Yet another set of functions $R_n(x)$, $C_n(x)$, $O_n(x)$ are like the functions $Q_n(x)$,B(x), and $M_n(x)$, except they multiply x by factors of 2, 3, 4, and so on.

$R_1(x) = \sin(x + \cos(2*x))$ (14)
$R_2(x) = \sin(x + \cos(2*x + \sin(3*x)))$ (15)
$C_1(x) = \sin(x - \cos(2*x))$ (16)
$C_2(x) = \sin(x - \cos(2*x + \sin(3*x)))$ (17)
$O_1(x) = \sin(x - \cos(2*x))$ (18)
$O_2(x) = \sin(x - \cos(2*x - \sin(3*x)))$ (19)

Functions $R_1(x)$, $C_1(x)$ and $O_1(x)$ are identical to $Q_1(x)$, $B_1(x)$ and $M_1(x)$ respectively. So, I will focus on functions $R_2(x)$, $C_2(x)$ and $O_2(x)$.

Figure 12 shows the graphs for function $R_2(x)$. This telescoping function shows multiple minima and maxima for $-1 < y < 1$.

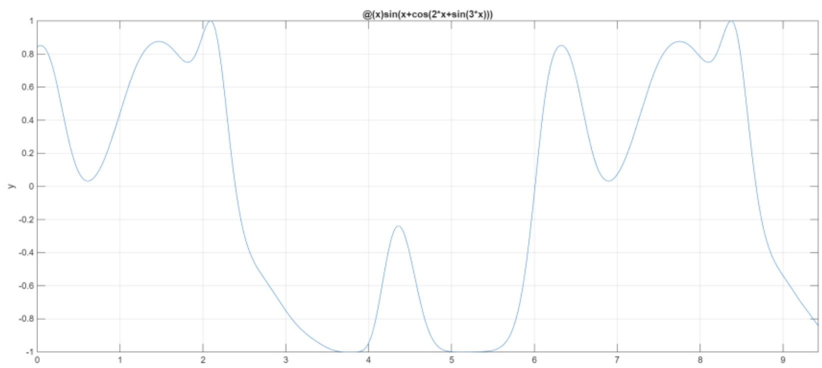

*Figure 12. The curve sin(x + cos(2*x + sin(3x))).*

Figure 13 shows the graphs for function $C_2(x)$. This telescoping function shows multiple minima and maxima for $-1 < y < 1$.

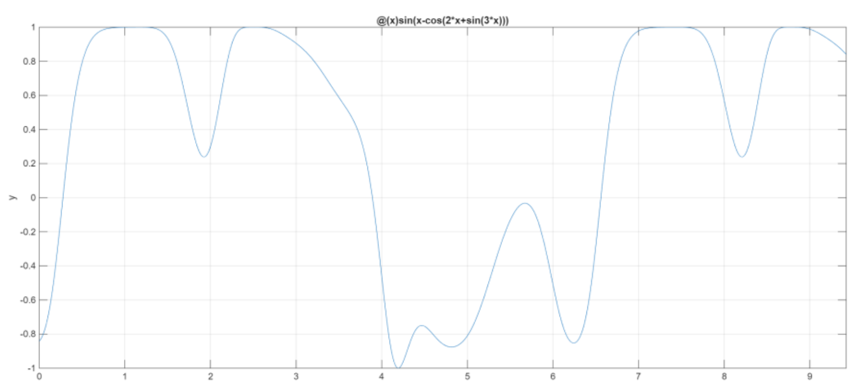

*Figure 13. The curve for sin(x – cos(2*x + sin(3*x))).*

Figure 14 shows the graphs for function $O_2(x)$. This telescoping function shows multiple minima and maxima for $-1 < y < 1$.

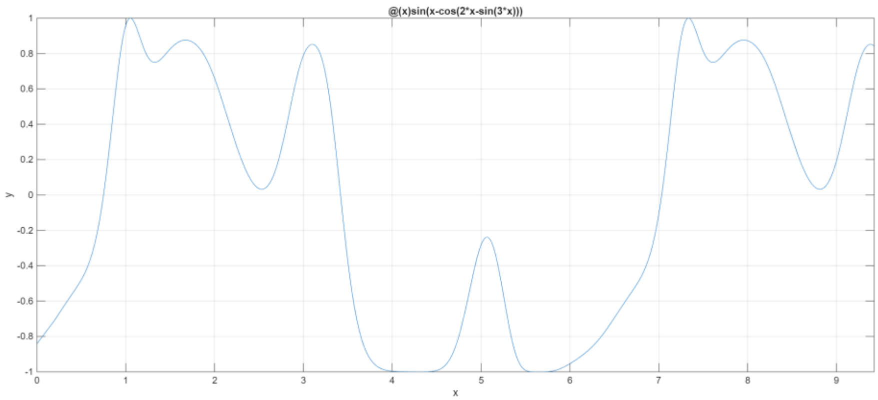

*Figure 14. The curve for sin(x – cos(2*x – sin(3*x))).*

Fourth Sample Telescoping Functions

A fourth set of telescoping functions $S_n(x)$, $D_n(x)$, $Z_n(x)$ are similar to the functions $R_n(x)$, $C_n(x)$, $O_n(x)$, except they use the square root of x and multiply it by factors of 2, 3, 4, and so on.

$$S_1(x) = \sin(\sqrt{x} + \cos(2*\sqrt{x})) \quad (20)$$
$$S_2(x) = \sin(\sqrt{x} + \cos(2*\sqrt{x} + \sin(3*\sqrt{x}))) \quad (21)$$
$$D_1(x) = \sin(\sqrt{x} - \cos(2*\sqrt{x})) \quad (22)$$
$$D_2(x) = \sin(\sqrt{x} - \cos(2*\sqrt{x} + \sin(3*\sqrt{x}))) \quad (23)$$
$$Z_1(x) = \sin(\sqrt{x} - \cos(2*\sqrt{x})) \quad (24)$$
$$Z_2(x) = \sin(\sqrt{x} - \cos(2*\sqrt{x} - \sin(3*\sqrt{x}))) \quad (25)$$

Figure 15 and 16 shows the graphs for functions $S_1(x)$ and $S_2(x)$. Using the factors of 2, 3, and 4 and also applying the square of x allows these functions to show fewer minima and maxima that the functions in the previous two sets.

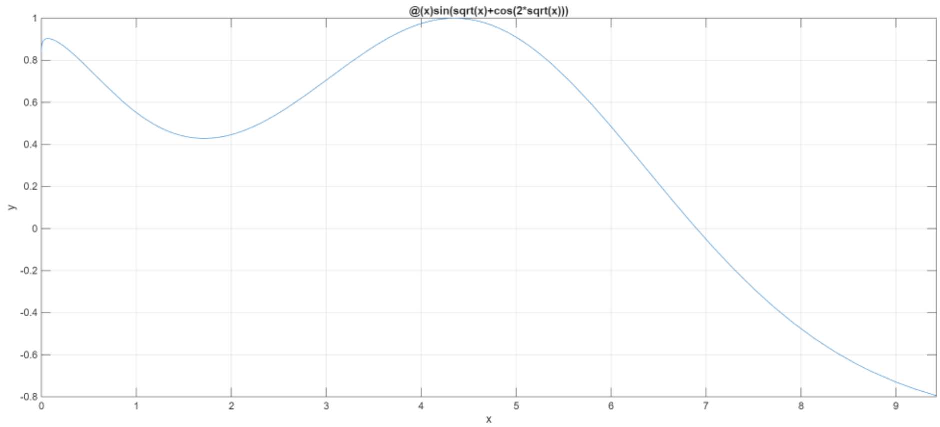

Figure 15. The curve for sin(√x + cos(2√x)).*

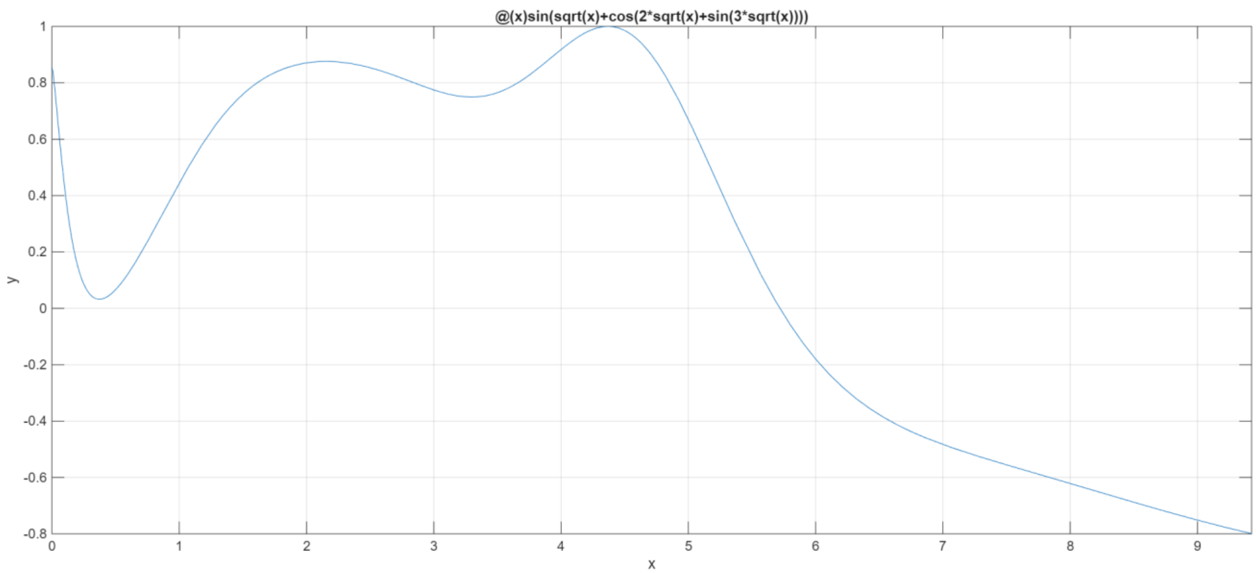

Figure 16. The curve for sin(√x + cos(2√x + sin(3*√x))).*

Figure 17 and 18 shows the graphs for functions $D_1(x)$ and $D_2(x)$. Using the factors of 2, 3, and 4 and also applying the square of x allows these functions to show fewer minima and maxima that the functions in the previous two sets.

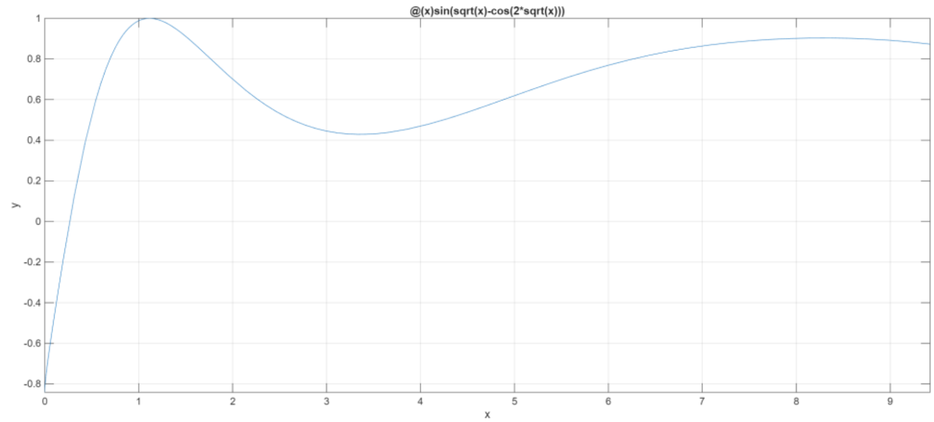

Figure 17. The curve for sin(√x – cos(2√x)).*

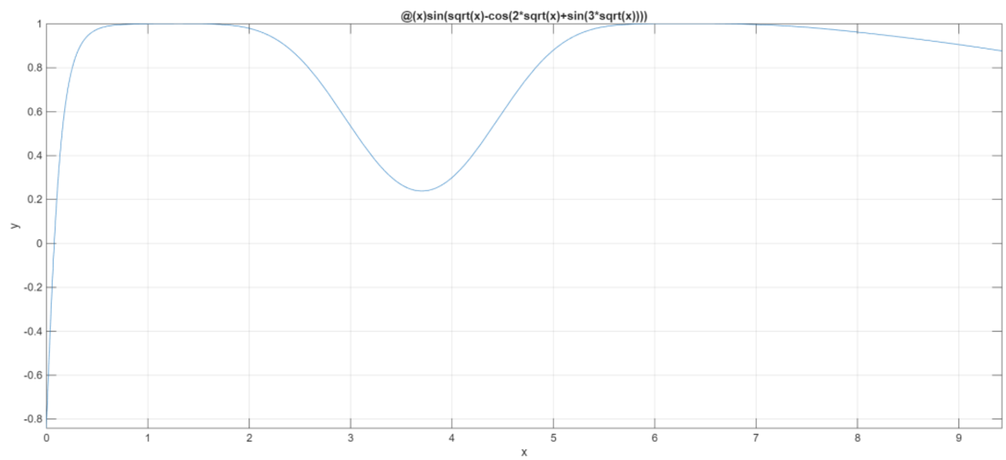

Figure 18. The curve for sin(√x – cos(2√x + sin(3*√x))).*

Figure 20 shows the graphs for functions $Z_2(x)$. Using the factors of 2, 3, and 4 and also applying the square of x allows these functions to show fewer minima and maxima that the functions in the previous two sets.

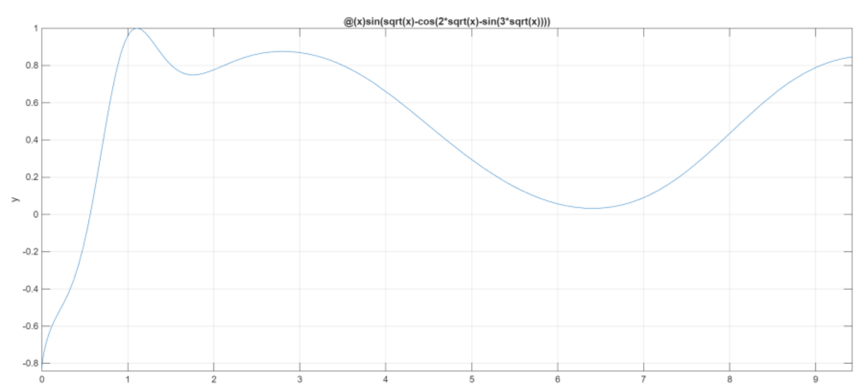

Figure 19. The curve for sin(√x – cos(2*√x – sin(3*√x)))

A 3D Sample Telescoping Functions

This last section gives you a heads up regarding bivariate telescoping trigonometric functions. Remember that the generality of equation (1) can lead to a vast number of single-variable telescoping trigonometric functions. Now add a second variable and the number of possible combinations of functions will explode in an overwhelming way. To keep things simple, I present a simple example, the function W(x,y) = sin(x + cos(y + sin(2*x + cos(2*y + sin(3*x + cos(3*y)))))). Figures 20, 21, and 22 give you the three-dimensional plot of W(x,y) as seen from different angles. The surface in these figures would be quite the challenge for 3D modeling!

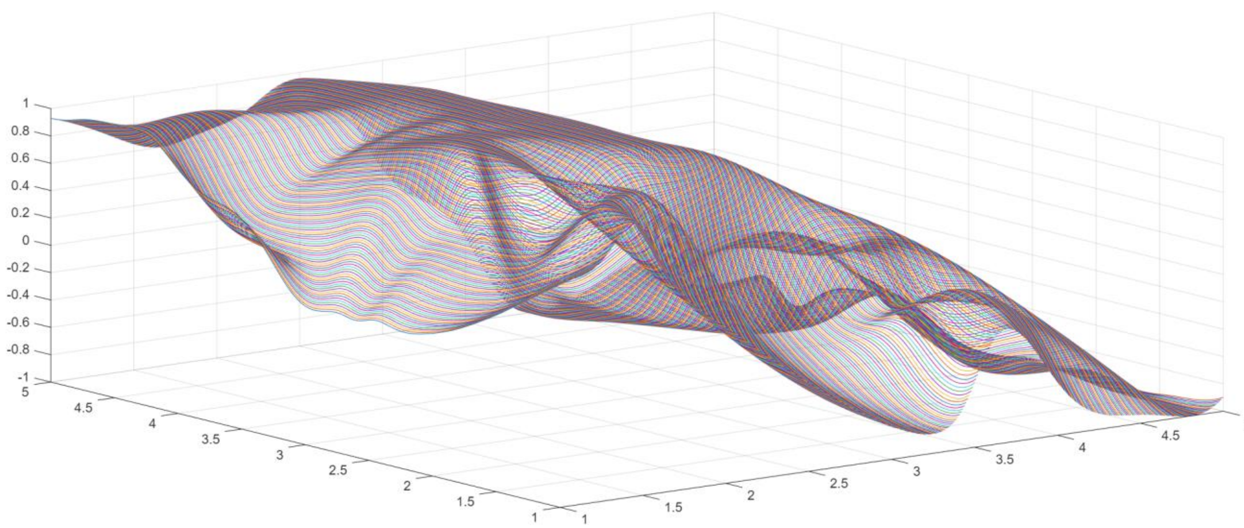

*Figure 20. A view of the curve for sin(x + cos(y + sin(2*x + cos(2*y + sin(3*x + cos(3*y)))))).*

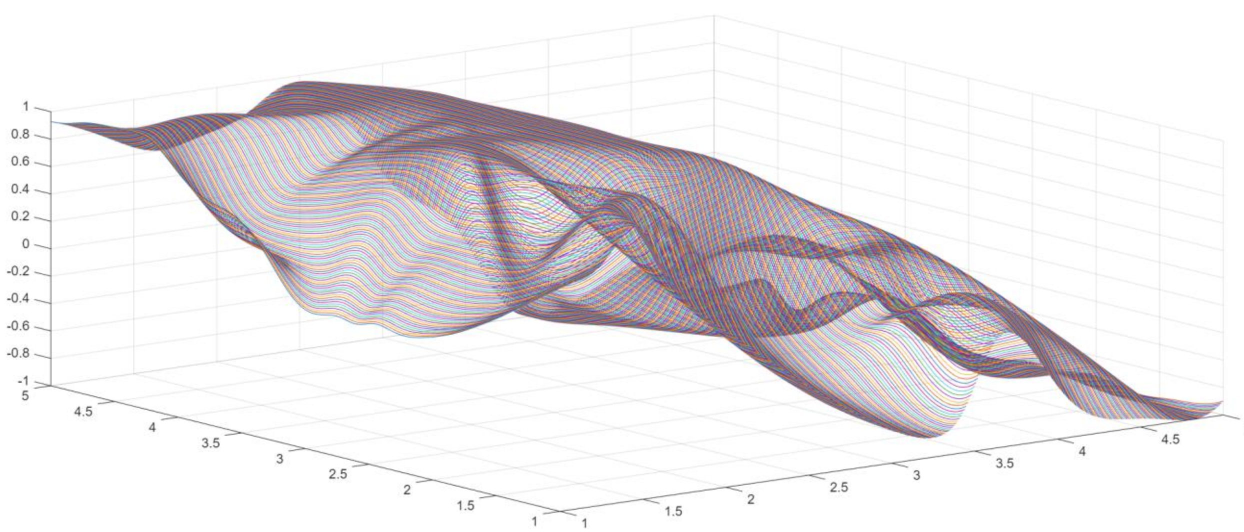

*Figure 21. A second view of the curve for sin(x + cos(y + sin(2*x + cos(2*y + sin(3*x + cos(3*y)))))).*

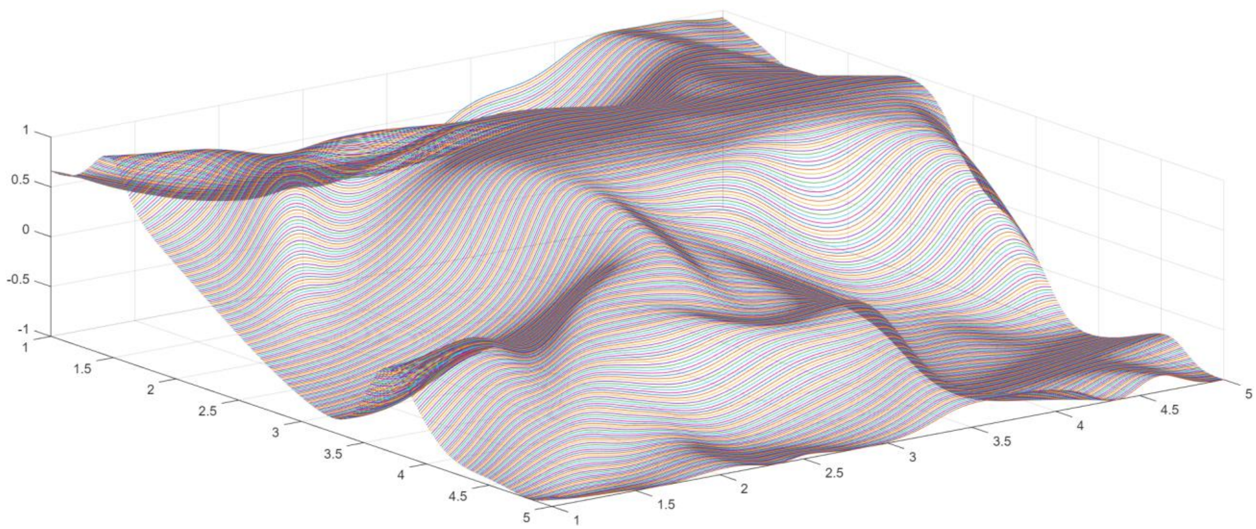

*Figure 22. A third view of the curve for sin(x + cos(y + sin(2*x + cos(2*y + sin(3*x + cos(3*y)))))).*

So, what happens to function W(x,y) if we divide the factors 2 and 3 instead of multiplying them. We get V(x,y) = sin(x + cos(y + sin(x/2 + cos(y/2 + sin(x/3 + cos(y/3)))))). Figure 23 shows the graph of the updated function V(x,y). This function has a smooth surface compared to that of function W(x,y). They would easily be fit by quadratic and cubic bivariate polynomials.

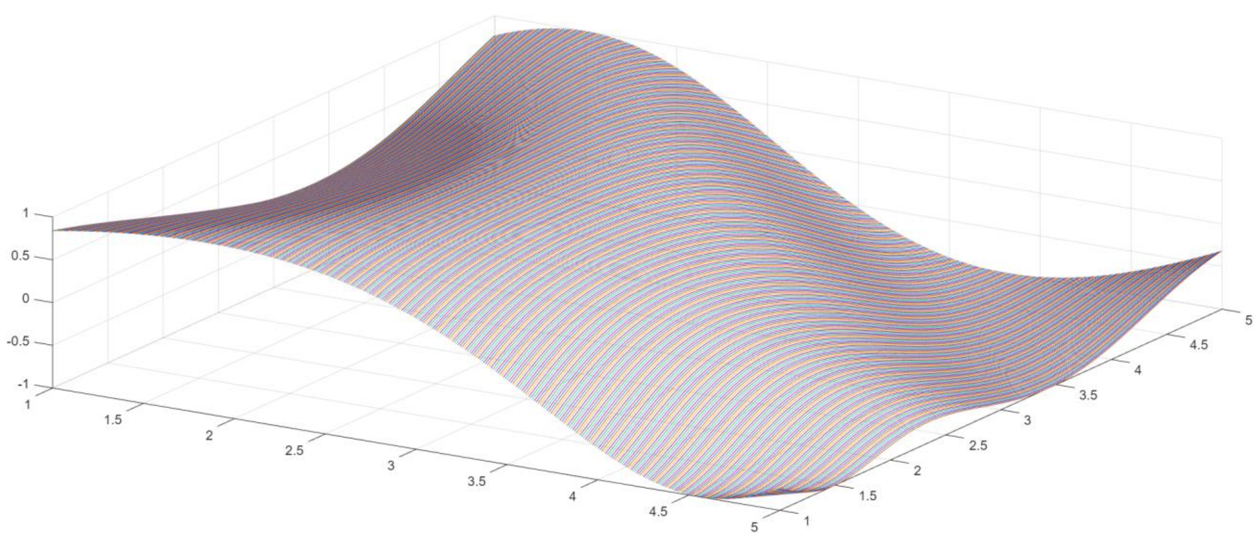

Figure 23. A third view of the curve for sin(x + cos(y + sin(x/3 + cos(y/2 + sin(x/3 + cos(y/3)))))).

Conclusions

The single-variable telescoping functions that I presented above calculate values of sine functions with expressions containing nested cosine and sine functions. You are welcome to reverse the order of using the trigonometric functions and make your custom functions calculate the cosine of expressions containing nested sine and cosine functions.

The second and third sets of single-variable telescoping trigonometric functions show more curvature that provides curves to challenge empirical curve fitting tools. Such curves may not be closely fit by lower (or even higher) order polynomials, which makes using empirical curve fit tools justifiable. The fourth set of single-variable telescoping trigonometric functions seems to lend itself to be fit with polynomials.

Keep in mind that equation (1) is always the go-to-equation if you desire to implement telescoping trigonometric functions other than the ones I presented. Fo example you can use F(x, [1, –1, –1], [2, 5, ¼]) to represent:

MyFx(x) = sin(x + cos(2*x – sin(5*x – cos(x/4))))