Pade-Shammas Polynomial Approximations Part IB

By

Namir C. Shammas

INTRODUCTION

This study looks at the collaboration of two mathematical frameworks—the Pade polynomials and the Shammas Polynomials. The study combines both frameworks to approximate several common functions. This study uses x and y data (calculated from the functions being approximated) without any data transformation. Please do not be initimidated by the page count of this study. Most of the pages contain

calculation results. You can quickly browse through this document. You are probably best served by focusing on the tables that summarize the results of fitting various functions with Pade-Shammas polynomials.

The Pade Polynomials

Pade polynomials are polynomial ratios defined as:

$$y = P_{m,n}(x) = \frac{Q_m(x)}{D_n(x)} \quad (1)$$

Where $Q_m(x)$ is defined as:

$$Q_m(x) = a_0 + \sum_{i=1}^{m} a_i\, x^i \quad (2)$$

And $D_n(x)$ is defined as:

$$D_n(x) = 1 + \sum_{i=1}^{n} b_i\, x^i \quad (3)$$

The Pade polynomials have more flexibility that ordinary legacy polynomials. The trick is to find the optimum values for m and n which are the orders of the polynomials $Q_m(x)$ and $D_n(x)$. This study uses swarm optimization to determine these optimum values within the range of (2, 7). The general multiple regression model used to fit Pade polynomials is:

$$y = a_0 + \sum_{i=1}^{m} a_i\, x^i - \sum_{i=1}^{n} b_i\, y\, x^i \quad (4)$$

Or,

$$y = a_0 + \sum_{i=1}^{m} a_i\, x^i + \sum_{i=1}^{n} c_i\, y\, x^i \quad (5)$$

Where c(i) = –b(i). This study uses the model in equation 5.

The Shammas Polynomials

In the HHC 2008 conference in Corvallis, Oregon, I introduced the Shammas polynomials as polynomials with non-integer powers. I explained that the powers of such polynomial change using some math expressions that involves the polynomial term sequence number. Moreover, the study explained that such expressions must never produce the same powers of two or more terms. In the 2008 presentation I showed that the Shammas polynomials can have multiple terms with a smaller power range compared to regular polynomials. This feature allows better curve fitting with Shammas polynomials that avoids the instability of raising large or small numbers to high powers. If you are not familiar with Shammas

polynomials, then you can download my 2008 presentation using the following link:

http://www.namirshammas.com/NEW/ShammasPolynomials.pdf.

A Shammas polynomial has the following general form:

$$y = a_0 + \sum_{i=1}^{n} a_i \, x^{gx(i,A,B,C)} \quad (6)$$

Where A, B, and C are fixed constants. Using C is optional. The function gx(i, A, B, C) supplies the math expressions used to calculate the powers of the Shammas polynomial. The powers can be integers. In most cases, the powers are non-integers.

The Pade-Shammas Polynomials

The Pade-Shammas polynomials combine Pade and Shammas polynomials concepts. They have the following multiple regression general form:

$$y = a_0 + \sum_{i=1}^{m} a_i \, x^{gx1(i,A,B,C)} - \sum_{i=1}^{n} b_i \, y \, x^{gx2(i,A,B,C)} \quad (7)$$

Notice that the integral powers in normal Pade polynoomials are replaced with power functions gx1() and gx2() that can be identical or different. The latter type gives the Pade-Shammas polynomials, in general, more flexibility in fitting a given set of (x, y) data. This statement is truer when the values of A, B, and C (when used) are different for the gx1() and gx2() functions. Part II of this study uses this scheme to give more flexibility to Pade-Shammas polynomials. Also note that Part IB conducts the Pade-Shammas polynomial fitting on the same functions but without normalizing the x and y data.

When using the Pade-Shammas polynomials to calculate new values employee the following equation:

$$y = [a_0 + \sum_{i=1}^{m} a_i \, x^{gx1(i,A,B,C)}] \,/ [1 + \sum_{i=1}^{n} b_i \, x^{gx2(i,A,B,C)}] \quad (8)$$

I am appending my last name to the name Pade in an effort to make sure it is a unique combined name. Using a more general descriptive name risks conflicts with other mathematicians that may use the same descriptive name for different types of Pade polynomials. I certainly would like to avoid unpleasant communications with these fine mathematicians!

This study looks at Pade-Shammas polynomials used to approximate common functions that include:

- Trigonometric functions and their inverses.
- Hyperbolic functions.
- Logarithmic functions.
- Exponential functions.
- Bessel functions $J_0(x)$ to $J_5(x)$.
- The sine and cosine integrals.
- The Fresnel sine and cosine.
- Inverse student-t functions.
- The common logarithm of the gamma function.
- The digamma function.
- The trigamma function.

The digamma function is defined as:

$$\psi(x) = \Gamma'(x) / \Gamma(x) = \frac{d \ln(\Gamma(x))}{dx} \tag{9}$$

The following Matlab function implements the code for the digamma function:

```
function y = digamma(x)
%DIGAMMA Summary of this function goes here
%   Detailed explanation goes here
  h = 0.001;
  fp = gammaln(x+h);
  fm = gammaln(x-h);
  y = (fp -fm)/2/h;
end
```

The above implementation of the digamma function was suggested by Albert Chan, a member of the hp museum web site, in a post he wrote on that site. The above code gives slightly more accurate results than the expression (gamma(x+h)-gamma(x-h)/(2*h)/gamma(x).

The trigamma function is defined as:

$$\psi_1(x) = \frac{d^2}{dx^2} \ln(\Gamma(x)) \tag{10}$$

The following Matlab function implements the code for the trigamma function:

```
function y = trigamma(x)
%DIGAMMA Summary of this function goes here
```



```
%   Detailed explanation goes here
  h = 0.001;
  fp = gammaln(x+h);
  fm = gammaln(x-h);
  f0 = gammaln(x);
  y = (fp -2*f0 + fm)/h/h;
end
```

The study works with Pade-Shammas Polynomials that use the following ten equations to calculate the powers of Shammas polynomial terms:

Power = A + B*i (11)

Power = A + B/i (12)

Power = A+B*sqrt(i) (13)

Power = A+B*sqrt(i+C) (14)

Power = A+B*i+C*(i–1) (15)

Power = A+B*i–C*(i–1) (16)

Power = A+B*i+C*sqrt(i–1) (17)

Power = A+B*sqrt(i)+C*(i-1) (18)

Power = A+B*sqrt(i)+C*sqrt(i–1) (19)

Power = A + B * $[\log_{10}(i)]^4$ (20)

Where A, B, and C are constants, and i a variable that represents the term number in a Pade-Shammas polynomial. The last equation introduces a new power expression not used in the Shammas Polynomial approximation study.

There are several choices that we can use to determine the best Pade-Shammas polynomial orders and the best values for parameters A, B, and C. The best one is to use an optimization function to determine the best Pade-Shammas polynomial order and the best values for parameters A, B, and C. The function would use ranges for the polynomial orders as well as the values for A, B, C.

The study uses the Matlab particle swarm optimization function to select the best Shammas polynomial orders (for $Q_n(x)$ and $D_m(x)$) along with the best values for A, B, and C.

Please note that I use the following model to perform the multiple regression:

$$y = a_0 + \sum_{i=1}^{m} a_i\, x^{gx1(i,A,B,C)} + \sum_{i=1}^{n} b_i\, y\, x^{gx2(i,A,B,C)} \quad (20)$$

I added the second set of summations instead of subtracting them (as in equation 6). Therefore, when you use the regression coefficients, use the following equation:

$$y = [a_0 + \sum_{i=1}^{m} a_i\, x^{gx1(i,A,B,C)}] / [1 - \sum_{i=1}^{n} b_i\, x^{gx2(i,A,B,C)}] \quad (21)$$

Notice the minus sign after the constant 1 in equation 21. I use equation 21 in the Matlab functions PadeShamPoly2() and PadeShamPoly3() that I present in the next section.

✍ The approximations that I obtain are typically for a defined and suitable interval. It is your responsibility to implement expanded versions of the approximation functions that take wider ranges if arguments and map them onto the interval used. For example, given that my approximation for $\log_{10}(x)$ uses the range (1, 10), to calculate $\log_{10}(235)$ use:

$\log_{10}(235) =$

$\log_{10}(2.35 * 100) =$

$\log_{10}(2.35) + \log_{10}(100) =$

$\log_{10}(2.35) + 2$

The argument of the $\log_{10}(x)$ function in the last line falls in the interval (1, 10).

MATLAB CODE

The algorithm in this study uses particle swarm optimization to obtain the best values for parameters A and B in a prespecified range of Pade-Shammas polynomial orders for $Q_n(x)$ and $D_m(x)$.

This study uses the function PadeShamPoly2 () to perform various Pade-Shammas polynomial curve fitting:

```
function PadeShamPoly2(fx,gx,xRange,Lb,Ub,runNum,sFxName,diaryFilename)
%PADESHAMPOLY2 Summary of this function goes here
%   Detailed explanation goes here
clc
global bDeleteIfExists
global bUseDiary
global xdata
global ydata
global orderA
```



```
global orderB
global ggx

warning('off','all')
if isempty(sFxName)
  sFxName = getFuncName(fx);
end
xdata = xRange';
ydata = xdata;
for i=1:length(xdata)
  ydata(i)=fx(xdata(i));
end
ggx = gx;

xmin = min(xdata);
xmax = max(xdata);
fprintf('Fitting %s in range (%f, %f)\n', sFxName, xmin ,xmax);

options = optimoptions('particleswarm', 'Display', 'iter');
[x,psAICc] = particleswarm(@optimFunc,4,Lb,Ub,options);
A = x(1);
B = x(2);
orderA = round(x(3));
orderB = round(x(4));
if bUseDiary
  diaryFilename = strrep(diaryFilename, ".txt", strcat("_",
num2str(orderA),"_", num2str(orderB),"_run", num2str(runNum),".txt"));
  if exist(diaryFilename, 'file')==2
    if bDeleteIfExists
      delete(diaryFilename);
    else
      return;
    end
  end
end
X = [];
for i=1:orderA
  xs = xdata.^gx(i,A,B);
  X = [X;xs'];
end
for i=1:orderB
  xs = ydata.*(xdata.^gx(i,A,B));
  X = [X;xs'];
end
X = X';
lm = fitlm(X,ydata);
if bUseDiary
  diary(diaryFilename)
end
fprintf('Fitting %s in range (%f, %f)\n', sFxName, xmin ,xmax);
sp = getFuncName(gx);
fprintf('Pade-Shammas polynomial power is %s\n', sp);
format long
disp(lm);
anva = anova(lm,'summary');
disp(anva);
format short
```



```
fprintf('A = %f, B = %f\n', A, B);
fprintf('order Q(x) = %f, order D(x) = %f\n', orderA, orderB);
fprintf("Xmin = %f and Xmax = %f\n", xmin, xmax);
fprintf("Ymin = %f and Ymax = %f\n", ymin, ymax);
lstPwrA = zeros(orderA,1);
for i=1:orderA
  lstPwrA(i) = gx(i,A,B);
end
lstPwrB = zeros(orderB,1);
for i=1:orderB
  lstPwrB(i) = gx(i,A,B);
end
fprintf('List of powers for Q(x): [');
for i=1:orderA-1
 fprintf('%f, ', lstPwrA(i));
end
fprintf('%f]\n', lstPwrA(orderA));
fprintf('List of powers for D(x): [');
for i=1:orderB-1
 fprintf('%f, ', lstPwrB(i));
end
fprintf('%f]\n', lstPwrB(orderB));
fprintf('Fitting %s in range (%f, %f)\n', sFxName, xmin ,xmax);
n = length(xdata);
sumsqr = 0;
for i=1:n
  qx = lm.Coefficients{1,1};
  for j=2:orderA+1
    qx = qx + lm.Coefficients{j,1} * xdata(i)^lstPwrA(j-1);
  end
  dx = 1;
  k = 1;
  for j=orderA+2:orderA+1+orderB
    dx = dx - lm.Coefficients{j,1} * xdata(i)^lstPwrB(k);
    k = k + 1;
  end
  yc = qx/dx;
  sumsqr = sumsqr + (ydata(i) - yc)^2;
end
k = orderA + orderB + 1;
fprintf('MSS of errors squared = %e\n', sqrt(sumsqr)/n);
fprintf('Corrected MSS of errors squared = %e\n', sqrt(sumsqr)/n*sqrt(2));
fprintf("R-Squared = %12.8f\n", lm.Rsquared.Ordinary);
fprintf("R-Squared Adjusted = %12.8f\n", lm.Rsquared.Adjusted);
AIC = lm.ModelCriterion.AIC;
AICc = AIC + 2*k*(1 + (k+1)/(n-k-1));
fprintf('Particle swarm AICc =  %e\n', psAICc);
fprintf('AIC =  %e\n', AIC);
fprintf('AICc =  %e\n', AICc);

if bUseDiary
  diary off
end
end

function AICc = optimFunc(x)
  global xdata
```



```
  global ydata
  global orderA
  global orderB
  global ggx

  A = x(1);
  B = x(2);
  orderA = round(x(3));
  orderB = round(x(4));
  X = [];
  for i=1:orderA
    xs = xdata.^ggx(i,A,B);
    X = [X;xs'];
  end
  for i=1:orderB
    xs = ydata.*(xdata.^ggx(i,A,B));
    X = [X;xs'];
  end
  X = X';
  lm = fitlm(X,ydata);

  n = length(xdata);
  k = orderA + orderB + 1;
  AIC = lm.ModelCriterion.AIC;
  AICc = AIC + 2*k*(1 + (k+1)/(n-k-1));
  if isinf(AICc), AICc = -1e+99; end
end

function sFx = getFuncName(fx)
  sFx = func2str(fx);
  if sFx(1:2)=="@("
    i = strfind(sFx,")");
    sFx = sFx(i(1)+1:end);
  elseif sFx(1)=="@"
    sFx = strcat(sFx(2:end),".m");
  else
    % return sFx as is
  end
end
```

The parameters of function PadeShamPoly2 () are:

- The parameter fx is the handle (or inline function) for the function being approximated. An example is @(x)cos(x) which also shows the ***recommended format*** for the argument of parameter fx.
- The parameter gx is the handle (or inline function) for the function that calculates the powers of the Shammas polynomial. An example is @(i,A,B)A+B*sqrt(i) which also shows the ***recommended format*** for the argument of parameter gx.
- The parameter xRange is the array that specifies the minimum value, increment value, and maximum value for the range of approximation.

- The parameter Lb is is the array of lower limits for the parameters A and B, and the Shammas polynomial order. An example is [0.1 1 3].
- The parameter Ub is is the array of upper limits for the parameters A and B, and the Shammas polynomial order. An example is [1 4 5].
- The parameter runNum specifies the run number. The arguments for this parameter have nothing to do with the calculations and serve in fine tuning the name of the diary files, when used.
- The optional parameter sFxName is the name of the approximated function. An example is “cos(x)”.
- The parameter diaryFilename is the name of the diary file. An example is “cos_1.txt”.

The above listing performs the following tasks:

1. Initialize the data for the curve fitting. The function uses the global variables xdata and ydata to store the data for the polynomial fitting.
2. Store the handle of function gx in the global handle ggx.
3. Set the optimization options and then call the Matlab function particleswarm(). The function call returns the optimized values of A and B and the optimum value for *corrected* Akaike information criterion (AICc). The arguments for this function call are:
    a. The handle to the local function optimFunc() that calculates the root mean sum of errors squared.
    b. The number of optimized variables which is 2.
    c. The lower and upper bounds arrays, Lb and Ub, respectively,
    d. The optimization parameters for function particleswarm().
4. Retrieve the optimum values and perform a Shammas polynomial fit for the best values of A and B.
5. Display the results of the regression and its associated ANOVA table.
6. Display the list of Shammas polynomial powers.
7. Display the range of the approximated function.
8. Calculate and display the value of the mean square root of the sum of squared errors. This statistic serves as a check that the Shammas polynomial performs well in checking the training data.
9. Calculate and display the value of the *corrected* mean square root of the sum of squared errors. This value equals the mean square root of the sum of squared errors multiplied by the square root of the number of Shammas

polynomial parameters (i.e. A, B, and C, if used). In the case of PadeShamPoly2(), the function uses square root of 2.

10. Calculate and display the *corrected* Akaike information criterion. This statistic is calculated using:

$$AIC = n * \ln(SSE/n) + 2*k \quad (8)$$

$$AICc = AIC + 2*k*(k+1)/(n-k-1) \quad (9)$$

Where n is the number of observations, k is the total number of regression coefficients (including the intercept), and SSE is the sum of squared errors. The program obtains the value of AIC using lm.ModelCriterion.AIC. The program uses equation (3) to calculate the value for AICc.

11. Close the diary file, if one is used.

The function optimFunc() obtains the array x containing the current values of A and B, and the best Shammas polynomial order. The function calculates the transformed variables needed to perform a curve fit for a Shammas polynomial. This task calls the Matlab function fitlm(). The optimFunc() function returns the AICc as its result. I am using this statistic since the optimization is dealing with different Shammas polynomial orders and thus a varying number of polynomial coefficients.

The function getFuncName() returns a string-type function name given a handle of a function. The best way to take advantage of this function is to supply arguments like @(x)cos(x) and @(x,A,B)A+B*sqrt(x). Such arguments allow the function to discard the part that declares the variable(s) and return the part that comes after the first closed parenthesis (e.g. cos(x) and A+B*sqrt(x) for the above examples). If you supply an argument like @fx1 which refers to the file fx1.m that defines the function fx1() then the function getFuncName() returns *fx1.m.* This string value indicates that you a rereferencing a separate Matlab file that implements the code for fx1.

It is worth going over the code that estmates the values of y to calculate the sum of error squared:

```
n = length(xdata);
sumsqr = 0;
for i=1:n
  qx = lm.Coefficients{1,1};
  for j=2:orderA+1
    qx = qx + lm.Coefficients{j,1} * xdata(i)^lstPwrA(j-1);
  end
```



```
  dx = 1;
  k = 1;
  for j=orderA+2:orderA+1+orderB
    dx = dx - lm.Coefficients{j,1} * xdata(i)^lstPwrB(k);
    k = k + 1;
  end
  yc = qx/dx;
  sumsqr = sumsqr + (ydata(i) - yc)^2;
end
```

The above code used the arrays lstPwrA and lstPwrB to store the powers used in the Shammas polynomials $Q_n(x)$ and $D_m(x)$, respectively. The first for loop calculate the value for Shammas polynomials $Q_n(x)$ using variable qx. Notice that the code initializes the value of qx using the intercept of the regression model. The loop accesses the powers of $Q_n(x)$ using lstPwrA(j-1). The second for loop calculates the value for Shammas polynomials $D_m(x)$using variable dx. Notice that the code initializes the value of dx using the constant 1. The loop accesses the powers of $Q_n(x)$ using lstPwrB(k). Here k is initialized with the value 1 and is incremented in the for loop. Also noticed that the value of variable dx is decremented with the expression lm.Coefficients{j,1} * xdata(i)^lstPwrB(k). I have explained the above code because you may need similar code to estimate other values of y given an array (or scalar) of x, as shown below:

```
n = length(your_xdata);
for i=1:n
  qx = lm.Coefficients{1,1};
  for j=2:orderA+1
    qx = qx + lm.Coefficients{j,1} * your_xdata(i)^lstPwrA(j-1);
  end
  dx = 1;
  k = 1;
  for j=orderA+2:orderA+1+orderB
    dx = dx - lm.Coefficients{j,1} * your_xdata(i)^lstPwrB(k);
    k = k + 1;
  end
  your_y(i) = qx/dx;
end
```

Under the current calculation scheme, note that, the function PadeShamPoly2 () does not explicitly iterate over different values of A and B. It delegates that task to the Matlab function particleswarm().

The study also uses the function PadeShamPoly3() to work with Shammas polynomial powers that use parameters A, B, and C:

```
function PadeShamPoly3(fx,gx,xRange,Lb,Ub,runNum,sFxName,diaryFilename)
%PADESHAMPOLY3 Summary of this function goes here
%   Detailed explanation goes here
clc
```



```
global bDeleteIfExists
global bUseDiary
global xdata
global ydata
global orderA
global orderB
global ggx

warning('off','all')
if isempty(sFxName)
  sFxName = getFuncName(fx);
end
xdata = xRange';
ydata = xdata;
for i=1:length(xdata)
  ydata(i)=fx(xdata(i));
end
ggx = gx;

xmin = min(xdata);
xmax = max(xdata);
fprintf('Fitting %s in range (%f, %f)\n', sFxName, xmin ,xmax);

options = optimoptions('particleswarm', 'Display', 'iter');
[x,psAICc] = particleswarm(@optimFunc,5,Lb,Ub,options);
A = x(1);
B = x(2);
C = x(3);
orderA = round(x(4));
orderB = round(x(5));
if bUseDiary
  diaryFilename = strrep(diaryFilename, ".txt", strcat("_",
num2str(orderA),"_", num2str(orderB),"_run", num2str(runNum),".txt"));
  if exist(diaryFilename, 'file')==2
    if bDeleteIfExists
      delete(diaryFilename);
    else
      return;
    end
  end
end
X = [];
for i=1:orderA
  xs = xdata.^gx(i,A,B,C);
  X = [X;xs'];
end
for i=1:orderB
  xs = ydata.*(xdata.^gx(i,A,B,C));
  X = [X;xs'];
end
X = X';
lm = fitlm(X,ydata);

if bUseDiary
  diary(diaryFilename)
end
fprintf('Fitting %s in range (%f, %f)\n', sFxName, xmin ,xmax);
```



```
sp = getFuncName(gx);
fprintf('Pade-Shammas polynomial power is %s\n', sp);
format long
disp(lm);
anva = anova(lm,'summary');
disp(anva);
format short
fprintf('A = %f, B = %f, C = %f\n', A, B, C);
fprintf('order Q(x) = %f, order D(x) = %f\n', orderA, orderB);
fprintf("Xmin = %f and Xmax = %f\n", xmin, xmax);
fprintf("Ymin = %f and Ymax = %f\n", ymin, ymax);
lstPwrA = zeros(orderA,1);
for i=1:orderA
  lstPwrA(i) = gx(i,A,B,C);
end
lstPwrB = zeros(orderB,1);
for i=1:orderB
  lstPwrB(i) = gx(i,A,B,C);
end
fprintf('List of powers for Q(x): [');
for i=1:orderA-1
 fprintf('%f, ', lstPwrA(i));
end
fprintf('%f]\n', lstPwrA(orderA));
fprintf('List of powers for D(x): [');
for i=1:orderB-1
 fprintf('%f, ', lstPwrB(i));
end
fprintf('%f]\n', lstPwrB(orderB));
fprintf('Fitting %s in range (%f, %f)\n', sFxName, xmin ,xmax);
n = length(xdata);
sumsqr = 0;
for i=1:n
  qx = lm.Coefficients{1,1};
  for j=2:orderA+1
    qx = qx + lm.Coefficients{j,1} * xdata(i)^lstPwrA(j-1);
  end
  dx = 1;
  k = 1;
  for j=orderA+2:orderA+1+orderB
    dx = dx - lm.Coefficients{j,1} * xdata(i)^lstPwrB(k));
    k = k + 1;
  end
  yc = qx/dx;
  sumsqr = sumsqr + (ydata(i) - yc)^2;
end
k = orderA + orderB + 1;
fprintf('MSS of errors squared = %e\n', sqrt(sumsqr)/n);
fprintf('Corrected MSS of errors squared = %e\n', sqrt(sumsqr)/n*sqrt(3));
fprintf("R-Squared = %12.8f\n", lm.Rsquared.Ordinary);
fprintf("R-Squared Adjusted = %12.8f\n", lm.Rsquared.Adjusted);
AIC = lm.ModelCriterion.AIC;
AICc = AIC + 2*k*(1 + (k+1)/(n-k-1));
fprintf('Particle swarm AICc =  %e\n', psAICc);
fprintf('AIC =  %e\n', AIC);
fprintf('AICc =  %e\n', AICc);
```



```
if bUseDiary
  diary off
end
end

function AICc = optimFunc(x)
  global xdata
  global ydata
  global orderA
  global orderB
  global ggx

  A = x(1);
  B = x(2);
  C = x(3);
  orderA = round(x(4));
  orderB = round(x(5));
  X = [];
  for i=1:orderA
    xs = xdata.^ggx(i,A,B,C);
    X = [X;xs'];
  end
  for i=1:orderB
    xs = ydata.*(xdata.^ggx(i,A,B,C));
    X = [X;xs'];
  end
  X = X';
  lm = fitlm(X,ydata);

  n = length(xdata);
  k = orderA + orderB + 1;
  AIC = lm.ModelCriterion.AIC;
  AICc = AIC + 2*k*(1 + (k+1)/(n-k-1));
  if isinf(AICc), AICc = -1e+99; end
end

function sFx = getFuncName(fx)
  sFx = func2str(fx);
  if sFx(1:2)=="@("
    i = strfind(sFx,")");
    sFx = sFx(i(1)+1:end);
  elseif sFx(1)=="@"
    sFx = strcat(sFx(2:end),".m");
  else
    % return sFx as is
  end
end
```

The parameters of function PadeShamPoly3() are:

- The parameter fx is the handle (or inline function) for the function being approximated. An example is @(x)cos(x) which also shows the ***recommended format*** for the argument of parameter fx.

- The parameter gx is the handle (or inline function) for the function that calculates the powers of the Shammas polynomial. An example is @(i,A,B,c)A+B*sqrt(i+C) which also shows the ***recommended format*** for the argument of parameter gx.
- The parameter xRange is the array that specifies the minimum value, increment value, and maximum value for the range of approximation.
- The parameter Lb is is the array of lower limits for the parameters A, B, C and the Shammas polynomial order. An example is [0.1 0 0 3].
- The parameter Ub is is the array of upper limits for the parameters A, B and C, and the Shammas polynomial order. An example is [1 4 3 7].
- The parameter runNum specifies the run number. The arguments for this parameter have nothing to do with the calculations and serve in fine tuning the name of the diary files, when used.
- The optional parameter sFxName is the name of the approximated function. An example is “cos(x)”.
- The parameter diaryFilename is the name of the diary file. An example is “cos_1.txt”.

The above listing performs basically the same tasks as function PadeShamPoly2 () with the exception that it optimizes the parameter C and manages values related to that parameter. The function optimFunc() is the function to be optimized to get the best values for the Shammas polynomial power parameters A, B, and C, and the best Shammas polynomial order. The function getFuncName() returns a string-type function name given a handle of a function.

Using the calculation scheme, note that function PadeShamPoly3() does not explicitly iterate over different values of A, B and C. It delegates that task to the Matlab function particleswarm().

The following Matlab script goAll (which is identical to goAll in Part I) performs the various Pade-Shammas polynomial fittings for the various tested functions:

```
% Version 1.0.0 8/9/2020
global bUseDiary
global bDeleteIfExists

bUseDiary = true;
bDeleteIfExists = true; % or false
selIdx = 0; % note a zero value will execute all the models
runNum = 1;
bShutdown = false;
```



```
tic;

if selIdx==0 || selIdx==1
  lstA = [0 0.5];
  lstB = [0.1 0.5];
  lstOrder = [3 7];
  Lb = [lstA(1) lstB(1) lstOrder(1) lstOrder(1)];
  Ub = [lstA(2) lstB(2) lstOrder(2) lstOrder(2)];
  gx = @(i,A,B)A+B*i;

PadeShamPoly2(@(x)acos(x),gx,[0:.01:1],Lb,Ub,runNum,"arccos(x)","arccos_1.txt
")

PadeShamPoly2(@(x)asin(x),gx,[0:.01:1],Lb,Ub,runNum,"arcsin(x)","arcsin_1.txt
")

PadeShamPoly2(@(x)atan(x),gx,[0:.01:1],Lb,Ub,runNum,"arctan(x)","arctan_1.txt
")

PadeShamPoly2(@(x)sin(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"sin(x)","sin_1.txt
")

PadeShamPoly2(@(x)cos(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"cos(x)","cos_1.txt
")
  PadeShamPoly2(@(x)tan(x),gx,[0:.01:1],Lb,Ub,runNum,"tan(x)","tan_1.txt")
  PadeShamPoly2(@(x)sinh(x),gx,[0:.01:5],Lb,Ub,runNum,"sinh(x)","sinh_1.txt")
  PadeShamPoly2(@(x)cosh(x),gx,[0:.01:5],Lb,Ub,runNum,"cosh(x)","cosh_1.txt")
  PadeShamPoly2(@(x)tanh(x),gx,[0:.01:3],Lb,Ub,runNum,"tanh(x)","tanh_1.txt")
  PadeShamPoly2(@(x)erf(x),gx,[0:.01:2.1],Lb,Ub,runNum,"erf(x)","erf_1.txt")
  PadeShamPoly2(@(x)exp(x),gx,[0:.01:2],Lb,Ub,runNum,"exp(x)","exp_1.txt")
  PadeShamPoly2(@(x)log(x),gx,[1:.01:10],Lb,Ub,runNum,"ln(x)","ln_1.txt")
  PadeShamPoly2(@(x)log10(x),gx,[1:.01:10],Lb,Ub,runNum,"log(x)","log_1.txt")
  PadeShamPoly2(@(x)10.^x,gx,[0:.01:1],Lb,Ub,runNum,"10^x","pwr10_1.txt")

PadeShamPoly2(@(x)tinv(0.95,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.95,x)","ti
nv1_1.txt")

PadeShamPoly2(@(x)tinv(0.975,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.975,x)","
tinv2_1.txt")

PadeShamPoly2(@(x)log10(gamma(x)),gx,[2:0.1:100],Lb,Ub,runNum,"log10Gamma(x)"
,"log10Gamma_1.txt")

PadeShamPoly2(@(x)digamma(x),gx,[2:0.1:100],Lb,Ub,runNum,"digamma(x)","digamm
a_1.txt")

PadeShamPoly2(@(x)trigamma(x),gx,[1:0.1:100],Lb,Ub,runNum,"tigamma(x)","triga
mma_1.txt")

PadeShamPoly2(@(x)besselj(0,x),gx,[2:0.1:30],Lb,Ub,runNum,"J0(x)","J0x_1.txt"
)

PadeShamPoly2(@(x)besselj(1,x),gx,[0:0.1:30],Lb,Ub,runNum,"J1(x)","J1x_1.txt"
)

PadeShamPoly2(@(x)besselj(2,x),gx,[0:0.1:30],Lb,Ub,runNum,"J2(x)","J2x_1.txt"
)
```



```
PadeShamPoly2(@(x)besselj(3,x),gx,[0:0.1:30],Lb,Ub,runNum,"J3(x)","J3x_1.txt"
)

PadeShamPoly2(@(x)besselj(4,x),gx,[0:0.1:30],Lb,Ub,runNum,"J4(x)","J4x_1.txt"
)

PadeShamPoly2(@(x)besselj(5,x),gx,[0:0.1:30],Lb,Ub,runNum,"J5(x)","J5x_1.txt"
)

PadeShamPoly2(@(x)integral(@(z)sin(z.^2),0,x),gx,[1:0.01:5],Lb,Ub,runNum,"Fre
snelSine(x)","FresnelSine_1.txt")

PadeShamPoly2(@(x)integral(@(z)cos(z.^2),0,x),gx,[0.5:0.01:5],Lb,Ub,runNum,"F
resnelCosine(x)","FresnelCosine_1.txt")

PadeShamPoly2(@(x)integral(@(z)sin(z)./z,0,x),gx,[1.3:0.1:20],Lb,Ub,runNum,"S
i(x)","Si_1.txt")
  PadeShamPoly2(@(x)0.57721566+log(x)-integral(@(z)(1-
cos(z))./z,0,x),gx,[0.5:0.1:20],Lb,Ub,runNum,"Ci(x)","CI_1.txt")

PadeShamPoly2(@(x)asinh(x),gx,[0:0.1:100],Lb,Ub,runNum,"asinh(x)","asinh_1.tx
t")

PadeShamPoly2(@(x)acosh(x),gx,[1:0.1:100],Lb,Ub,runNum,"acosh(x)","acosh_1.tx
t")

PadeShamPoly2(@(x)atanh(x),gx,[0:0.001:0.999],Lb,Ub,runNum,"atanh(x)","atanh_
1.txt")
end

if selIdx==0 || selIdx==2
  lstA = [0 2];
  lstB = [0.1 2];
  lstOrder = [3 7];
  Lb = [lstA(1) lstB(1) lstOrder(1) lstOrder(1)];
  Ub = [lstA(2) lstB(2) lstOrder(2) lstOrder(2)];
  gx = @(i,A,B)A+B/i;

PadeShamPoly2(@(x)acos(x),gx,[0:.01:1],Lb,Ub,runNum,"arccos(x)","arccos_2.txt
")

PadeShamPoly2(@(x)asin(x),gx,[0:.01:1],Lb,Ub,runNum,"arcsin(x)","arcsin_2.txt
")

PadeShamPoly2(@(x)atan(x),gx,[0:.01:1],Lb,Ub,runNum,"arctan(x)","arctan_2.txt
")

PadeShamPoly2(@(x)sin(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"sin(x)","sin_2.txt
")

PadeShamPoly2(@(x)cos(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"cos(x)","cos_2.txt
")
  PadeShamPoly2(@(x)tan(x),gx,[0:.01:1],Lb,Ub,runNum,"tan(x)","tan_2.txt")
  PadeShamPoly2(@(x)sinh(x),gx,[0:.01:5],Lb,Ub,runNum,"sinh(x)","sinh_2.txt")
  PadeShamPoly2(@(x)cosh(x),gx,[0:.01:5],Lb,Ub,runNum,"cosh(x)","cosh_2.txt")
  PadeShamPoly2(@(x)tanh(x),gx,[0:.01:3],Lb,Ub,runNum,"tanh(x)","tanh_2.txt")
```



```
  PadeShamPoly2(@(x)erf(x),gx,[0:.01:2.1],Lb,Ub,runNum,"erf(x)","erf_2.txt")
  PadeShamPoly2(@(x)exp(x),gx,[0:.01:2],Lb,Ub,runNum,"exp(x)","exp_2.txt")
  PadeShamPoly2(@(x)log(x),gx,[1:.01:10],Lb,Ub,runNum,"ln(x)","ln_2.txt")
  PadeShamPoly2(@(x)log10(x),gx,[1:.01:10],Lb,Ub,runNum,"log(x)","log_2.txt")
  PadeShamPoly2(@(x)10.^x,gx,[0:.01:1],Lb,Ub,runNum,"10^x","pwr10_2.txt")

PadeShamPoly2(@(x)tinv(0.95,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.95,x)","ti
nv1_2.txt")

PadeShamPoly2(@(x)tinv(0.975,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.975,x)","
tinv2_2.txt")

PadeShamPoly2(@(x)log10(gamma(x)),gx,[2:0.1:100],Lb,Ub,runNum,"log10Gamma(x)"
,"log10Gamma_2.txt")

PadeShamPoly2(@(x)digamma(x),gx,[2:0.1:100],Lb,Ub,runNum,"digamma(x)","digamm
a_2.txt")

PadeShamPoly2(@(x)trigamma(x),gx,[1:0.1:100],Lb,Ub,runNum,"trigamma(x)","trig
amma_2.txt")

PadeShamPoly2(@(x)besselj(0,x),gx,[2:0.1:30],Lb,Ub,runNum,"J0(x)","J0x_2.txt"
)

PadeShamPoly2(@(x)besselj(1,x),gx,[0:0.1:30],Lb,Ub,runNum,"J1(x)","J1x_2.txt"
)

PadeShamPoly2(@(x)besselj(2,x),gx,[0:0.1:30],Lb,Ub,runNum,"J2(x)","J2x_2.txt"
)

PadeShamPoly2(@(x)besselj(3,x),gx,[0:0.1:30],Lb,Ub,runNum,"J3(x)","J3x_2.txt"
)

PadeShamPoly2(@(x)besselj(4,x),gx,[0:0.1:30],Lb,Ub,runNum,"J4(x)","J4x_2.txt"
)

PadeShamPoly2(@(x)besselj(5,x),gx,[0:0.1:30],Lb,Ub,runNum,"J5(x)","J5x_2.txt"
)

PadeShamPoly2(@(x)integral(@(z)sin(z.^2),0,x),gx,[1:0.01:5],Lb,Ub,runNum,"Fre
snelSine(x)","FresnelSine_2.txt")

PadeShamPoly2(@(x)integral(@(z)cos(z.^2),0,x),gx,[0.5:0.01:5],Lb,Ub,runNum,"F
resnelCosine(x)","FresnelCosine_2.txt")

PadeShamPoly2(@(x)integral(@(z)sin(z)./z,0,x),gx,[1.3:0.1:20],Lb,Ub,runNum,"S
i(x)","Si_2.txt")
  PadeShamPoly2(@(x)0.57721566+log(x)-integral(@(z)(1-
cos(z))./z,0,x),gx,[0.5:0.1:20],Lb,Ub,runNum,"Ci(x)","Ci_2.txt")

PadeShamPoly2(@(x)asinh(x),gx,[0:0.1:100],Lb,Ub,runNum,"asinh(x)","asinh_2.tx
t")

PadeShamPoly2(@(x)acosh(x),gx,[1:0.1:100],Lb,Ub,runNum,"acosh(x)","acosh_2.tx
t")
```



```
PadeShamPoly2(@(x)atanh(x),gx,[0:0.001:0.999],Lb,Ub,runNum,"atanh(x)","atanh_
2.txt")
end

if selIdx==0 || selIdx==3
  lstA = [0 1.4];
  lstB = [0.1 1];
  lstOrder = [3 7];
  Lb = [lstA(1) lstB(1) lstOrder(1) lstOrder(1)];
  Ub = [lstA(2) lstB(2) lstOrder(2) lstOrder(2)];
  gx = @(i,A,B)A+B*sqrt(i);

PadeShamPoly2(@(x)acos(x),gx,[0:.01:1],Lb,Ub,runNum,"arccos(x)","arccos_3.txt
")

PadeShamPoly2(@(x)asin(x),gx,[0:.01:1],Lb,Ub,runNum,"arcsin(x)","arcsin_3.txt
")

PadeShamPoly2(@(x)atan(x),gx,[0:.01:1],Lb,Ub,runNum,"arctan(x)","arctan_3.txt
")

PadeShamPoly2(@(x)sin(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"sin(x)","sin_3.txt
")

PadeShamPoly2(@(x)cos(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"cos(x)","cos_3.txt
")
  PadeShamPoly2(@(x)tan(x),gx,[0:.01:1],Lb,Ub,runNum,"tan(x)","tan_3.txt")
  PadeShamPoly2(@(x)sinh(x),gx,[0:.01:5],Lb,Ub,runNum,"sinh(x)","sinh_3.txt")
  PadeShamPoly2(@(x)cosh(x),gx,[0:.01:5],Lb,Ub,runNum,"cosh(x)","cosh_3.txt")
  PadeShamPoly2(@(x)tanh(x),gx,[0:.01:3],Lb,Ub,runNum,"tanh(x)","tanh_3.txt")
  PadeShamPoly2(@(x)erf(x),gx,[0:.01:2.1],Lb,Ub,runNum,"erf(x)","erf_3.txt")
  PadeShamPoly2(@(x)exp(x),gx,[0:.01:2],Lb,Ub,runNum,"exp(x)","exp_3.txt")
  PadeShamPoly2(@(x)log(x),gx,[1:.01:10],Lb,Ub,runNum,"ln(x)","ln_3.txt")
  PadeShamPoly2(@(x)log10(x),gx,[1:.01:10],Lb,Ub,runNum,"log(x)","log_3.txt")
  PadeShamPoly2(@(x)10.^x,gx,[0:.01:1],Lb,Ub,runNum,"10^x","pwr10_3.txt")

PadeShamPoly2(@(x)tinv(0.95,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.95,x)","ti
nv1_3.txt")

PadeShamPoly2(@(x)tinv(0.975,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.975,x)","
tinv2_3.txt")

PadeShamPoly2(@(x)log10(gamma(x)),gx,[2:0.1:100],Lb,Ub,runNum,"log10Gamma(x)"
,"log10Gamma_3.txt")

PadeShamPoly2(@(x)digamma(x),gx,[2:0.1:100],Lb,Ub,runNum,"digamma(x)","digamm
a_3.txt")

PadeShamPoly2(@(x)trigamma(x),gx,[1:0.1:100],Lb,Ub,runNum,"trigamma(x)","trig
amma_3.txt")

PadeShamPoly2(@(x)besselj(0,x),gx,[2:0.1:30],Lb,Ub,runNum,"J0(x)","J0x_3.txt"
)

PadeShamPoly2(@(x)besselj(1,x),gx,[0:0.1:30],Lb,Ub,runNum,"J1(x)","J1x_3.txt"
)
```



```
PadeShamPoly2(@(x)besselj(2,x),gx,[0:0.1:30],Lb,Ub,runNum,"J2(x)","J2x_3.txt"
)

PadeShamPoly2(@(x)besselj(3,x),gx,[0:0.1:30],Lb,Ub,runNum,"J3(x)","J3x_3.txt"
)

PadeShamPoly2(@(x)besselj(4,x),gx,[0:0.1:30],Lb,Ub,runNum,"J4(x)","J4x_3.txt"
)

PadeShamPoly2(@(x)besselj(5,x),gx,[0:0.1:30],Lb,Ub,runNum,"J5(x)","J5x_3.txt"
)

PadeShamPoly2(@(x)integral(@(z)sin(z.^2),0,x),gx,[1:0.01:5],Lb,Ub,runNum,"Fre
snelSine(x)","FresnelSine_3.txt")

PadeShamPoly2(@(x)integral(@(z)cos(z.^2),0,x),gx,[0.5:0.01:5],Lb,Ub,runNum,"F
resnelCosine(x)","FresnelCosine_3.txt")

PadeShamPoly2(@(x)integral(@(z)sin(z)./z,0,x),gx,[1.3:0.1:20],Lb,Ub,runNum,"S
i(x)","Si_3.txt")
  PadeShamPoly2(@(x)0.57721566+log(x)-integral(@(z)(1-
cos(z))./z,0,x),gx,[0.5:0.1:20],Lb,Ub,runNum,"Ci(x)","Ci_3.txt")

PadeShamPoly2(@(x)asinh(x),gx,[0:0.1:100],Lb,Ub,runNum,"asinh(x)","asinh_3.tx
t")

PadeShamPoly2(@(x)acosh(x),gx,[1:0.1:100],Lb,Ub,runNum,"acosh(x)","acosh_3.tx
t")

PadeShamPoly2(@(x)atanh(x),gx,[0:0.001:0.999],Lb,Ub,runNum,"atanh(x)","atanh_
3.txt")
end

if selIdx==0 || selIdx==4
  lstA = [0 1];
  lstB = [0.1 1];
  lstC = [0 2];
  lstOrder = [3 7];
  Lb = [lstA(1) lstB(1) lstC(1) lstOrder(1) lstOrder(1)];
  Ub = [lstA(2) lstB(2) lstC(2) lstOrder(2) lstOrder(2)];
  gx = @(i,A,B,C)A+B*sqrt(i+C);

PadeShamPoly3(@(x)acos(x),gx,[0:.01:1],Lb,Ub,runNum,"arccos(x)","arccos_4.txt
")

PadeShamPoly3(@(x)asin(x),gx,[0:.01:1],Lb,Ub,runNum,"arcsin(x)","arcsin_4.txt
")

PadeShamPoly3(@(x)atan(x),gx,[0:.01:1],Lb,Ub,runNum,"arctan(x)","arctan_4.txt
")

PadeShamPoly3(@(x)sin(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"sin(x)","sin_4.txt
")

PadeShamPoly3(@(x)cos(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"cos(x)","cos_4.txt
")
```



```
  PadeShamPoly3(@(x)tan(x),gx,[0:.01:1],Lb,Ub,runNum,"tan(x)","tan_4.txt")
  PadeShamPoly3(@(x)sinh(x),gx,[0:.01:5],Lb,Ub,runNum,"sinh(x)","sinh_4.txt")
  PadeShamPoly3(@(x)cosh(x),gx,[0:.01:5],Lb,Ub,runNum,"cosh(x)","cosh_4.txt")
  PadeShamPoly3(@(x)tanh(x),gx,[0:.01:3],Lb,Ub,runNum,"tanh(x)","tanh_4.txt")
  PadeShamPoly3(@(x)erf(x),gx,[0:.01:2.1],Lb,Ub,runNum,"erf(x)","erf_4.txt")
  PadeShamPoly3(@(x)exp(x),gx,[0:.01:2],Lb,Ub,runNum,"exp(x)","exp_4.txt")
  PadeShamPoly3(@(x)log(x),gx,[1:.01:10],Lb,Ub,runNum,"ln(x)","ln_4.txt")
  PadeShamPoly3(@(x)log10(x),gx,[1:.01:10],Lb,Ub,runNum,"log(x)","log_4.txt")
  PadeShamPoly3(@(x)10.^x,gx,[0:.01:1],Lb,Ub,runNum,"10^x","pwr10_4.txt")

PadeShamPoly3(@(x)tinv(0.95,x),gx,[2:100],Lb,Ub,runNum,"tinv(0.95,x)","tinv1_
4.txt")

PadeShamPoly3(@(x)tinv(0.975,x),gx,[2:100],Lb,Ub,runNum,"tinv(0.975,x)","tinv
2_4.txt")

PadeShamPoly3(@(x)log10(gamma(x)),gx,[2:100],Lb,Ub,runNum,"log10Gamma(x)","lo
g10Gamma_4.txt")

PadeShamPoly3(@(x)digamma(x),gx,[2:100],Lb,Ub,runNum,"digamma(x)","digamma_4.
txt")

PadeShamPoly3(@(x)trigamma(x),gx,[1:100],Lb,Ub,runNum,"trigamma(x)","trigamma
_4.txt")

PadeShamPoly3(@(x)besselj(0,x),gx,[2:0.1:30],Lb,Ub,runNum,"J0(x)","J0x_4.txt"
)

PadeShamPoly3(@(x)besselj(1,x),gx,[0:0.1:30],Lb,Ub,runNum,"J1(x)","J1x_4.txt"
)

PadeShamPoly3(@(x)besselj(2,x),gx,[0:0.1:30],Lb,Ub,runNum,"J2(x)","J2x_4.txt"
)

PadeShamPoly3(@(x)besselj(3,x),gx,[0:0.1:30],Lb,Ub,runNum,"J3(x)","J3x_4.txt"
)

PadeShamPoly3(@(x)besselj(4,x),gx,[0:0.1:30],Lb,Ub,runNum,"J4(x)","J4x_4.txt"
)

PadeShamPoly3(@(x)besselj(5,x),gx,[0:0.1:30],Lb,Ub,runNum,"J5(x)","J5x_4.txt"
)

PadeShamPoly3(@(x)integral(@(z)sin(z.^2),0,x),gx,[1:0.01:5],Lb,Ub,runNum,"Fre
snelSine(x)","FresnelSine_4.txt")

PadeShamPoly3(@(x)integral(@(z)cos(z.^2),0,x),gx,[0.5:0.01:5],Lb,Ub,runNum,"F
resnelCosine(x)","FresnelCosine_4.txt")

PadeShamPoly3(@(x)integral(@(z)sin(z)./z,0,x),gx,[1.3:0.1:20],Lb,Ub,runNum,"S
i(x)","Si_4.txt")
  PadeShamPoly3(@(x)0.57721566+log(x)-integral(@(z)(1-
cos(z))./z,0,x),gx,[0.5:0.1:20],Lb,Ub,runNum,"Ci(x)","Ci_4.txt")

PadeShamPoly3(@(x)asinh(x),gx,[0:0.1:100],Lb,Ub,runNum,"asinh(x)","asinh_4.tx
t")
```



```
PadeShamPoly3(@(x)acosh(x),gx,[1:0.1:100],Lb,Ub,runNum,"acosh(x)","acosh_4.tx
t")

PadeShamPoly3(@(x)atanh(x),gx,[0:0.001:0.999],Lb,Ub,runNum,"atanh(x)","atanh_
4.txt")
end

if selIdx==0 || selIdx==5
  lstA = [0 0.5];
  lstB = [0.1 0.27];
  lstC = [0 0.27];
  lstOrder = [3 7];
  Lb = [lstA(1) lstB(1) lstC(1) lstOrder(1) lstOrder(1)];
  Ub = [lstA(2) lstB(2) lstC(2) lstOrder(2) lstOrder(2)];
  gx = @(i,A,B,C)A+B*i+C*(i-1);

PadeShamPoly3(@(x)acos(x),gx,[0:.01:1],Lb,Ub,runNum,"arccos(x)","arccos_5.txt
")

PadeShamPoly3(@(x)asin(x),gx,[0:.01:1],Lb,Ub,runNum,"arcsin(x)","arcsin_5.txt
")

PadeShamPoly3(@(x)atan(x),gx,[0:.01:1],Lb,Ub,runNum,"arctan(x)","arctan_5.txt
")

PadeShamPoly3(@(x)sin(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"sin(x)","sin_5.txt
")

PadeShamPoly3(@(x)cos(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"cos(x)","cos_5.txt
")
  PadeShamPoly3(@(x)tan(x),gx,[0:.01:1],Lb,Ub,runNum,"tan(x)","tan_5.txt")
  PadeShamPoly3(@(x)sinh(x),gx,[0:.01:5],Lb,Ub,runNum,"sinh(x)","sinh_5.txt")
  PadeShamPoly3(@(x)cosh(x),gx,[0:.01:5],Lb,Ub,runNum,"cosh(x)","cosh_5.txt")
  PadeShamPoly3(@(x)tanh(x),gx,[0:.01:3],Lb,Ub,runNum,"tanh(x)","tanh_5.txt")
  PadeShamPoly3(@(x)erf(x),gx,[0:.01:2.1],Lb,Ub,runNum,"erf(x)","erf_5.txt")
  PadeShamPoly3(@(x)exp(x),gx,[0:.01:2],Lb,Ub,runNum,"exp(x)","exp_5.txt")
  PadeShamPoly3(@(x)log(x),gx,[1:.01:10],Lb,Ub,runNum,"ln(x)","ln_5.txt")
  PadeShamPoly3(@(x)log10(x),gx,[1:.01:10],Lb,Ub,runNum,"log(x)","log_5.txt")
  PadeShamPoly3(@(x)10.^x,gx,[0:.01:1],Lb,Ub,runNum,"10^x","pwr10_5.txt")

PadeShamPoly3(@(x)tinv(0.95,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.95,x)","ti
nv1_5.txt")

PadeShamPoly3(@(x)tinv(0.975,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.975,x)","
tinv2_5.txt")

PadeShamPoly3(@(x)log10(gamma(x)),gx,[2:0.1:100],Lb,Ub,runNum,"log10Gamma(x)"
,"log10Gamma_5.txt")

PadeShamPoly3(@(x)digamma(x),gx,[2:0.1:100],Lb,Ub,runNum,"digamma(x)","digamm
a_5.txt")

PadeShamPoly3(@(x)trigamma(x),gx,[1:0.1:100],Lb,Ub,runNum,"trigamma(x)","trig
amma_5.txt")
```



```
PadeShamPoly3(@(x)besselj(0,x),gx,[2:0.1:30],Lb,Ub,runNum,"J0(x)","J0x_5.txt"
)

PadeShamPoly3(@(x)besselj(1,x),gx,[0:0.1:30],Lb,Ub,runNum,"J1(x)","J1x_5.txt"
)

PadeShamPoly3(@(x)besselj(2,x),gx,[0:0.1:30],Lb,Ub,runNum,"J2(x)","J2x_5.txt"
)

PadeShamPoly3(@(x)besselj(3,x),gx,[0:0.1:30],Lb,Ub,runNum,"J3(x)","J3x_5.txt"
)

PadeShamPoly3(@(x)besselj(4,x),gx,[0:0.1:30],Lb,Ub,runNum,"J4(x)","J4x_5.txt"
)

PadeShamPoly3(@(x)besselj(5,x),gx,[0:0.1:30],Lb,Ub,runNum,"J5(x)","J5x_5.txt"
)

PadeShamPoly3(@(x)integral(@(z)sin(z.^2),0,x),gx,[1:0.01:5],Lb,Ub,runNum,"Fre
snelSine(x)","FresnelSine_5.txt")

PadeShamPoly3(@(x)integral(@(z)cos(z.^2),0,x),gx,[0.5:0.01:5],Lb,Ub,runNum,"F
resnelCosine(x)","FresnelCosine_5.txt")

PadeShamPoly3(@(x)integral(@(z)sin(z)./z,0,x),gx,[1.3:0.1:20],Lb,Ub,runNum,"S
i(x)","Si_5.txt")
  PadeShamPoly3(@(x)0.57721566+log(x)-integral(@(z)(1-
cos(z))./z,0,x),gx,[0.5:0.1:20],Lb,Ub,runNum,"Ci(x)","Ci_5.txt")

PadeShamPoly3(@(x)asinh(x),gx,[0:0.1:100],Lb,Ub,runNum,"asinh(x)","asinh_5.tx
t")

PadeShamPoly3(@(x)acosh(x),gx,[1:0.1:100],Lb,Ub,runNum,"acosh(x)","acosh_5.tx
t")

PadeShamPoly3(@(x)atanh(x),gx,[0:0.001:0.999],Lb,Ub,runNum,"atanh(x)","atanh_
5.txt")
end

if selIdx==0 || selIdx==6
  lstA = [0 0.75];
  lstB = [0.1 0.25];
  lstC = [0 0.25];
  lstOrder = [3 7];
  Lb = [lstA(1) lstB(1) lstC(1) lstOrder(1) lstOrder(1)];
  Ub = [lstA(2) lstB(2) lstC(2) lstOrder(2) lstOrder(2)];
  gx = @(i,A,B,C)A+B*i-C*(i-1);

PadeShamPoly3(@(x)acos(x),gx,[0:.01:1],Lb,Ub,runNum,"arccos(x)","arccos_6.txt
")

PadeShamPoly3(@(x)asin(x),gx,[0:.01:1],Lb,Ub,runNum,"arcsin(x)","arcsin_6.txt
")

PadeShamPoly3(@(x)atan(x),gx,[0:.01:1],Lb,Ub,runNum,"arctan(x)","arctan_6.txt
")
```



```
PadeShamPoly3(@(x)sin(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"sin(x)","sin_6.txt
")

PadeShamPoly3(@(x)cos(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"cos(x)","cos_6.txt
")
  PadeShamPoly3(@(x)tan(x),gx,[0:.01:1],Lb,Ub,runNum,"tan(x)","tan_6.txt")
  PadeShamPoly3(@(x)sinh(x),gx,[0:.01:5],Lb,Ub,runNum,"sinh(x)","sinh_6.txt")
  PadeShamPoly3(@(x)cosh(x),gx,[0:.01:5],Lb,Ub,runNum,"cosh(x)","cosh_6.txt")
  PadeShamPoly3(@(x)tanh(x),gx,[0:.01:3],Lb,Ub,runNum,"tanh(x)","tanh_6.txt")
  PadeShamPoly3(@(x)erf(x),gx,[0:.01:2.1],Lb,Ub,runNum,"erf(x)","erf_6.txt")
  PadeShamPoly3(@(x)exp(x),gx,[0:.01:2],Lb,Ub,runNum,"exp(x)","exp_6.txt")
  PadeShamPoly3(@(x)log(x),gx,[1:.01:10],Lb,Ub,runNum,"ln(x)","ln_6.txt")
  PadeShamPoly3(@(x)log10(x),gx,[1:.01:10],Lb,Ub,runNum,"log(x)","log_6.txt")
  PadeShamPoly3(@(x)10.^x,gx,[0:.01:1],Lb,Ub,runNum,"10^x","pwr10_6.txt")

PadeShamPoly3(@(x)tinv(0.95,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.95,x)","ti
nv1_6.txt")

PadeShamPoly3(@(x)tinv(0.975,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.975,x)","
tinv2_6.txt")

PadeShamPoly3(@(x)log10(gamma(x)),gx,[2:0.1:100],Lb,Ub,runNum,"log10Gamma(x)"
,"log10Gamma_6.txt")

PadeShamPoly3(@(x)digamma(x),gx,[2:0.1:100],Lb,Ub,runNum,"digamma(x)","digamm
a_6.txt")

PadeShamPoly3(@(x)trigamma(x),gx,[1:0.1:100],Lb,Ub,runNum,"trigamma(x)","trig
amma_6.txt")

PadeShamPoly3(@(x)besselj(0,x),gx,[2:0.1:30],Lb,Ub,runNum,"J0(x)","J0x_6.txt"
)

PadeShamPoly3(@(x)besselj(1,x),gx,[0:0.1:30],Lb,Ub,runNum,"J1(x)","J1x_6.txt"
)

PadeShamPoly3(@(x)besselj(2,x),gx,[0:0.1:30],Lb,Ub,runNum,"J2(x)","J2x_6.txt"
)

PadeShamPoly3(@(x)besselj(3,x),gx,[0:0.1:30],Lb,Ub,runNum,"J3(x)","J3x_6.txt"
)

PadeShamPoly3(@(x)besselj(4,x),gx,[0:0.1:30],Lb,Ub,runNum,"J4(x)","J4x_6.txt"
)

PadeShamPoly3(@(x)besselj(5,x),gx,[0:0.1:30],Lb,Ub,runNum,"J5(x)","J5x_6.txt"
)

PadeShamPoly3(@(x)integral(@(z)sin(z.^2),0,x),gx,[1:0.01:5],Lb,Ub,runNum,"Fre
snelSine(x)","FresnelSine_6.txt")

PadeShamPoly3(@(x)integral(@(z)cos(z.^2),0,x),gx,[0.5:0.01:5],Lb,Ub,runNum,"F
resnelCosine(x)","FresnelCosine_6.txt")

PadeShamPoly3(@(x)integral(@(z)sin(z)./z,0,x),gx,[1.3:0.1:20],Lb,Ub,runNum,"S
i(x)","Si_6.txt")
```



```
  PadeShamPoly3(@(x)0.57721566+log(x)-integral(@(z)(1-
cos(z))./z,0,x),gx,[0.5:0.1:20],Lb,Ub,runNum,"Ci(x)","Ci_6.txt")

PadeShamPoly3(@(x)asinh(x),gx,[0:0.1:100],Lb,Ub,runNum,"asinh(x)","asinh_6.tx
t")

PadeShamPoly3(@(x)acosh(x),gx,[1:0.1:100],Lb,Ub,runNum,"acosh(x)","acosh_6.tx
t")

PadeShamPoly3(@(x)atanh(x),gx,[0:0.001:0.999],Lb,Ub,runNum,"atanh(x)","atanh_
6.txt")
end

if selIdx==0 || selIdx==7
  lstA = [0 0.34];
  lstB = [0.1 0.39];
  lstC = [0 0.36];
  lstOrder = [3 7];
  Lb = [lstA(1) lstB(1) lstC(1) lstOrder(1) lstOrder(1)];
  Ub = [lstA(2) lstB(2) lstC(2) lstOrder(2) lstOrder(2)];
  gx = @(i,A,B,C)A+B*i+C*sqrt(i-1);

PadeShamPoly3(@(x)acos(x),gx,[0:.01:1],Lb,Ub,runNum,"arccos(x)","arccos_7.txt
")

PadeShamPoly3(@(x)asin(x),gx,[0:.01:1],Lb,Ub,runNum,"arcsin(x)","arcsin_7.txt
")

PadeShamPoly3(@(x)atan(x),gx,[0:.01:1],Lb,Ub,runNum,"arctan(x)","arctan_7.txt
")

PadeShamPoly3(@(x)sin(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"sin(x)","sin_7.txt
")

PadeShamPoly3(@(x)cos(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"cos(x)","cos_7.txt
")
  PadeShamPoly3(@(x)tan(x),gx,[0:.01:1],Lb,Ub,runNum,"tan(x)","tan_7.txt")
  PadeShamPoly3(@(x)sinh(x),gx,[0:.01:5],Lb,Ub,runNum,"sinh(x)","sinh_7.txt")
  PadeShamPoly3(@(x)cosh(x),gx,[0:.01:5],Lb,Ub,runNum,"cosh(x)","cosh_7.txt")
  PadeShamPoly3(@(x)tanh(x),gx,[0:.01:3],Lb,Ub,runNum,"tanh(x)","tanh_7.txt")
  PadeShamPoly3(@(x)erf(x),gx,[0:.01:2.1],Lb,Ub,runNum,"erf(x)","erf_7.txt")
  PadeShamPoly3(@(x)exp(x),gx,[0:.01:2],Lb,Ub,runNum,"exp(x)","exp_7.txt")
  PadeShamPoly3(@(x)log(x),gx,[1:.01:10],Lb,Ub,runNum,"ln(x)","ln_7.txt")
  PadeShamPoly3(@(x)log10(x),gx,[1:.01:10],Lb,Ub,runNum,"log(x)","log_7.txt")
  PadeShamPoly3(@(x)10.^x,gx,[0:.01:1],Lb,Ub,runNum,"10^x","pwr10_7.txt")

PadeShamPoly3(@(x)tinv(0.95,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.95,x)","ti
nv1_7.txt")

PadeShamPoly3(@(x)tinv(0.975,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.975,x)","
tinv2_7.txt")

PadeShamPoly3(@(x)log10(gamma(x)),gx,[2:0.1:100],Lb,Ub,runNum,"log10Gamma(x)"
,"log10Gamma_7.txt")

PadeShamPoly3(@(x)digamma(x),gx,[2:0.1:100],Lb,Ub,runNum,"digamma(x)","digamm
a_7.txt")
```



```
PadeShamPoly3(@(x)trigamma(x),gx,[1:0.1:100],Lb,Ub,runNum,"trigamma(x)","trig
amma_7.txt")

PadeShamPoly3(@(x)besselj(0,x),gx,[2:0.1:30],Lb,Ub,runNum,"J0(x)","J0x_7.txt"
)

PadeShamPoly3(@(x)besselj(1,x),gx,[0:0.1:30],Lb,Ub,runNum,"J1(x)","J1x_7.txt"
)

PadeShamPoly3(@(x)besselj(2,x),gx,[0:0.1:30],Lb,Ub,runNum,"J2(x)","J2x_7.txt"
)

PadeShamPoly3(@(x)besselj(3,x),gx,[0:0.1:30],Lb,Ub,runNum,"J3(x)","J3x_7.txt"
)

PadeShamPoly3(@(x)besselj(4,x),gx,[0:0.1:30],Lb,Ub,runNum,"J4(x)","J4x_7.txt"
)

PadeShamPoly3(@(x)besselj(5,x),gx,[0:0.1:30],Lb,Ub,runNum,"J5(x)","J5x_7.txt"
)

PadeShamPoly3(@(x)integral(@(z)sin(z.^2),0,x),gx,[1:0.01:5],Lb,Ub,runNum,"Fre
snelSine(x)","FresnelSine_7.txt")

PadeShamPoly3(@(x)integral(@(z)cos(z.^2),0,x),gx,[0.5:0.01:5],Lb,Ub,runNum,"F
resnelCosine(x)","FresnelCosine_7.txt")

PadeShamPoly3(@(x)integral(@(z)sin(z)./z,0,x),gx,[1.3:0.1:20],Lb,Ub,runNum,"S
i(x)","Si_7.txt")
  PadeShamPoly3(@(x)0.57721566+log(x)-integral(@(z)(1-
cos(z))./z,0,x),gx,[0.5:0.1:20],Lb,Ub,runNum,"Ci(x)","Ci_7.txt")

PadeShamPoly3(@(x)asinh(x),gx,[0:0.1:100],Lb,Ub,runNum,"asinh(x)","asinh_7.tx
t")

PadeShamPoly3(@(x)acosh(x),gx,[1:0.1:100],Lb,Ub,runNum,"acosh(x)","acosh_7.tx
t")

PadeShamPoly3(@(x)atanh(x),gx,[0:0.001:0.999],Lb,Ub,runNum,"atanh(x)","atanh_
7.txt")
end

if selIdx==0 || selIdx==8
  lstA = [0 0.55];
  lstB = [0.1 0.4];
  lstC = [0 0.4];
  lstOrder = [3 7];
  Lb = [lstA(1) lstB(1) lstC(1) lstOrder(1) lstOrder(1)];
  Ub = [lstA(2) lstB(2) lstC(2) lstOrder(2) lstOrder(2)];
  gx = @(i,A,B,C)A+B*sqrt(i)+C*(i-1);

PadeShamPoly3(@(x)acos(x),gx,[0:.01:1],Lb,Ub,runNum,"arccos(x)","arccos_8.txt
")

PadeShamPoly3(@(x)asin(x),gx,[0:.01:1],Lb,Ub,runNum,"arcsin(x)","arcsin_8.txt
")
```



```
PadeShamPoly3(@(x)atan(x),gx,[0:.01:1],Lb,Ub,runNum,"arctan(x)","arctan_8.txt
")

PadeShamPoly3(@(x)sin(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"sin(x)","sin_8.txt
")

PadeShamPoly3(@(x)cos(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"cos(x)","cos_8.txt
")
  PadeShamPoly3(@(x)tan(x),gx,[0:.01:1],Lb,Ub,runNum,"tan(x)","tan_8.txt")
  PadeShamPoly3(@(x)sinh(x),gx,[0:.01:5],Lb,Ub,runNum,"sinh(x)","sinh_8.txt")
  PadeShamPoly3(@(x)cosh(x),gx,[0:.01:5],Lb,Ub,runNum,"cosh(x)","cosh_8.txt")
  PadeShamPoly3(@(x)tanh(x),gx,[0:.01:3],Lb,Ub,runNum,"tanh(x)","tanh_8.txt")
  PadeShamPoly3(@(x)erf(x),gx,[0:.01:2.1],Lb,Ub,runNum,"erf(x)","erf_8.txt")
  PadeShamPoly3(@(x)exp(x),gx,[0:.01:2],Lb,Ub,runNum,"exp(x)","exp_8.txt")
  PadeShamPoly3(@(x)log(x),gx,[1:.01:10],Lb,Ub,runNum,"ln(x)","ln_8.txt")
  PadeShamPoly3(@(x)log10(x),gx,[1:.01:10],Lb,Ub,runNum,"log(x)","log_8.txt")
  PadeShamPoly3(@(x)10.^x,gx,[0:.01:1],Lb,Ub,runNum,"10^x","pwr10_8.txt")

PadeShamPoly3(@(x)tinv(0.95,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.95,x)","ti
nv1_8.txt")

PadeShamPoly3(@(x)tinv(0.975,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.975,x)","
tinv2_8.txt")

PadeShamPoly3(@(x)log10(gamma(x)),gx,[2:0.1:100],Lb,Ub,runNum,"log10Gamma(x)"
,"log10Gamma_8.txt")

PadeShamPoly3(@(x)digamma(x),gx,[2:0.1:100],Lb,Ub,runNum,"digamma(x)","digamm
a_8.txt")

PadeShamPoly3(@(x)trigamma(x),gx,[1:0.1:100],Lb,Ub,runNum,"trigamma(x)","trig
amma_8.txt")

PadeShamPoly3(@(x)besselj(0,x),gx,[2:0.1:30],Lb,Ub,runNum,"J0(x)","J0x_8.txt"
)

PadeShamPoly3(@(x)besselj(1,x),gx,[0:0.1:30],Lb,Ub,runNum,"J1(x)","J1x_8.txt"
)

PadeShamPoly3(@(x)besselj(2,x),gx,[0:0.1:30],Lb,Ub,runNum,"J2(x)","J2x_8.txt"
)

PadeShamPoly3(@(x)besselj(3,x),gx,[0:0.1:30],Lb,Ub,runNum,"J3(x)","J3x_8.txt"
)

PadeShamPoly3(@(x)besselj(4,x),gx,[0:0.1:30],Lb,Ub,runNum,"J4(x)","J4x_8.txt"
)

PadeShamPoly3(@(x)besselj(5,x),gx,[0:0.1:30],Lb,Ub,runNum,"J5(x)","J5x_8.txt"
)

PadeShamPoly3(@(x)integral(@(z)sin(z.^2),0,x),gx,[1:0.01:5],Lb,Ub,runNum,"Fre
snelSine(x)","FresnelSine_8.txt")

PadeShamPoly3(@(x)integral(@(z)cos(z.^2),0,x),gx,[0.5:0.01:5],Lb,Ub,runNum,"F
resnelCosine(x)","FresnelCosine_8.txt")
```



```
PadeShamPoly3(@(x)integral(@(z)sin(z)./z,0,x),gx,[1.3:0.1:20],Lb,Ub,runNum,"S
i(x)","Si_8.txt")
  PadeShamPoly3(@(x)0.57721566+log(x)-integral(@(z)(1-
cos(z))./z,0,x),gx,[0.5:0.1:20],Lb,Ub,runNum,"Ci(x)","Ci_8.txt")

PadeShamPoly3(@(x)asinh(x),gx,[0:0.1:100],Lb,Ub,runNum,"asinh(x)","asinh_8.tx
t")

PadeShamPoly3(@(x)acosh(x),gx,[1:0.1:100],Lb,Ub,runNum,"acosh(x)","acosh_8.tx
t")

PadeShamPoly3(@(x)atanh(x),gx,[0:0.001:0.999],Lb,Ub,runNum,"atanh(x)","atanh_
8.txt")
end

if selIdx==0 || selIdx==9
  lstA = [0 0.5];
  lstB = [0.1 0.7];
  lstC = [0 0.7];
  lstOrder = [3 7];
  Lb = [lstA(1) lstB(1) lstC(1) lstOrder(1) lstOrder(1)];
  Ub = [lstA(2) lstB(2) lstC(2) lstOrder(2) lstOrder(2)];
  gx = @(i,A,B,C)A+B*sqrt(i)+C*sqrt(i-1);

PadeShamPoly3(@(x)acos(x),gx,[0:.01:1],Lb,Ub,runNum,"arccos(x)","arccos_9.txt
")

PadeShamPoly3(@(x)asin(x),gx,[0:.01:1],Lb,Ub,runNum,"arcsin(x)","arcsin_9.txt
")

PadeShamPoly3(@(x)atan(x),gx,[0:.01:1],Lb,Ub,runNum,"arctan(x)","arctan_9.txt
")

PadeShamPoly3(@(x)sin(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"sin(x)","sin_9.txt
")

PadeShamPoly3(@(x)cos(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"cos(x)","cos_9.txt
")
  PadeShamPoly3(@(x)tan(x),gx,[0:.01:1],Lb,Ub,runNum,"tan(x)","tan_9.txt")
  PadeShamPoly3(@(x)sinh(x),gx,[0:.01:5],Lb,Ub,runNum,"sinh(x)","sinh_9.txt")
  PadeShamPoly3(@(x)cosh(x),gx,[0:.01:5],Lb,Ub,runNum,"cosh(x)","cosh_9.txt")
  PadeShamPoly3(@(x)tanh(x),gx,[0:.01:3],Lb,Ub,runNum,"tanh(x)","tanh_9.txt")
  PadeShamPoly3(@(x)erf(x),gx,[0:.01:2.1],Lb,Ub,runNum,"erf(x)","erf_9.txt")
  PadeShamPoly3(@(x)exp(x),gx,[0:.01:2],Lb,Ub,runNum,"exp(x)","exp_9.txt")
  PadeShamPoly3(@(x)log(x),gx,[1:.01:10],Lb,Ub,runNum,"ln(x)","ln_9.txt")
  PadeShamPoly3(@(x)log10(x),gx,[1:.01:10],Lb,Ub,runNum,"log(x)","log_9.txt")
  PadeShamPoly3(@(x)10.^x,gx,[0:.01:1],Lb,Ub,runNum,"10^x","pwr10_9.txt")

PadeShamPoly3(@(x)tinv(0.95,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.95,x)","ti
nv1_9.txt")

PadeShamPoly3(@(x)tinv(0.975,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.975,x)","
tinv2_9.txt")

PadeShamPoly3(@(x)log10(gamma(x)),gx,[2:0.1:100],Lb,Ub,runNum,"log10Gamma(x)"
,"log10Gamma_9.txt")
```



```
PadeShamPoly3(@(x)digamma(x),gx,[2:0.1:100],Lb,Ub,runNum,"digamma(x)","digamm
a_9.txt")

PadeShamPoly3(@(x)trigamma(x),gx,[1:0.1:100],Lb,Ub,runNum,"trigamma(x)","trig
amma_9.txt")

PadeShamPoly3(@(x)besselj(0,x),gx,[2:0.1:30],Lb,Ub,runNum,"J0(x)","J0x_9.txt"
)

PadeShamPoly3(@(x)besselj(1,x),gx,[0:0.1:30],Lb,Ub,runNum,"J1(x)","J1x_9.txt"
)

PadeShamPoly3(@(x)besselj(2,x),gx,[0:0.1:30],Lb,Ub,runNum,"J2(x)","J2x_9.txt"
)

PadeShamPoly3(@(x)besselj(3,x),gx,[0:0.1:30],Lb,Ub,runNum,"J3(x)","J3x_9.txt"
)

PadeShamPoly3(@(x)besselj(4,x),gx,[0:0.1:30],Lb,Ub,runNum,"J4(x)","J4x_9.txt"
)

PadeShamPoly3(@(x)besselj(5,x),gx,[0:0.1:30],Lb,Ub,runNum,"J5(x)","J5x_9.txt"
)

PadeShamPoly3(@(x)integral(@(z)sin(z.^2),0,x),gx,[1:0.01:5],Lb,Ub,runNum,"Fre
snelSine(x)","FresnelSine_9.txt")

PadeShamPoly3(@(x)integral(@(z)cos(z.^2),0,x),gx,[0.5:0.01:5],Lb,Ub,runNum,"F
resnelCosine(x)","FresnelCosine_9.txt")

PadeShamPoly3(@(x)integral(@(z)sin(z)./z,0,x),gx,[1.3:0.1:20],Lb,Ub,runNum,"S
i(x)","Si_9.txt")
  PadeShamPoly3(@(x)0.57721566+log(x)-integral(@(z)(1-
cos(z))./z,0,x),gx,[0.5:0.1:20],Lb,Ub,runNum,"Ci(x)","Ci_9.txt")

PadeShamPoly3(@(x)asinh(x),gx,[0:0.1:100],Lb,Ub,runNum,"asinh(x)","asinh_9.tx
t")

PadeShamPoly3(@(x)acosh(x),gx,[1:0.1:100],Lb,Ub,runNum,"acosh(x)","acosh_9.tx
t")

PadeShamPoly3(@(x)atanh(x),gx,[0:0.001:0.999],Lb,Ub,runNum,"atanh(x)","atanh_
9.txt")
end

if selIdx==0 || selIdx==10
  lstA = [0 2.2];
  lstB = [0.1 3.5];
  lstOrder = [3 7];
  Lb = [lstA(1) lstB(1) lstOrder(1) lstOrder(1)];
  Ub = [lstA(2) lstB(2) lstOrder(2) lstOrder(2)];
  gx = @(i,A,B)A+B*log10(i)^4;

PadeShamPoly2(@(x)acos(x),gx,[0:.01:1],Lb,Ub,runNum,"arccos(x)","arccos_10.tx
t")
```



```
PadeShamPoly2(@(x)asin(x),gx,[0:.01:1],Lb,Ub,runNum,"arcsin(x)","arcsin_10.tx
t")

PadeShamPoly2(@(x)atan(x),gx,[0:.01:1],Lb,Ub,runNum,"arctan(x)","arctan_10.tx
t")

PadeShamPoly2(@(x)sin(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"sin(x)","sin_10.tx
t")

PadeShamPoly2(@(x)cos(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"cos(x)","cos_10.tx
t")
  PadeShamPoly2(@(x)tan(x),gx,[0:.01:1],Lb,Ub,runNum,"tan(x)","tan_10.txt")

PadeShamPoly2(@(x)sinh(x),gx,[0:.01:5],Lb,Ub,runNum,"sinh(x)","sinh_10.txt")

PadeShamPoly2(@(x)cosh(x),gx,[0:.01:5],Lb,Ub,runNum,"cosh(x)","cosh_10.txt")

PadeShamPoly2(@(x)tanh(x),gx,[0:.01:3],Lb,Ub,runNum,"tanh(x)","tanh_10.txt")
  PadeShamPoly2(@(x)erf(x),gx,[0:.01:2.1],Lb,Ub,runNum,"erf(x)","erf_10.txt")
  PadeShamPoly2(@(x)exp(x),gx,[0:.01:2],Lb,Ub,runNum,"exp(x)","exp_10.txt")
  PadeShamPoly2(@(x)log(x),gx,[1:.01:10],Lb,Ub,runNum,"ln(x)","ln_10.txt")

PadeShamPoly2(@(x)log10(x),gx,[1:.01:10],Lb,Ub,runNum,"log(x)","log_10.txt")
  PadeShamPoly2(@(x)10.^x,gx,[0:.01:1],Lb,Ub,runNum,"10^x","pwr10_10.txt")

PadeShamPoly2(@(x)tinv(0.95,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.95,x)","ti
nv1_10.txt")

PadeShamPoly2(@(x)tinv(0.975,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.975,x)","
tinv2_10.txt")

PadeShamPoly2(@(x)log10(gamma(x)),gx,[2:0.1:100],Lb,Ub,runNum,"log10Gamma(x)"
,"log10Gamma_10.txt")

PadeShamPoly2(@(x)digamma(x),gx,[2:0.1:100],Lb,Ub,runNum,"digamma(x)","digamm
a_10.txt")

PadeShamPoly2(@(x)trigamma(x),gx,[1:0.1:100],Lb,Ub,runNum,"trigamma(x)","trig
amma_10.txt")

PadeShamPoly2(@(x)besselj(0,x),gx,[2:0.1:30],Lb,Ub,runNum,"J0(x)","J0x_10.txt
")

PadeShamPoly2(@(x)besselj(1,x),gx,[0:0.1:30],Lb,Ub,runNum,"J1(x)","J1x_10.txt
")

PadeShamPoly2(@(x)besselj(2,x),gx,[0:0.1:30],Lb,Ub,runNum,"J2(x)","J2x_10.txt
")

PadeShamPoly2(@(x)besselj(3,x),gx,[0:0.1:30],Lb,Ub,runNum,"J3(x)","J3x_10.txt
")

PadeShamPoly2(@(x)besselj(4,x),gx,[0:0.1:30],Lb,Ub,runNum,"J4(x)","J4x_10.txt
")
```



```
PadeShamPoly2(@(x)besselj(5,x),gx,[0:0.1:30],Lb,Ub,runNum,"J5(x)","J5x_10.txt
")

PadeShamPoly2(@(x)integral(@(z)sin(z.^2),0,x),gx,[1:0.01:5],Lb,Ub,runNum,"Fre
snelSine(x)","FresnelSine_10.txt")

PadeShamPoly2(@(x)integral(@(z)cos(z.^2),0,x),gx,[0.5:0.01:5],Lb,Ub,runNum,"F
resnelCosine(x)","FresnelCosine_10.txt")

PadeShamPoly2(@(x)integral(@(z)sin(z)./z,0,x),gx,[1.3:0.1:20],Lb,Ub,runNum,"S
i(x)","Si_10.txt")
  PadeShamPoly2(@(x)0.57721566+log(x)-integral(@(z)(1-
cos(z))./z,0,x),gx,[0.5:0.1:20],Lb,Ub,runNum,"Ci(x)","Ci_10.txt")

PadeShamPoly2(@(x)asinh(x),gx,[0:0.1:100],Lb,Ub,runNum,"asinh(x)","asinh_10.t
xt")

PadeShamPoly2(@(x)acosh(x),gx,[1:0.1:100],Lb,Ub,runNum,"acosh(x)","acosh_10.t
xt")

PadeShamPoly2(@(x)atanh(x),gx,[0:0.001:0.999],Lb,Ub,runNum,"atanh(x)","atanh_
10.txt")
end

toc;

% make some sound
playmynotes()

if bShutdown
  system('shutdown -s');
else
  fprintf("\n\nDone!\n\n");
end
```

The above listing has the following global and operational variabls:

- The global variable bUseDiary is a Boolean flag used to tell the functions PadeShamPoly2() and PadeShamPoly3() whether you want to copy the screen output to diary text files.
- The global variable bDeleteIfExists is a Boolean flag used to tell the functions PadeShamPoly2() and PadeShamPoly3() whether you want to delete diary files if they exist.
- The variable selIdx allows you to select calculations for one of the seven groups (when set to the targeted group number) or all of the groups (when set to 0). I am using this scheme to reduce calculation time which can be done by working with a specific set of approximations.
- The Boolean variable bShutdown tells the Matlab script whether to shut down the computer when done.

When the script goAll is done with the calculation it executes the script playmynotes.m to play ten random sinusoidal notes. You can comment out the call to the audible script to keep the script quiet. I wrote the script for playmynotes.m based on the code I found in the fine book *Programming Fundamentals Using MATLAB* written by M. Weeks and published in 2020 by Mercury Learning. The book comes with a CD that containes all the Matlab source code and figures in that book. The source code for playmynotes.m is:

```
n = 10;
frq = 8192;
notes = 220 + fix(220*rand(n,1));

for i =1:n
  znotes(i,1:frq) = 0.5*sin(2*pi*notes(i)*(0:frq-1)/frq);
end

for i = 1:n
    sound(znotes(i,1:frq));
    pause(1)
end
```

You can edit the script in goAll.m to perform Pade-Shammas polynomial fitting for other functions you are interested in or for other Shammas polynomial orders.

General Comments on Results

The next sections show the results of fitting various common functions each with a variety of Pade-Shammas polynomials. There is a summary table for each fitted function showing the Shammas polynomial parameters, the F statistic, the AICc statistic, the root mean sum of errors squared, and orders of the fitted Pade-Shammas polynomials. I use the the root mean sum of errors squared to select the best Shammas polynomial powers. I have deliberately not included the coefficient of determination and its adjusted value in the table. Why? These values are very close to 1 and we are better served to look at other statistics. I have included the values for the F and AICc statistics, even thought they are infinity and minus infinity.

The next sections display the results of the Pade-Shammas polynomials. The regression coefficients for the $Q_n(x)$ and $D_m(x)$ Shammas polynomials appear as one long list labeled x1 and up. The results include the orders for the $Q_n(x)$ and $D_m(x)$ Shammas polynomials which should easily indicate with regression coefficients belong to $Q_n(x)$ Shammas polynomials and which one belong to $D_m(x)$ Shammas polynomials. The results also include the list of powers associated with

$Q_n(x)$ and $D_m(x)$ Shammas polynomials. Here is an example. Looking at the results for the inverse hyperbolic cosine:


```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)      -7.65714007212127     0     -Inf       0
    x1                33.3385052174594     0      Inf       0
    x2               -57.1709436312584     0     -Inf       0
    x3                49.1740680668373     0      Inf       0
    x4               -22.7142392781885     0     -Inf       0
    x5                5.89936499013805     0      Inf       0
    x6               -0.936296960435223    0     -Inf       0
    x7               0.0666816376160401    0      Inf       0
    x8                2.76468166667437     0      Inf       0
    x9                -2.5074894781991     0     -Inf       0
    x10               0.742729123722548    0      Inf       0


Number of observations: 991, Error degrees of freedom: 980
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF          MeanSq            F     pValue
                  _______________    ___    ________________    ___    ______

    Total        804.145957758438    990    0.812268644200442
    Model        804.145957758438     10     80.4145957758438    Inf       0
    Residual                    0    980                     0

A = 0.000000, B = 0.152812
order Q(x) = 7.000000, order D(x) = 3.000000
List of powers for Q(x): [0.152812, 0.305624, 0.458436, 0.611248, 0.764060, 0.916872,
1.069684]
List of powers for D(x): [0.152812, 0.305624, 0.458436]
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 9.351430e-07
Corrected MSS of errors squared = 1.322492e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

First look at the orders of the $Q_n(x)$ and $D_m(x)$ Shammas polynomials. They are 7 and 3, respectively. Then look at the list of regression coefficients labeled x1

through x10. Since $Q_n(x)$ has the order of seven, its regression coefficients are the ones labeled x1 through x7. The remaining three regression coefficients are labeled x8 through x10. You can find the list of the $Q_n(x)$ and $D_m(x)$ Shammas polynomials right below the line showing the polynomial orders.

RESULTS FOR THE INVERSE HYPERBOLIC COSINE

Using Power A+B* i

```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     -7.65714007212127     0     -Inf       0
    x1               33.3385052174594     0      Inf       0
    x2              -57.1709436312584     0     -Inf       0
    x3               49.1740680668373     0      Inf       0
    x4              -22.7142392781885     0     -Inf       0
    x5               5.89936499013805     0      Inf       0
    x6              -0.936296960435223    0     -Inf       0
    x7              0.0666816376160401    0      Inf       0
    x8               2.76468166667437     0      Inf       0
    x9              -2.5074894781991      0     -Inf       0
    x10              0.742729123722548    0      Inf       0


Number of observations: 991, Error degrees of freedom: 980
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF          MeanSq           F     pValue
                  _______________    ___    ________________    ___    ______

    Total         804.145957758438    990    0.812268644200442
    Model         804.145957758438     10     80.4145957758438    Inf       0
    Residual                     0    980                     0

A = 0.000000, B = 0.152812
order Q(x) = 7.000000, order D(x) = 3.000000
List of powers for Q(x): [0.152812, 0.305624, 0.458436, 0.611248, 0.764060, 0.916872,
1.069684]
List of powers for D(x): [0.152812, 0.305624, 0.458436]
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 9.351430e-07
Corrected MSS of errors squared = 1.322492e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```


Using Power A+B/i

```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      0.231597869387483     0      Inf        0
    x1              -0.0996254818328427    0     -Inf        0
    x2                3.15921408159802     0      Inf        0
    x3               -15.0525332981518     0     -Inf        0
    x4                18.9287330553568     0      Inf        0
    x5                               0     0      NaN      NaN
    x6               -7.16738622611794     0     -Inf        0
    x7              0.00553988904295155    0      Inf        0
    x8               0.593896562343958     0      Inf        0
    x9               -10.5430509644483     0     -Inf        0
    x10               42.5858817192991     0      Inf        0
    x11              -51.0039089534056     0     -Inf        0
    x12                              0     0      NaN      NaN
    x13               19.3616369653409     0      Inf        0


Number of observations: 991, Error degrees of freedom: 979
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______

    Total       804.145957758685    990    0.812268644200692
    Model       804.145957758685     11     73.1041779780622    Inf       0
    Residual                   0    979                    0

A = 0.000000, B = 0.694529
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.694529, 0.347265, 0.231510, 0.173632, 0.138906, 0.115755]
List of powers for D(x): [0.694529, 0.347265, 0.231510, 0.173632, 0.138906, 0.115755,
0.099218]
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 9.444387e-07
Corrected MSS of errors squared = 1.335638e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
```



```
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ____________________    __    _____    ______

    (Intercept)    -0.607068943091106     0     -Inf        0
    x1               6.08742152279404     0      Inf        0
    x2              -8.67635920106154     0     -Inf        0
    x3                             0     0      NaN       NaN
    x4                             0     0      NaN       NaN
    x5               5.54807755286479     0      Inf        0
    x6                             0     0      NaN       NaN
    x7              -2.35207093204665     0     -Inf        0
    x8               53.1300786375054     0      Inf        0
    x9              -354.890621972371     0     -Inf        0
    x10              1007.97937579804     0      Inf        0
    x11             -1510.59156519966     0     -Inf        0
    x12              1257.16525789763     0      Inf        0
    x13             -551.795117408575     0     -Inf        0
    x14              100.002591640815     0      Inf        0


Number of observations: 991, Error degrees of freedom: 979
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                 _______________    ___    ____________________    ___    ______

    Total        804.145957753158    990    0.812268644195109
    Model        804.145957753158     11     73.1041779775598    Inf       0
    Residual                    0    979                    0

A = 0.000000, B = 0.100000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 1.039883e-06
Corrected MSS of errors squared = 1.470616e-06
R-Squared =   1.00000000
```



```
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)    -20.5711606525456    0      -Inf        0
    x1              684.804235361524    0       Inf        0
    x2             -2988.58926517397    0      -Inf        0
    x3              4970.82413045046    0       Inf        0
    x4             -3256.71758171713    0      -Inf        0
    x5                             0    0       NaN      NaN
    x6                864.8317433432    0       Inf        0
    x7             -254.582101610785    0      -Inf        0
    x8              16.6651672231044    0       Inf        0
    x9             -46.8023871335487    0      -Inf        0
    x10             47.7017367008104    0       Inf        0
    x11            -16.5645243526515    0      -Inf        0


Number of observations: 991, Error degrees of freedom: 980
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F      pValue
                  _______________    ___    ____________________    ___    ______

    Total        804.14595775528    990    0.812268644197253
    Model        804.14595775528     10      80.414595775528    Inf        0
    Residual                   0    980                    0

A = 0.002999, B = 0.106413, C = 0.000000
order Q(x) = 7.000000, order D(x) = 4.000000
List of powers for Q(x): [0.109412, 0.153490, 0.187312, 0.215825, 0.240946, 0.263656,
0.284541]
List of powers for D(x): [0.109412, 0.153490, 0.187312, 0.215825]
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 6.013680e-07
Corrected MSS of errors squared = 1.041600e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
```



```
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      -2.09556452004188     0     -Inf        0
    x1                3.60785831589973     0      Inf        0
    x2               -1.7791951229907      0     -Inf        0
    x3                                0     0      NaN      NaN
    x4               -0.920144144697184    0     -Inf        0
    x5                2.10152193433366     0      Inf        0
    x6               -0.850246205258021    0     -Inf        0
    x7               -0.064230254475905    0     -Inf        0
    x8                1.47356102391029     0      Inf        0
    x9               -0.659740259855103    0     -Inf        0
    x10               0.611420382836852    0      Inf        0
    x11              -0.648199169457652    0     -Inf        0
    x12               0.0986372241645598   0      Inf        0
    x13               0.120358013372217    0      Inf        0
    x14              0.00394012458783382   0      Inf        0


Number of observations: 991, Error degrees of freedom: 977
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                  _____________    ___    ________________    ___    ______

    Total       804.145957758919    990    0.812268644200928
    Model       804.145957758919     13     61.8573813660707    Inf       0
    Residual                   0    977                     0

A = 0.000000, B = 0.117525, C = 0.202333
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.117525, 0.437384, 0.757243, 1.077102, 1.396960, 1.716819,
2.036678]
List of powers for D(x): [0.117525, 0.437384, 0.757243, 1.077102, 1.396960, 1.716819,
2.036678]
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 3.140402e-07
```



```
Corrected MSS of errors squared = 5.439335e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      0.275356235819997     0      Inf       0
    x1              -0.126570823822444     0     -Inf       0
    x2               0.395272677752026     0      Inf       0
    x3              -0.544058089263608     0     -Inf       0
    x4              0.00739665652561899    0      Inf       0
    x5              -0.0253692173058622    0     -Inf       0
    x6               0.340191283343167     0      Inf       0
    x7               0.942540619884829     0      Inf       0
    x8              -0.384900077436039     0     -Inf       0
    x9               0.147466913036586     0      Inf       0
    x10             -0.0273230096580313    0     -Inf       0


Number of observations: 991, Error degrees of freedom: 980
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                  _____________    ___    _________________    ___    ______

    Total        804.14595775857    990    0.812268644200576
    Model        804.14595775857     10     80.4145957758571    Inf       0
    Residual                   0    980                    0

A = 0.259057, B = 0.100000, C = 0.242635
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.359057, 0.216422, 0.073787]
List of powers for D(x): [0.359057, 0.216422, 0.073787, -0.068847, -0.211482, -
0.354117, -0.496752]
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 3.337864e-07
Corrected MSS of errors squared = 5.781351e-07
R-Squared =   1.00000000
```



```
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     -6.54897489320197    0     -Inf       0
    x1                16.342927014663    0      Inf       0
    x2               -36.8062381819599    0     -Inf       0
    x3                31.6616161304753    0      Inf       0
    x4                25.9511249025635    0      Inf       0
    x5               -49.1379437435576    0     -Inf       0
    x6                18.0964037761085    0      Inf       0
    x7               0.441085003257686    0      Inf       0
    x8                 1.5416459144726    0      Inf       0
    x9                2.09159169895671    0      Inf       0
    x10              -9.00371841449092    0     -Inf       0
    x11               8.59601062646321    0      Inf       0
    x12              -0.910368492982139    0     -Inf       0
    x13              -1.31572962012511    0     -Inf       0


Number of observations: 991, Error degrees of freedom: 977
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq           F     pValue
                  ___________________    ___    ____________________    ___    ______

    Total       804.145957762918    990    0.812268644204967
    Model       804.145957762918     13     61.8573813663783    Inf       0
    Residual                   0    977                     0

A = 0.092614, B = 0.103277, C = 0.360000
order Q(x) = 7.000000, order D(x) = 6.000000
List of powers for Q(x): [0.195891, 0.659168, 0.911561, 1.129260, 1.328998, 1.517259,
1.697368]
List of powers for D(x): [0.195891, 0.659168, 0.911561, 1.129260, 1.328998, 1.517259]
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 1.443199e-07
Corrected MSS of errors squared = 2.499693e-07
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ____________________    __    _____    ______

    (Intercept)    -2.31403543248461     0     -Inf        0
    x1              16.0774547253833     0      Inf        0
    x2             -23.8127816751693     0     -Inf        0
    x3                           0       0      NaN       NaN
    x4              18.8896833972755     0      Inf        0
    x5             -8.84032101445198     0     -Inf        0
    x6              14.4171435946962     0      Inf        0
    x7              -59.529967650162     0     -Inf        0
    x8              119.653891805069     0      Inf        0
    x9             -134.281132174544     0     -Inf        0
    x10             85.9896074153046     0      Inf        0
    x11            -29.5014650868546     0     -Inf        0
    x12             4.25190952858039     0      Inf        0


Number of observations: 991, Error degrees of freedom: 979
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq          F     pValue
                  _____________    ___    _________________    ___    ______

    Total        804.145957758762    990    0.81226864420077
    Model        804.145957758762     11    73.1041779780693    Inf       0
    Residual                    0    979                   0

A = 0.000000, B = 0.131595, C = 0.021625
order Q(x) = 5.000000, order D(x) = 7.000000
List of powers for Q(x): [0.131595, 0.207728, 0.271178, 0.328063, 0.380753]
List of powers for D(x): [0.131595, 0.207728, 0.271178, 0.328063, 0.380753, 0.430463,
0.477915]
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 1.244453e-05
Corrected MSS of errors squared = 2.155455e-05
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      -4.21440131813467    0     -Inf       0
    x1                8.44989853382429    0      Inf       0
    x2               -19.7207171795824    0     -Inf       0
    x3                30.4139749047284    0      Inf       0
    x4               -19.1645195476864    0     -Inf       0
    x5                4.23576462518046    0      Inf       0
    x6                1.40604225858568    0      Inf       0
    x7               0.272780520197806    0      Inf       0
    x8               -2.46763455369908    0     -Inf       0
    x9                2.80221201859251    0      Inf       0
    x10              -1.23473872947692    0     -Inf       0
    x11               0.247530179334794    0      Inf       0
    x12              -0.0266405726177113    0     -Inf       0


Number of observations: 991, Error degrees of freedom: 978
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq           F     pValue
                  _____________    ___    ________________    ___    ______

    Total       804.145957758374    990    0.812268644200377
    Model       804.145957758374     12     67.0121631465311    Inf       0
    Residual                   0    978                     0

A = 0.061638, B = 0.100000, C = 0.483502
order Q(x) = 5.000000, order D(x) = 7.000000
List of powers for Q(x): [0.161638, 0.686561, 0.918618, 1.099088, 1.252249]
List of powers for D(x): [0.161638, 0.686561, 0.918618, 1.099088, 1.252249, 1.387730,
1.510547]
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 4.964112e-07
Corrected MSS of errors squared = 8.598094e-07
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate           SE     tStat     pValue
                    ____________________    __    _____    _______

    (Intercept)       -2.1836711556097      0     -Inf        0
    x1                 6.49506796220532     0      Inf        0
    x2                                0     0      NaN       NaN
    x3                -6.27076797669487     0     -Inf        0
    x4                 2.25765939612071     0      Inf        0
    x5                -0.298288225429545    0     -Inf        0
    x6                 15.5892519011767     0      Inf        0
    x7                -16.7848874769485     0     -Inf        0
    x8                 2.5360366711197      0      Inf        0
    x9                -0.392852832656103    0     -Inf        0
    x10                0.0548718801276958   0      Inf        0
    x11               -0.00250109164412567  0     -Inf        0
    x12               7.07847148139394e-05  0      Inf        0


Number of observations: 991, Error degrees of freedom: 979
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq          F     pValue
                  _____________    ___    _________________    ___    ______

    Total       804.145957758614    990    0.81226864420062
    Model       804.145957758614     11    73.1041779780558    Inf       0
    Residual                   0    979                   0

A = 0.080462, B = 1.763951
order Q(x) = 5.000000, order D(x) = 7.000000
List of powers for Q(x): [0.080462, 0.094947, 0.171873, 0.312225, 0.501499]
List of powers for D(x): [0.080462, 0.094947, 0.171873, 0.312225, 0.501499, 0.727221,
0.980197]
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 5.406037e-07
Corrected MSS of errors squared = 7.645291e-07
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the inverse hyperbolic cosine function. The results are sorted in ascending order using the MSSE values.

Power	*A*	*B*	*C*	*F*	*AICc*	*MSSE*	*Q/D Orders*
A+B*i+C*sqrt(i-1)	0.092	0.103	0.36	Inf	-Inf	1.44319900E-07	7, 6
A+B*i+C*(i-1)	0	0.117	0.202	Inf	-Inf	3.14040200E-07	7, 7
A+B*i-C*(i-1)	0.259	0.1	0.242	Inf	-Inf	3.33786400E-07	3, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.061	0.1	0.483	Inf	-Inf	4.96411200E-07	5, 7
A+B*log10(i)^4	0.08	1.763		Inf	-Inf	5.40603700E-07	5, 7
A+B*sqrt(i+C)	0.002	0.106	0	Inf	-Inf	6.01368000E-07	7, 4
A+B*i	0	0.152		Inf	-Inf	9.35143000E-07	7, 3
A+B/i	0	0.694		Inf	-Inf	9.44438700E-07	6, 7
A+B*sqrt(i)	0	0.1		Inf	-Inf	1.03988300E-06	7, 7
A+B*sqrt(i)+C*(i-1)	0	0.131	0.021	Inf	-Inf	1.24445300E-05	5, 7

RESULTS FOR THE ARC COSINE

Using Power A+B* i

```
Fitting arccos(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ________________    __    _____    ______

    (Intercept)     1.57079632679126    0       Inf       0
    x1               -43.456724171806   0      -Inf       0
    x2              -90.6908551599143   0      -Inf       0
    x3               761.486649383669   0       Inf       0
    x4              -1490.18831588691   0      -Inf       0
    x5               1380.95075672938   0       Inf       0
    x6              -639.703606198682   0      -Inf       0
    x7               120.031298977697   0       Inf       0
    x8                 28.91069463147   0       Inf       0
    x9               46.2094276835702   0       Inf       0
    x10             -440.593949332277   0      -Inf       0
    x11              859.550127319194   0       Inf       0
    x12             -780.442097674202   0      -Inf       0
    x13              350.674361491544   0       Inf       0
    x14             -63.3085648227558   0      -Inf       0


Number of observations: 101, Error degrees of freedom: 86
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F      pValue
                  _____________    ___    _________________    ___    ______

    Total        14.8796738812058    100    0.148796738812058
    Model        14.8796738812058     14     1.06283384865756    Inf       0
    Residual                    0     86                    0

A = 0.357763, B = 0.100000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.457763, 0.557763, 0.657763, 0.757763, 0.857763, 0.957763,
1.057763]
List of powers for D(x): [0.457763, 0.557763, 0.657763, 0.757763, 0.857763, 0.957763,
1.057763]
Fitting arccos(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.131438e-05
Corrected MSS of errors squared = 1.600096e-05
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting arccos(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)        1.57079632679476     0      Inf        0
    x1              -0.00341315302811853    0     -Inf        0
    x2                -0.521909287043382    0     -Inf        0
    x3                 12.6914812888227     0      Inf        0
    x4                -57.6689726203883     0     -Inf        0
    x5                 73.0645069547046     0      Inf        0
    x6                                0     0      NaN      NaN
    x7                -29.1324895098621     0     -Inf        0
    x8               0.00362027529560987    0      Inf        0
    x9                0.108918902499885     0      Inf        0
    x10               -4.18917781987594     0     -Inf        0
    x11                15.8191491088084     0      Inf        0
    x12                               0     0      NaN      NaN
    x13               -45.4368590516594     0     -Inf        0
    x14                34.6943485848124     0      Inf        0


Number of observations: 101, Error degrees of freedom: 88
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq           F     pValue
                 ___________________    ___    ____________________    ___    ______

    Total        14.8796738812107     100    0.148796738812107
    Model        14.8796738812107      12     1.23997282343423     Inf       0
    Residual                    0      88                    0

A = 0.000000, B = 0.229127
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.229127, 0.114564, 0.076376, 0.057282, 0.045825, 0.038188,
0.032732]
List of powers for D(x): [0.229127, 0.114564, 0.076376, 0.057282, 0.045825, 0.038188,
0.032732]
```



```
Fitting arccos(x) in range (0.000000, 1.000000)
MSS of errors squared = 4.514345e-05
Corrected MSS of errors squared = 6.384248e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting arccos(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                   ____________________    __    _____    ______

    (Intercept)     1.57079632679329    0      Inf        0
    x1             -815.730583980435    0     -Inf        0
    x2              5019.38300557651    0      Inf        0
    x3             -10714.6281925562    0     -Inf        0
    x4              8772.40840730565    0      Inf        0
    x5                             0    0      NaN      NaN
    x6             -3669.41578495688    0     -Inf        0
    x7              1406.4123522846     0      Inf        0
    x8              537.876186938449    0      Inf        0
    x9             -3435.92987718983    0     -Inf        0
    x10             7883.82331234217    0      Inf        0
    x11            -7831.32359060907    0     -Inf        0
    x12             2491.26500427837    0      Inf        0
    x13             933.743599271783    0      Inf        0
    x14            -578.454635032125    0     -Inf        0


Number of observations: 101, Error degrees of freedom: 87
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq           DF           MeanSq            F      pValue
                   _________________    ___    ___________________    ___    ______

    Total       14.8796738812314     100    0.148796738812314
    Model       14.8796738812314      13     1.14459029855626     Inf        0
    Residual                   0      87                    0

A = 0.134094, B = 0.100000
order Q(x) = 7.000000, order D(x) = 7.000000
```



```
List of powers for Q(x): [0.234094, 0.275515, 0.307299, 0.334094, 0.357700, 0.379042,
0.398669]
List of powers for D(x): [0.234094, 0.275515, 0.307299, 0.334094, 0.357700, 0.379042,
0.398669]
Fitting arccos(x) in range (0.000000, 1.000000)
MSS of errors squared = 8.918658e-05
Corrected MSS of errors squared = 1.261289e-04
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting arccos(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)      1.57079632686646     0       Inf        0
    x1              -2469.68948023539     0      -Inf        0
    x2               14842.6884644499     0       Inf        0
    x3              -31401.2759957678     0      -Inf        0
    x4               25711.5074530851     0       Inf        0
    x5                             0     0       NaN      NaN
    x6              -10925.6555420795     0      -Inf        0
    x7               4240.85430422263     0       Inf        0
    x8               1593.61470050916     0       Inf        0
    x9              -9819.19300680217     0      -Inf        0
    x10              22061.371290224     0       Inf        0
    x11             -21727.9402450845     0      -Inf        0
    x12              7074.93331314213     0       Inf        0
    x13              2323.75939078687     0       Inf        0
    x14             -1505.54544827077     0      -Inf        0


Number of observations: 101, Error degrees of freedom: 87
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq           DF           MeanSq            F      pValue
                  ____________________    ___    ____________________    ___    ______

    Total        14.8796738812256     100    0.148796738812256
    Model        14.8796738812256      13     1.14459029855582     Inf        0
    Residual                    0      87                    0
```



```
A = 0.645185, B = 0.247036, C = 0.236532
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.919888, 1.014629, 1.089612, 1.153656, 1.210490, 1.262110,
1.309733]
List of powers for D(x): [0.919888, 1.014629, 1.089612, 1.153656, 1.210490, 1.262110,
1.309733]
Fitting arccos(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.174722e-05
Corrected MSS of errors squared = 2.034678e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting arccos(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                      Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      1.57079632679508     0      Inf       0
    x1              -9.69232271284722     0     -Inf       0
    x2               25.226325070692      0      Inf       0
    x3              -37.7970473857247     0     -Inf       0
    x4               37.2962303498598     0      Inf       0
    x5              -24.9910836889691     0     -Inf       0
    x6               10.3083064185215     0      Inf       0
    x7              -1.92120437841514     0     -Inf       0
    x8               6.16809710507429     0      Inf       0
    x9              -16.0133742957663     0     -Inf       0
    x10              23.5921508689123     0      Inf       0
    x11             -22.0836757929774     0     -Inf       0
    x12              13.3564032572472     0      Inf       0
    x13             -4.77140691190244     0     -Inf       0
    x14             0.751806500002245     0      Inf       0


Number of observations: 101, Error degrees of freedom: 86
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq          DF          MeanSq            F     pValue
                    ____________________    ___    ____________________    ___    ______
```



```
    Total       14.8796738812114    100    0.148796738812114
    Model       14.8796738812114     14     1.06283384865796    Inf       0
    Residual                   0     86                    0

A = 0.000000, B = 0.270000, C = 0.000000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.270000, 0.540000, 0.810000, 1.080000, 1.350000, 1.620000,
1.890000]
List of powers for D(x): [0.270000, 0.540000, 0.810000, 1.080000, 1.350000, 1.620000,
1.890000]
Fitting arccos(x) in range (0.000000, 1.000000)
MSS of errors squared = 6.357835e-06
Corrected MSS of errors squared = 1.101209e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting arccos(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      1.57079632679406     0      Inf        0
    x1              -2202.29288484209     0     -Inf        0
    x2               5899.7624467663      0      Inf        0
    x3                             0     0      NaN       NaN
    x4              -16350.1688653628     0     -Inf        0
    x5               22816.573075345      0      Inf        0
    x6              -12897.7451044397     0     -Inf        0
    x7               2732.30053620819     0      Inf        0
    x8               1453.53393850745     0      Inf        0
    x9              -4195.03491075626     0     -Inf        0
    x10              1547.56555802709     0      Inf        0
    x11              7520.52216900646     0      Inf        0
    x12             -11530.0820415162     0     -Inf        0
    x13              6581.42159526681     0      Inf        0
    x14             -1376.92631393491     0     -Inf        0


Number of observations: 101, Error degrees of freedom: 87
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```


	SumSq	DF	MeanSq	F	pValue
Total	14.8796738813592	100	0.148796738813592		
Model	14.8796738813592	13	1.1445902985661	Inf	0
Residual	0	87	0		

```
A = 0.707567, B = 0.209068, C = 0.139269
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.916635, 0.986434, 1.056233, 1.126032, 1.195832, 1.265631,
1.335430]
List of powers for D(x): [0.916635, 0.986434, 1.056233, 1.126032, 1.195832, 1.265631,
1.335430]
Fitting arccos(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.142964e-05
Corrected MSS of errors squared = 1.979672e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting arccos(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
```

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	1.57079632679578	0	Inf	0
x1	-3.03417517678718	0	-Inf	0
x2	5.21144570913736	0	Inf	0
x3	-8.76621233366959	0	-Inf	0
x4	8.9559820546881	0	Inf	0
x5	-5.22160063961711	0	-Inf	0
x6	1.28376405983404	0	Inf	0
x7	1.93150670648911	0	Inf	0
x8	-3.30138349646443	0	-Inf	0
x9	5.0218038804782	0	Inf	0
x10	-3.96606625478548	0	-Inf	0
x11	1.34958012375382	0	Inf	0
x12	0.089634059895191	0	Inf	0
x13	-0.125076261160044	0	-Inf	0

```
Number of observations: 101, Error degrees of freedom: 87
```



```
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq            F     pValue
                ___________________    ___    ___________________    ___    ______

    Total       14.8796738812115    100    0.148796738812115
    Model       14.8796738812115     13     1.14459029855473    Inf       0
    Residual                   0     87                    0

A = 0.046510, B = 0.132984, C = 0.360000
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.179493, 0.672477, 0.954577, 1.201982, 1.431427, 1.649396]
List of powers for D(x): [0.179493, 0.672477, 0.954577, 1.201982, 1.431427, 1.649396,
1.859211]
Fitting arccos(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.150549e-05
Corrected MSS of errors squared = 1.992809e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting arccos(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      1.57079632679466    0      Inf       0
    x1              -29.3088653274174    0     -Inf       0
    x2               126.530904668925    0      Inf       0
    x3              -253.090128538486    0     -Inf       0
    x4               268.517863976157    0      Inf       0
    x5              -146.876360615805    0     -Inf       0
    x6               32.655789510409     0      Inf       0
    x7               18.6624486821928    0      Inf       0
    x8              -80.7493867261917    0     -Inf       0
    x9               163.246976520823    0      Inf       0
    x10             -180.001898281916    0     -Inf       0
    x11              110.068534608595    0      Inf       0
    x12             -34.2593518249202    0     -Inf       0
    x13              4.03267572214619    0      Inf       0
```



```
Number of observations: 101, Error degrees of freedom: 87
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq            F      pValue
                ___________________    ___    ___________________    ___    ______

    Total       14.8796738812114    100    0.148796738812114
    Model       14.8796738812114     13     1.14459029855472    Inf       0
    Residual                   0     87                    0

A = 0.000000, B = 0.400000, C = 0.031418
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.400000, 0.597104, 0.755657, 0.894255, 1.020100, 1.136887]
List of powers for D(x): [0.400000, 0.597104, 0.755657, 0.894255, 1.020100, 1.136887,
1.246810]
Fitting arccos(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.542243e-05
Corrected MSS of errors squared = 2.671243e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting arccos(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ____________________    __    _____    ______

    (Intercept)     1.57079632679402     0      Inf       0
    x1             -44.3734955639346     0     -Inf       0
    x2              207.988045656669     0      Inf       0
    x3             -130.706372369087     0     -Inf       0
    x4             -567.378467880046     0     -Inf       0
    x5              1084.89287948406     0      Inf       0
    x6             -721.797769962765     0     -Inf       0
    x7              169.804384309032     0      Inf       0
    x8              30.0366506908317     0      Inf       0
    x9             -181.784773741438     0     -Inf       0
    x10             355.352350367167     0      Inf       0
    x11            -285.553792926247     0     -Inf       0
    x12             82.9495652290495     0      Inf       0
```



```
Number of observations: 101, Error degrees of freedom: 88
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq            F     pValue
                ________________    ___    ________________    ___    ______

    Total       14.8796738812183    100    0.148796738812183
    Model       14.8796738812183     12     1.23997282343486    Inf       0
    Residual                   0     88                    0

A = 0.113229, B = 0.100000, C = 0.049748
order Q(x) = 7.000000, order D(x) = 5.000000
List of powers for Q(x): [0.213229, 0.304398, 0.356788, 0.399394, 0.436331, 0.469417,
0.499660]
List of powers for D(x): [0.213229, 0.304398, 0.356788, 0.399394, 0.436331]
Fitting arccos(x) in range (0.000000, 1.000000)
MSS of errors squared = 5.058315e-05
Corrected MSS of errors squared = 8.761259e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting arccos(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      1.57079632679486    0      Inf       0
    x1              -28.5348974072509    0     -Inf       0
    x2               28.5466780947089    0      Inf       0
    x3              -1.16019860110646    0     -Inf       0
    x4              -0.422378413545505    0     -Inf       0
    x5               18.6659921378593    0      Inf       0
    x6              -18.9159549861201    0     -Inf       0
    x7               1.17905116831604    0      Inf       0
    x8              -0.134343486460277    0     -Inf       0
    x9               0.340071672411207    0      Inf       0
    x10             -0.166991378804352    0     -Inf       0
    x11             0.0321732082647812    0      Inf       0


Number of observations: 101, Error degrees of freedom: 89
```



```
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq            F      pValue
                ___________________    ___    ____________________    ___    ______

    Total       14.8796738812115    100    0.148796738812115
    Model       14.8796738812115     11     1.35269762556469    Inf       0
    Residual                   0     89                     0

A = 0.199118, B = 2.632339
order Q(x) = 4.000000, order D(x) = 7.000000
List of powers for Q(x): [0.199118, 0.220734, 0.335531, 0.544978]
List of powers for D(x): [0.199118, 0.220734, 0.335531, 0.544978, 0.827430, 1.164274,
1.541790]
Fitting arccos(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.789853e-05
Corrected MSS of errors squared = 2.531234e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the arc cosine function. The results are sorted in ascending order using the MSSE values.

Power	*A*	*B*	*C*	*F*	*AICc*	*MSSE*	*Q/D Orders*
A+B*i+C*(i-1)	0	0.27	0	Inf	-Inf	6.35783500E-06	7, 7
A+B*i	0.357	0.1		Inf	-Inf	1.13143800E-05	7, 7
A+B*i-C*(i-1)	0.707	0.209	0.139	Inf	-Inf	1.14296400E-05	7, 7
A+B*i+C*sqrt(i-1)	0.046	0.132	0.36	Inf	-Inf	1.15054900E-05	6, 7
A+B*sqrt(i+C)	0.645	0.247	0.236	Inf	-Inf	1.17472200E-05	7, 7
A+B*sqrt(i)+C*(i-1)	0	0.4	0.031	Inf	-Inf	1.54224300E-05	6, 7
A+B*log10(i)^4	0.199	2.632		Inf	-Inf	1.78985300E-05	4, 7
A+B/i	0	0.229		Inf	-Inf	4.51434500E-05	7, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.113	0.1	0.049	Inf	-Inf	5.05831500E-05	7, 5
A+B*sqrt(i)	0.134	0.1		Inf	-Inf	8.91865800E-05	7, 7

RESULTS FOR THE ARC SINE

Using Power A+B* i

```
Fitting arcsin(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    2.61596085506494e-14     0      Inf       0
    x1              0.00271322209040669     0      Inf       0
    x2              -0.0212366191302404     0     -Inf       0
    x3               0.0634094226772156     0      Inf       0
    x4              -0.0877803816537527     0     -Inf       0
    x5               0.0567467471111257     0      Inf       0
    x6              -0.0138523636686379     0     -Inf       0
    x7                 6.52409971196353     0      Inf       0
    x8                -18.5789264227971     0     -Inf       0
    x9                 29.9078447561642     0      Inf       0
    x10               -29.3538693093213     0     -Inf       0
    x11                17.5401285350343     0      Inf       0
    x12               -5.90001864657175     0     -Inf       0
    x13                0.860741358059224     0      Inf       0


Number of observations: 101, Error degrees of freedom: 87
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                  _____________    ___    ________________    ___    ______

    Total        14.8796738812115    100    0.148796738812115
    Model        14.8796738812115     13     1.14459029855473    Inf       0
    Residual                    0     87                    0

A = 0.000000, B = 0.117349
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.117349, 0.234697, 0.352046, 0.469394, 0.586743, 0.704091]
List of powers for D(x): [0.117349, 0.234697, 0.352046, 0.469394, 0.586743, 0.704091,
0.821440]
Fitting arcsin(x) in range (0.000000, 1.000000)
MSS of errors squared = 2.649912e-05
Corrected MSS of errors squared = 3.747541e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
```



```
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting arcsin(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -5.97981254580512e-14    0     -Inf       0
    x1              -0.00619394340410247    0     -Inf       0
    x2                  1.03365133798345    0      Inf       0
    x3                 -16.8768890269518    0     -Inf       0
    x4                  86.8192161829786    0      Inf       0
    x5                 -187.812426503391    0     -Inf       0
    x6                  179.796020068434    0      Inf       0
    x7                 -62.9533780420367    0     -Inf       0
    x8                0.0529008435447592    0      Inf       0
    x9                 -3.48547976089073    0     -Inf       0
    x10                   26.55974564912    0      Inf       0
    x11                -55.4127993336421    0     -Inf       0
    x12                 33.2856325549378    0      Inf       0


Number of observations: 101, Error degrees of freedom: 88
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF          MeanSq            F     pValue
                  _______________    ___    ________________    ___    ______

    Total        14.8796738812122    100    0.148796738812122
    Model        14.8796738812122     12     1.23997282343435    Inf       0
    Residual                    0     88                     0

A = 0.015819, B = 0.484141
order Q(x) = 7.000000, order D(x) = 5.000000
List of powers for Q(x): [0.499960, 0.257890, 0.177200, 0.136855, 0.112648, 0.096510,
0.084982]
List of powers for D(x): [0.499960, 0.257890, 0.177200, 0.136855, 0.112648]
Fitting arcsin(x) in range (0.000000, 1.000000)
MSS of errors squared = 5.836815e-05
Corrected MSS of errors squared = 8.254502e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
```



```
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting arcsin(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    6.84391921000774e-13     0      Inf       0
    x1               -0.496274642089966     0     -Inf       0
    x2                 7.99341760128281     0      Inf       0
    x3                -41.5026712263319     0     -Inf       0
    x4                 99.9326393878796     0      Inf       0
    x5                -123.58340338145      0     -Inf       0
    x6                 76.3878928249908     0      Inf       0
    x7                -18.731600544699      0     -Inf       0
    x8                 41.0987188690704     0      Inf       0
    x9                -226.010525777887     0     -Inf       0
    x10                514.869427737801     0      Inf       0
    x11               -584.651444815693     0     -Inf       0
    x12                329.245108538796     0      Inf       0
    x13               -73.5512845645559     0     -Inf       0


Number of observations: 101, Error degrees of freedom: 87
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                  _____________    ___    ____________________    ___    ______

    Total        14.8796738812096    100    0.148796738812096
    Model        14.8796738812096     13     1.14459029855459    Inf       0
    Residual                    0     87                    0

A = 0.000000, B = 0.222740
order Q(x) = 7.000000, order D(x) = 6.000000
List of powers for Q(x): [0.222740, 0.315003, 0.385798, 0.445481, 0.498063, 0.545600,
0.589316]
List of powers for D(x): [0.222740, 0.315003, 0.385798, 0.445481, 0.498063, 0.545600]
Fitting arcsin(x) in range (0.000000, 1.000000)
MSS of errors squared = 8.047159e-06
Corrected MSS of errors squared = 1.138040e-05
R-Squared =   1.00000000
```



```
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting arcsin(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ______________________    __    _____    ______

    (Intercept)    2.91041906504263e-10    0      Inf       0
    x1              0.00138593782211927    0      Inf       0
    x2             -0.00321106261770551    0     -Inf       0
    x3              0.0018250289133776     0      Inf       0
    x4                 6988.9438620978     0      Inf       0
    x5                -54152.0888449538    0     -Inf       0
    x6                 169205.910717085    0      Inf       0
    x7                -275131.455865906    0     -Inf       0
    x8                 246874.299036357    0      Inf       0
    x9                -116343.076687597    0     -Inf       0
    x10                22558.4677829779    0      Inf       0


Number of observations: 101, Error degrees of freedom: 90
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F      pValue
                ___________________    ___    ___________________    ___    ______

    Total       14.8796738784001    100    0.148796738784001
    Model       14.8796738784001     10     1.48796738784001    Inf       0
    Residual                   0     90                     0

A = 0.071412, B = 0.100000, C = 1.526897
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.230374, 0.259212, 0.284177]
List of powers for D(x): [0.230374, 0.259212, 0.284177, 0.306505, 0.326890, 0.345764,
0.363420]
Fitting arcsin(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.346224e-04
Corrected MSS of errors squared = 2.331728e-04
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
```



```
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting arcsin(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -9.75594320238577e-13    0     -Inf       0
    x1              0.000434366159101534    0      Inf       0
    x2               -0.0137837116367913    0     -Inf       0
    x3                  0.44204459598833    0      Inf       0
    x4                 -1.16948463964352    0     -Inf       0
    x5                  1.06785711484264    0      Inf       0
    x6                 -0.327064228188532    0     -Inf       0
    x7                  2.00595464225188    0      Inf       0
    x8                 -1.42501572911665    0     -Inf       0
    x9                  0.270423271645076    0      Inf       0
    x10                 0.512608076134365    0      Inf       0
    x11                -0.663070570206695    0     -Inf       0
    x12                 0.382593044855972    0      Inf       0
    x13                -0.0834949625865144    0     -Inf       0


Number of observations: 101, Error degrees of freedom: 87
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _____________    ___    ________________    ___    ______

    Total        14.8796738812115    100    0.148796738812115
    Model        14.8796738812115     13     1.14459029855473    Inf       0
    Residual                    0     87                    0

A = 0.000000, B = 0.201107, C = 0.140871
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.201107, 0.543085, 0.885063, 1.227041, 1.569020, 1.910998]
List of powers for D(x): [0.201107, 0.543085, 0.885063, 1.227041, 1.569020, 1.910998,
2.252976]
Fitting arcsin(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.129378e-05
Corrected MSS of errors squared = 1.956141e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
```



```
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting arcsin(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -2.10705041628771e-11    0     -Inf       0
    x1                0.0113057193815671    0      Inf       0
    x2                 -0.18943732794813    0     -Inf       0
    x3                  1.65387820050886    0      Inf       0
    x4                 -3.80510692180437    0     -Inf       0
    x5                  3.35185536084946    0      Inf       0
    x6                 -1.0224932006763     0     -Inf       0
    x7                  8.30662744927207    0      Inf       0
    x8                 -19.8123424968093    0     -Inf       0
    x9                  23.817737606036     0      Inf       0
    x10                -16.9525035190402    0     -Inf       0
    x11                 6.82001273647953    0      Inf       0
    x12                -1.17953294079936    0     -Inf       0


Number of observations: 101, Error degrees of freedom: 88
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq           F     pValue
                 _______________    ___    _________________    ___    ______

    Total        14.8796738812112    100    0.148796738812112
    Model        14.8796738812112     12     1.23997282343427    Inf       0
    Residual                    0     88                    0

A = 0.162267, B = 0.248338, C = 0.000000
order Q(x) = 6.000000, order D(x) = 6.000000
List of powers for Q(x): [0.410605, 0.658943, 0.907281, 1.155620, 1.403958, 1.652296]
List of powers for D(x): [0.410605, 0.658943, 0.907281, 1.155620, 1.403958, 1.652296]
Fitting arcsin(x) in range (0.000000, 1.000000)
MSS of errors squared = 6.338664e-06
Corrected MSS of errors squared = 1.097889e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
```



```
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting arcsin(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     2.37557526297789e-13    0      Inf       0
    x1             -0.000279306061204712    0     -Inf       0
    x2                0.0237706221485688    0      Inf       0
    x3                  1.59007330008951    0      Inf       0
    x4                 -5.64235191662946    0     -Inf       0
    x5                  7.23881894183045    0      Inf       0
    x6                 -4.02678503191755    0     -Inf       0
    x7                 0.816754752186117    0      Inf       0
    x8                  5.32265639803189    0      Inf       0
    x9                 -8.94833848057112    0     -Inf       0
    x10                 6.60586771142996    0      Inf       0
    x11                -1.91982127102818    0     -Inf       0
    x12               -0.627946113303209    0     -Inf       0
    x13                 0.79600113384975    0      Inf       0
    x14               -0.228420245293392    0     -Inf       0


Number of observations: 101, Error degrees of freedom: 86
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                 _______________    ___    ________________    ___    ______

    Total        14.8796738812116    100    0.148796738812116
    Model        14.8796738812116     14     1.06283384865797    Inf       0
    Residual                    0     86                    0

A = 0.124825, B = 0.312658, C = 0.000000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.437483, 0.750141, 1.062799, 1.375458, 1.688116, 2.000774,
2.313432]
List of powers for D(x): [0.437483, 0.750141, 1.062799, 1.375458, 1.688116, 2.000774,
2.313432]
Fitting arcsin(x) in range (0.000000, 1.000000)
MSS of errors squared = 2.937046e-06
Corrected MSS of errors squared = 5.087114e-06
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting arcsin(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      1.6738614390179e-13    0      Inf       0
    x1              -5.19900829738961e-05   0     -Inf       0
    x2                0.00267682144178296   0      Inf       0
    x3                   0.46019520385434   0      Inf       0
    x4                  -1.43530993085193   0     -Inf       0
    x5                   1.57237323986969   0      Inf       0
    x6                  -0.692425818095225  0     -Inf       0
    x7                  0.0925447778355341  0      Inf       0
    x8                   0.935974488819787  0      Inf       0
    x9                   1.03743874527483   0      Inf       0
    x10                 -2.16108549216508   0     -Inf       0
    x11                  2.03289432223419   0      Inf       0
    x12                 -1.30341804422162   0     -Inf       0
    x13                  0.576023074291449  0      Inf       0
    x14                 -0.117828561075799  0     -Inf       0


Number of observations: 101, Error degrees of freedom: 86
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq            F     pValue
                ________________    ___    ________________    ___    ______

    Total       14.8796738812115    100    0.148796738812115
    Model       14.8796738812115     14     1.06283384865797    Inf       0
    Residual                   0     86                    0

A = 0.000000, B = 0.100000, C = 0.400000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.541421, 0.973205, 1.400000, 1.823607, 2.244949,
2.664575]
List of powers for D(x): [0.100000, 0.541421, 0.973205, 1.400000, 1.823607, 2.244949,
2.664575]
```



```
Fitting arcsin(x) in range (0.000000, 1.000000)
MSS of errors squared = 4.306815e-06
Corrected MSS of errors squared = 7.459622e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting arcsin(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                      ____________________    __    _____    ______

    (Intercept)    -3.34842989448312e-12     0     -Inf        0
    x1                  -32.486681030906     0     -Inf        0
    x2                   440.986916480103    0      Inf        0
    x3                  -2017.68223626807    0     -Inf        0
    x4                   4388.63699660571    0      Inf        0
    x5                  -4985.9013334182     0     -Inf        0
    x6                   2866.59718079558    0      Inf        0
    x7                  -660.150843149063    0     -Inf        0
    x8                   467.525307473131    0      Inf        0
    x9                  -2512.0352390757     0     -Inf        0
    x10                  3968.97565445065    0      Inf        0
    x11                                0     0      NaN      NaN
    x12                 -5700.05837843928    0     -Inf        0
    x13                  5284.18255797108    0      Inf        0
    x14                 -1507.58990238953    0     -Inf        0


Number of observations: 101, Error degrees of freedom: 87
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq           F     pValue
                  _________________    ___    _____________________    ___    ______

    Total       14.8796738812052     100    0.148796738812052
    Model       14.8796738812052      13     1.14459029855425    Inf       0
    Residual                   0      87                    0

A = 0.182794, B = 0.127247, C = 0.000000
order Q(x) = 7.000000, order D(x) = 7.000000
```



```
List of powers for Q(x): [0.310040, 0.362747, 0.403191, 0.437287, 0.467326, 0.494483,
0.519457]
List of powers for D(x): [0.310040, 0.362747, 0.403191, 0.437287, 0.467326, 0.494483,
0.519457]
Fitting arcsin(x) in range (0.000000, 1.000000)
MSS of errors squared = 6.357235e-06
Corrected MSS of errors squared = 1.101105e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting arcsin(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      2.81306747358741e-16    0      Inf       0
    x1              -1.95224407940193e-07    0     -Inf       0
    x2               2.49800407665756e-07    0      Inf       0
    x3              -6.44074727363659e-08    0     -Inf       0
    x4                9.8314371124856e-09    0      Inf       0
    x5                    1.0326056626056    0      Inf       0
    x6                 -0.0341432476201254    0     -Inf       0
    x7                 0.00169196043763387    0      Inf       0
    x8              -0.000166172286825789    0     -Inf       0
    x9               1.17968637370506e-05    0      Inf       0


Number of observations: 101, Error degrees of freedom: 91
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                 _______________    ___    ________________    ___    ______

    Total        14.8796738812115    100    0.148796738812115
    Model        14.8796738812115      9     1.65329709791239    Inf       0
    Residual                    0     91                    0

A = 0.000069, B = 0.326167
order Q(x) = 4.000000, order D(x) = 5.000000
List of powers for Q(x): [0.000069, 0.002747, 0.016972, 0.042924]
List of powers for D(x): [0.000069, 0.002747, 0.016972, 0.042924, 0.077922]
```



```
Fitting arcsin(x) in range (0.000000, 1.000000)
MSS of errors squared = 3.547909e-04
Corrected MSS of errors squared = 5.017500e-04
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the arc sine function. The results are sorted in ascending order using the MSSE values.

Power	*A*	*B*	*C*	*F*	*AICc*	*MSSE*	*Q/D Orders*
A+B*i+C*sqrt(i-1)	0.124	0.312	0	Inf	-Inf	2.93704600E-06	7, 7
A+B*sqrt(i)+C*(i-1)	0	0.1	0.4	Inf	-Inf	4.30681500E-06	7, 7
A+B*i-C*(i-1)	0.162	0.248	0	Inf	-Inf	6.33866400E-06	6, 6
A+B*sqrt(i)+C*sqrt(i-1)	0.182	0.127	0	Inf	-Inf	6.35723500E-06	7, 7
A+B*sqrt(i)	0	0.222		Inf	-Inf	8.04715900E-06	7, 6
A+B*i+C*(i-1)	0	0.201	0.14	Inf	-Inf	1.12937800E-05	6, 7
A+B*i	0	0.117		Inf	-Inf	2.64991200E-05	6, 7
A+B/i	0.015	0.484		Inf	-Inf	5.83681500E-05	7, 5
A+B*sqrt(i+C)	0.071	0.1	1.526	Inf	-Inf	1.34622400E-04	3, 7
A+B*log10(i)^4	0	0.326		Inf	-Inf	3.54790900E-04	4, 5

RESULTS FOR THE ARC TANGENT

Using Power A+B* i

```
Fitting arctan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                      ____________________    __    _____    ______

    (Intercept)    -1.68269914460934e-13     0     -Inf       0
    x1                0.0036629077505218     0      Inf       0
    x2                 0.972197283088851     0      Inf       0
    x3                 0.537561656818993     0      Inf       0
    x4                 -4.71941377000884     0     -Inf       0
    x5                  7.14991382023113     0      Inf       0
    x6                 -4.89798883206892     0     -Inf       0
    x7                  1.69662339015496     0      Inf       0
    x8                  7.13114580525943     0      Inf       0
    x9                 -20.0572981937859     0     -Inf       0
    x10                 28.8296848542177     0      Inf       0
    x11                -26.3810101870686     0     -Inf       0
    x12                 14.3325054064096     0      Inf       0
    x13                -3.80047992909191     0     -Inf       0


Number of observations: 101, Error degrees of freedom: 87
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F     pValue
                  _______________    ___    ____________________    ___    ______

    Total       5.42875862162701     100    0.0542875862162701
    Model       5.42875862162701      13     0.417596817048231    Inf       0
    Residual                   0      87                     0

A = 0.445341, B = 0.279331
order Q(x) = 7.000000, order D(x) = 6.000000
List of powers for Q(x): [0.724672, 1.004004, 1.283335, 1.562667, 1.841998, 2.121330,
2.400661]
List of powers for D(x): [0.724672, 1.004004, 1.283335, 1.562667, 1.841998, 2.121330]
Fitting arctan(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.324055e-11
Corrected MSS of errors squared = 1.872497e-11
R-Squared =   1.00000000
```



```
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting arctan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                      ____________________    __    _____    ______

    (Intercept)    -3.88852141565674e-14     0     -Inf       0
    x1                0.0123876016021156     0      Inf       0
    x2               -0.0467335156031249     0     -Inf       0
    x3                0.0944996293794637     0      Inf       0
    x4                -0.064166406656884     0     -Inf       0
    x5                0.0190549027225154     0      Inf       0
    x6               -0.0153456510108964     0     -Inf       0
    x7                 0.919776399402454     0      Inf       0
    x8                 -13.1230701090025     0     -Inf       0
    x9                  56.5767445066933     0      Inf       0
    x10                -89.6289177342893     0     -Inf       0
    x11                 46.2516602487727     0      Inf       0


Number of observations: 101, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq          DF           MeanSq           F     pValue
                   _______________    ___    ____________________    ___    ______

    Total       5.42875862162683     100    0.0542875862162683
    Model       5.42875862162683      11     0.493523511056985    Inf       0
    Residual                   0      89                     0

A = 0.000000, B = 1.572490
order Q(x) = 5.000000, order D(x) = 6.000000
List of powers for Q(x): [1.572490, 0.786245, 0.524163, 0.393122, 0.314498]
List of powers for D(x): [1.572490, 0.786245, 0.524163, 0.393122, 0.314498, 0.262082]
Fitting arctan(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.190903e-09
Corrected MSS of errors squared = 1.684191e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
```



```
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting arctan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    4.29865999967428e-13     0      Inf       0
    x1              0.00818726963755553     0      Inf       0
    x2              -0.0566144530755467     0     -Inf       0
    x3                0.127766666249993     0      Inf       0
    x4               -0.118426501253949     0     -Inf       0
    x5               0.0390897517958973     0      Inf       0
    x6                 47.8956674632241     0      Inf       0
    x7                -278.130313080059     0     -Inf       0
    x8                  657.96915747878     0      Inf       0
    x9                -769.138380443966     0     -Inf       0
    x10                 443.41287157103     0      Inf       0
    x11                -101.00900646923     0     -Inf       0


Number of observations: 101, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______

    Total       5.42875862162629    100    0.0542875862162629
    Model       5.42875862162629     11     0.493523511056935    Inf       0
    Residual                   0     89                     0

A = 0.000000, B = 0.100000
order Q(x) = 5.000000, order D(x) = 6.000000
List of powers for Q(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949]
Fitting arctan(x) in range (0.000000, 1.000000)
MSS of errors squared = 2.734315e-08
Corrected MSS of errors squared = 3.866905e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```


Using Power A+B*sqrt(i+C)

```
Fitting arctan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -5.09051842066691e-14    0     -Inf       0
    x1                 0.234803892058013    0      Inf       0
    x2                  -1.4962308782923    0     -Inf       0
    x3                  3.24491520141193    0      Inf       0
    x4                  -2.9473323785139    0     -Inf       0
    x5                 0.964129208829856    0      Inf       0
    x6                  81.5760825135422    0      Inf       0
    x7                 -354.364902764779    0     -Inf       0
    x8                  585.288627225634    0      Inf       0
    x9                 -428.444811403738    0     -Inf       0
    x10                 116.944641498878    0      Inf       0


Number of observations: 101, Error degrees of freedom: 90
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______

    Total       5.42875862163027     100    0.0542875862163027
    Model       5.42875862163027      10     0.542875862163027     Inf       0
    Residual                   0      90                       0

A = 0.023622, B = 0.100000, C = 0.379636
order Q(x) = 5.000000, order D(x) = 5.000000
List of powers for Q(x): [0.141080, 0.177882, 0.207459, 0.232897, 0.255562]
List of powers for D(x): [0.141080, 0.177882, 0.207459, 0.232897, 0.255562]
Fitting arctan(x) in range (0.000000, 1.000000)
MSS of errors squared = 4.264933e-08
Corrected MSS of errors squared = 7.387081e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting arctan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
```



```
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    5.77473358755586e-12     0      Inf       0
    x1             -0.00325463602591543     0     -Inf       0
    x2                0.181625254562198     0      Inf       0
    x3                 1.66196730718277     0      Inf       0
    x4                -3.32943604674479     0     -Inf       0
    x5                 2.82281237189652     0      Inf       0
    x6                -0.844904227155495    0     -Inf       0
    x7                 0.155418581937233    0      Inf       0
    x8                 0.705264835510118    0      Inf       0
    x9                 1.51189667213782     0      Inf       0
    x10               -3.86589584185699     0     -Inf       0
    x11                2.88096217799884     0      Inf       0
    x12               -1.05248518337807     0     -Inf       0


Number of observations: 101, Error degrees of freedom: 88
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______

    Total        5.42875862162703     100    0.0542875862162703
    Model        5.42875862162703      12     0.452396551802252    Inf       0
    Residual                    0      88                      0

A = 0.211802, B = 0.260733, C = 0.084322
order Q(x) = 7.000000, order D(x) = 5.000000
List of powers for Q(x): [0.472536, 0.817591, 1.162646, 1.507702, 1.852757, 2.197813,
2.542868]
List of powers for D(x): [0.472536, 0.817591, 1.162646, 1.507702, 1.852757]
Fitting arctan(x) in range (0.000000, 1.000000)
MSS of errors squared = 2.374420e-10
Corrected MSS of errors squared = 4.112616e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting arctan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
```



```
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)       0.046118100496827     0      Inf       0
    x1               0.00140036977491076    0      Inf       0
    x2               -0.00749496397954672   0     -Inf       0
    x3                0.0188902726142371    0      Inf       0
    x4               -0.0405229909465196    0     -Inf       0
    x5               -0.0193001824572527    0     -Inf       0
    x6               0.000909614455708207   0      Inf       0
    x7               -0.0111195612936673    0     -Inf       0
    x8                0.0773592386915951    0      Inf       0
    x9                -0.257627159690966    0     -Inf       0
    x10               0.741685607031552     0      Inf       0
    x11               0.480984232868382     0      Inf       0
    x12              -0.0312826376665883    0     -Inf       0


Number of observations: 100, Error degrees of freedom: 87
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                ____________________    __    ____________________    ___    ______

    Total       5.23467420800659     99    0.0528754970505716
    Model       5.23467420800659     12     0.436222850667216    Inf       0
    Residual                   0     87                     0

A = 0.000000, B = 0.195111, C = 0.250000
order Q(x) = 6.000000, order D(x) = 6.000000
List of powers for Q(x): [0.195111, 0.140223, 0.085334, 0.030446, -0.024443, -
0.079331]
List of powers for D(x): [0.195111, 0.140223, 0.085334, 0.030446, -0.024443, -
0.079331]
Fitting arctan(x) in range (0.000000, 1.000000)
MSS of errors squared = NaN
Corrected MSS of errors squared = NaN
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting arctan(x) in range (0.000000, 1.000000)
```



```
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    3.7471871707687e-14     0      Inf       0
    x1              0.0868340450242604     0      Inf       0
    x2              -0.881368913458402     0     -Inf       0
    x3               3.77348656198704      0      Inf       0
    x4              -8.76512976411322      0     -Inf       0
    x5               11.694187533678       0      Inf       0
    x6              -8.28546029648666      0     -Inf       0
    x7               2.40708900780335      0      Inf       0
    x8               20.7632382038392      0      Inf       0
    x9              -72.3762002203612      0     -Inf       0
    x10              118.366721342762      0      Inf       0
    x11             -107.094738052899      0     -Inf       0
    x12              51.8886459327458      0      Inf       0
    x13             -10.5854037020872      0     -Inf       0


Number of observations: 101, Error degrees of freedom: 87
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  ___________________    ___    ____________________    ___    ______

    Total       5.42875862162693    100    0.0542875862162693
    Model       5.42875862162693     13     0.417596817048225    Inf       0
    Residual                   0     87                     0

A = 0.153138, B = 0.118140, C = 0.000000
order Q(x) = 7.000000, order D(x) = 6.000000
List of powers for Q(x): [0.271278, 0.389418, 0.507558, 0.625698, 0.743837, 0.861977,
0.980117]
List of powers for D(x): [0.271278, 0.389418, 0.507558, 0.625698, 0.743837, 0.861977]
Fitting arctan(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.643939e-10
Corrected MSS of errors squared = 2.847386e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)


```
Fitting arctan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                         Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    4.45420361808355e-12     0      Inf       0
    x1                0.307984839789493     0      Inf       0
    x2                -4.53607521341613     0     -Inf       0
    x3                 27.4626167325061     0      Inf       0
    x4                -53.0650323036789     0     -Inf       0
    x5                 40.3265802102785     0      Inf       0
    x6                -10.2488487380259     0     -Inf       0
    x7                 215.057761743011     0      Inf       0
    x8                -1785.76430673101     0     -Inf       0
    x9                 5942.18187949407     0      Inf       0
    x10               -10205.0632967501     0     -Inf       0
    x11                9606.18784966117     0      Inf       0
    x12               -4721.91774297849     0     -Inf       0
    x13                950.003078243233     0      Inf       0


Number of observations: 101, Error degrees of freedom: 87
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F     pValue
                  _______________    ___    ____________________    ___    ______

    Total        5.42875862166878    100    0.0542875862166878
    Model        5.42875862166878     13     0.417596817051445    Inf       0
    Residual                    0     87                     0

A = 0.432901, B = 0.400000, C = 0.000000
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.832901, 0.998586, 1.125721, 1.232901, 1.327328, 1.412697]
List of powers for D(x): [0.832901, 0.998586, 1.125721, 1.232901, 1.327328, 1.412697,
1.491201]
Fitting arctan(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.950609e-11
Corrected MSS of errors squared = 3.378553e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```


Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting arctan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -3.02603039450755e-13    0     -Inf        0
    x1                -0.206121301791514    0     -Inf        0
    x2                  2.46202611581532    0      Inf        0
    x3                 -9.44605232462911    0     -Inf        0
    x4                  16.0536427153765    0      Inf        0
    x5                 -12.6005747296089    0     -Inf        0
    x6                  3.73796126186351    0      Inf        0
    x7                  47.1101551574384    0      Inf        0
    x8                 -238.667419519637    0     -Inf        0
    x9                  459.113356830472    0      Inf        0
    x10                -351.366555644646    0     -Inf        0
    x11                                0    0      NaN      NaN
    x12                 135.333048239902    0      Inf        0
    x13                -50.5237077260214    0     -Inf        0


Number of observations: 101, Error degrees of freedom: 88
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______

    Total       5.42875862162473     100    0.0542875862162473
    Model       5.42875862162473      12     0.452396551802061    Inf       0
    Residual                   0      88                     0

A = 0.020934, B = 0.175344, C = 0.000000
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.196277, 0.268907, 0.324638, 0.371621, 0.413014, 0.450436]
List of powers for D(x): [0.196277, 0.268907, 0.324638, 0.371621, 0.413014, 0.450436,
0.484849]
Fitting arctan(x) in range (0.000000, 1.000000)
MSS of errors squared = 7.460719e-10
Corrected MSS of errors squared = 1.292235e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
```



```
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting arctan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     1.62226714089074e-16    0      Inf        0
    x1                                 0    0      NaN      NaN
    x2              2.25480607721411e-12    0      Inf        0
    x3             -6.05372307898265e-12    0     -Inf        0
    x4              6.82084565609973e-12    0      Inf        0
    x5             -2.80709757748797e-12    0     -Inf        0
    x6                 0.999999998650833    0      Inf        0
    x7              1.69224495998002e-09    0      Inf        0
    x8             -3.94665407882824e-10    0     -Inf        0
    x9              5.13143697398584e-11    0      Inf        0


Number of observations: 101, Error degrees of freedom: 92
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _______________    ___    ____________________    ___    ______

    Total       5.42875862162705    100    0.0542875862162705
    Model       5.42875862162705      8     0.678594827703381    Inf       0
    Residual                   0     92                     0

A = 0.000000, B = 1.774843
order Q(x) = 5.000000, order D(x) = 4.000000
List of powers for Q(x): [0.000000, 0.014575, 0.091976, 0.233195, 0.423637]
List of powers for D(x): [0.000000, 0.014575, 0.091976, 0.233195]
Fitting arctan(x) in range (0.000000, 1.000000)
MSS of errors squared = 2.269805e-05
Corrected MSS of errors squared = 3.209989e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the arc tangent function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i	0.445	0.279		Inf	-Inf	1.32405500E-11	7, 6
A+B*sqrt(i)+C*(i-1)	0.432	0.4	0	Inf	-Inf	1.95060900E-11	6, 7
A+B*i+C*sqrt(i-1)	0.153	0.118	0	Inf	-Inf	1.64393900E-10	7, 6
A+B*i+C*(i-1)	0.211	0.26	0.084	Inf	-Inf	2.37442000E-10	7, 5
A+B*sqrt(i)+C*sqrt(i-1)	0.02	0.175	0	Inf	-Inf	7.46071900E-10	6, 7
A+B/i	0	1.572		Inf	-Inf	1.19090300E-09	5, 6
A+B*sqrt(i)	0	0.1		Inf	-Inf	2.73431500E-08	5, 6
A+B*sqrt(i+C)	0.023	0.1	0.379	Inf	-Inf	4.26493300E-08	5, 5
A+B*log10(i)^4	0	1.774		Inf	-Inf	2.26980500E-05	5, 4
A+B*i-C*(i-1)	0	0.195	0.25	Inf	-Inf	NaN	6, 6

RESULTS FOR INEVRSE HYPERBOLIC SINE

Using Power A+B* i

```
Fitting asinh(x) in range (0.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                      ____________________    __    _____    ______

    (Intercept)    -2.82862398226334e-12     0     -Inf       0
    x1                 -3.13536390549341     0     -Inf       0
    x2                  18.8336524896147     0      Inf       0
    x3                 -45.3937629411297     0     -Inf       0
    x4                  54.9017735809945     0      Inf       0
    x5                 -33.3357117688985     0     -Inf       0
    x6                  8.13736171793895     0      Inf       0
    x7                  4.40150149685205     0      Inf       0
    x8                 -8.37516814620709     0     -Inf       0
    x9                  9.28168092860719     0      Inf       0
    x10                -6.96509212368284     0     -Inf       0
    x11                 3.64994714861973     0      Inf       0
    x12                -1.08273678573532     0     -Inf       0
    x13               0.0808483797271489     0      Inf       0


Number of observations: 1001, Error degrees of freedom: 987
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq            DF           MeanSq            F      pValue
                    ____________________    ____    ____________________    ___    ______
```



```
    Total       923.674148234117    1000    0.923674148234117
    Model       923.674148234117      13     71.0518575564706    Inf       0
    Residual                   0     987                    0

A = 0.000000, B = 0.100000
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000]
List of powers for D(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
Fitting asinh(x) in range (0.000000, 100.000000)
MSS of errors squared = 2.985649e-08
Corrected MSS of errors squared = 4.222345e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting asinh(x) in range (0.000000, 100.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                      ____________________    __    _____    ______

    (Intercept)    7.53778357502033e-13    0      Inf        0
    x1                0.144613489559498    0      Inf        0
    x2                  -3.358469778396    0     -Inf        0
    x3                 10.2896598196742    0      Inf        0
    x4                                0    0      NaN      NaN
    x5                -15.4766054369795    0     -Inf        0
    x6                                0    0      NaN      NaN
    x7                 8.40080794728198    0      Inf        0
    x8               0.0259277832529297    0      Inf        0
    x9                -2.18213730899047    0     -Inf        0
    x10                17.8038689421109    0      Inf        0
    x11               -38.7992275285957    0     -Inf        0
    x12                24.1515612583752    0      Inf        0


Number of observations: 1001, Error degrees of freedom: 990
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq            DF           MeanSq            F      pValue
                  _____________    _____    ____________________    ___    ______
```


Total	923.674148234163	1000	0.923674148234163		
Model	923.674148234163	10	92.3674148234163	Inf	0
Residual	0	990	0		

```
A = 0.000000, B = 0.100000
order Q(x) = 7.000000, order D(x) = 5.000000
List of powers for Q(x): [0.100000, 0.050000, 0.033333, 0.025000, 0.020000, 0.016667,
0.014286]
List of powers for D(x): [0.100000, 0.050000, 0.033333, 0.025000, 0.020000]
Fitting asinh(x) in range (0.000000, 100.000000)
MSS of errors squared = 6.406461e-07
Corrected MSS of errors squared = 9.060104e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting asinh(x) in range (0.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	5.48538068119977e-10	0	Inf	0
x1	-0.000718799208452402	0	-Inf	0
x2	0.00196043823681871	0	Inf	0
x3	-0.00122295275821207	0	-Inf	0
x4	93.0524914494886	0	Inf	0
x5	-666.999878337955	0	-Inf	0
x6	1968.50150976774	0	Inf	0
x7	-3031.38888494404	0	-Inf	0
x8	2578.01984908563	0	Inf	0
x9	-1152.22888015617	0	-Inf	0
x10	212.043771917614	0	Inf	0

```
Number of observations: 1001, Error degrees of freedom: 990
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```

	SumSq	DF	MeanSq	F	pValue
Total	923.674148238661	1000	0.923674148238661		
Model	923.674148238661	10	92.3674148238661	Inf	0


```
    Residual                   0     990                   0

A = 0.019139, B = 0.100062
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.119201, 0.160648, 0.192452]
List of powers for D(x): [0.119201, 0.160648, 0.192452, 0.219263, 0.242885, 0.264240,
0.283879]
Fitting asinh(x) in range (0.000000, 100.000000)
MSS of errors squared = 8.588162e-05
Corrected MSS of errors squared = 1.214549e-04
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting asinh(x) in range (0.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                          Estimate           SE     tStat    pValue
                      ____________________    __    _____    ______

    (Intercept)    2.98317484895983e-10     0      Inf        0
    x1                 34.6429184524044     0      Inf        0
    x2                -150.36893967632      0     -Inf        0
    x3                 182.464282089494     0      Inf        0
    x4                                0     0      NaN      NaN
    x5                -93.1882440961996     0     -Inf        0
    x6                                0     0      NaN      NaN
    x7                 26.4566581315686     0      Inf        0
    x8                 151.304718308149     0      Inf        0
    x9                -731.744377748127     0     -Inf        0
    x10                1449.94683882091     0      Inf        0
    x11               -1452.30657053255     0     -Inf        0
    x12                732.576764460442     0      Inf        0
    x13               -148.784946288615     0     -Inf        0


Number of observations: 1001, Error degrees of freedom: 989
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq            DF           MeanSq             F      pValue
                  _____________    ____    ____________________    ___    ______

    Total       923.674148239864    1000    0.923674148239864
```



```
    Model       923.674148239864      11     83.9703771127149    Inf       0
    Residual                   0     989                    0

A = 0.000000, B = 0.124618, C = 1.150341
order Q(x) = 7.000000, order D(x) = 6.000000
List of powers for Q(x): [0.182740, 0.221187, 0.253877, 0.282813, 0.309051, 0.333230,
0.355770]
List of powers for D(x): [0.182740, 0.221187, 0.253877, 0.282813, 0.309051, 0.333230]
Fitting asinh(x) in range (0.000000, 100.000000)
MSS of errors squared = 2.683964e-07
Corrected MSS of errors squared = 4.648762e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting asinh(x) in range (0.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -3.67782427860001e-12    0     -Inf       0
    x1                  6.99507027163459    0      Inf       0
    x2                 -41.8562688586649    0     -Inf       0
    x3                  101.164905975435    0      Inf       0
    x4                 -124.428837810732    0     -Inf       0
    x5                  79.5615289320447    0      Inf       0
    x6                 -23.1257847690067    0     -Inf       0
    x7                  1.68489962354545    0      Inf       0
    x8                  5.53030500887262    0      Inf       0
    x9                 -12.7772345026042    0     -Inf       0
    x10                 15.3765837080249    0      Inf       0
    x11                -9.56341182130823    0     -Inf       0
    x12                 2.43884851235093    0      Inf       0


Number of observations: 1001, Error degrees of freedom: 988
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq            DF          MeanSq            F      pValue
                  _____________    _____    _____________________    ___    ______

    Total       923.674148233829    1000    0.923674148233829
```



```
    Model       923.674148233829      12     76.9728456861524    Inf       0
    Residual                   0     988                    0

A = 0.001607, B = 0.100000, C = 0.000000
order Q(x) = 7.000000, order D(x) = 5.000000
List of powers for Q(x): [0.101607, 0.201607, 0.301607, 0.401607, 0.501607, 0.601607,
0.701607]
List of powers for D(x): [0.101607, 0.201607, 0.301607, 0.401607, 0.501607]
Fitting asinh(x) in range (0.000000, 100.000000)
MSS of errors squared = 2.005677e-06
Corrected MSS of errors squared = 3.473934e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting asinh(x) in range (0.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    2.04142816200501e-10     0      Inf        0
    x1                  198.96391292703     0      Inf        0
    x2                 -817.975874877945    0     -Inf        0
    x3                  1261.16729554166    0      Inf        0
    x4                 -864.289411997328    0     -Inf        0
    x5                  222.134569493261    0      Inf        0
    x6                  549.272534429659    0      Inf        0
    x7                 -1405.15556476168    0     -Inf        0
    x8                                0    0      NaN      NaN
    x9                  3506.13744564056    0      Inf        0
    x10                -4602.78854602989    0     -Inf        0
    x11                 2432.22421788679    0      Inf        0
    x12                -478.690644318773    0     -Inf        0


Number of observations: 1001, Error degrees of freedom: 989
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq            DF           MeanSq             F      pValue
                ____________________    ____    ____________________    ___    ______

    Total       923.674148242552    1000    0.923674148242552
```



```
    Model       923.674148242552      11     83.9703771129593    Inf       0
    Residual                   0     989                    0

A = 0.000000, B = 0.121424, C = 0.107812
order Q(x) = 5.000000, order D(x) = 7.000000
List of powers for Q(x): [0.121424, 0.135036, 0.148648, 0.162260, 0.175872]
List of powers for D(x): [0.121424, 0.135036, 0.148648, 0.162260, 0.175872, 0.189484,
0.203096]
Fitting asinh(x) in range (0.000000, 100.000000)
MSS of errors squared = 2.613156e-07
Corrected MSS of errors squared = 4.526119e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting asinh(x) in range (0.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                    ____________________    ___    _____    ______

    (Intercept)    -2.21354904164089e-11     0     -Inf       0
    x1                 -6.26373485524914     0     -Inf       0
    x2                  45.4310721264682     0      Inf       0
    x3                 -119.839322775812     0     -Inf       0
    x4                  152.920631915371     0      Inf       0
    x5                 -96.5152115541371     0     -Inf       0
    x6                  24.2969644458831     0      Inf       0
    x7                  5.25314166953493     0      Inf       0
    x8                 -11.4049918894635     0     -Inf       0
    x9                  13.9435358545079     0      Inf       0
    x10                -11.9584056412844     0     -Inf       0
    x11                 7.50401773279905     0      Inf       0
    x12                -2.54429561142129     0     -Inf       0
    x13                 0.17250690903212     0      Inf       0


Number of observations: 1001, Error degrees of freedom: 987
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq            DF           MeanSq             F      pValue
                  _____________    _____    ____________________    ___    ______
```



```
    Total       923.674148233379    1000    0.923674148233379
    Model       923.674148233379      13     71.0518575564138    Inf       0
    Residual                   0     987                    0

A = 0.046272, B = 0.100000, C = 0.020685
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.146272, 0.266958, 0.375526, 0.482100, 0.587643, 0.692526]
List of powers for D(x): [0.146272, 0.266958, 0.375526, 0.482100, 0.587643, 0.692526,
0.796941]
Fitting asinh(x) in range (0.000000, 100.000000)
MSS of errors squared = 3.221321e-08
Corrected MSS of errors squared = 5.579491e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting asinh(x) in range (0.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    6.7667786882989e-09    0      Inf       0
    x1                80.2771794047143    0      Inf       0
    x2               -1185.47071671643    0     -Inf       0
    x3                5826.97536634251    0      Inf       0
    x4               -13539.499660355     0     -Inf       0
    x5                16384.6501316253    0      Inf       0
    x6               -10018.5185773153    0     -Inf       0
    x7                2451.90842166659    0      Inf       0
    x8                29.9707843465489    0      Inf       0
    x9               -186.771060489881    0     -Inf       0
    x10               482.995284465815    0      Inf       0
    x11              -567.718716458842    0     -Inf       0
    x12               222.219918377583    0      Inf       0
    x13               89.3953596694193    0      Inf       0
    x14              -69.4570732994412    0     -Inf       0


Number of observations: 1001, Error degrees of freedom: 986
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq            DF           MeanSq            F      pValue
```


Total	923.674148457221	1000	0.923674148457221		
Model	923.674148457221	14	65.9767248898015	Inf	0
Residual	0	986	0		

A = 0.000000, B = 0.400000, C = 0.000000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.400000, 0.565685, 0.692820, 0.800000, 0.894427, 0.979796, 1.058301]
List of powers for D(x): [0.400000, 0.565685, 0.692820, 0.800000, 0.894427, 0.979796, 1.058301]
Fitting asinh(x) in range (0.000000, 100.000000)
MSS of errors squared = 3.263043e-09
Corrected MSS of errors squared = 5.651757e-09
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*sqrt(i)+C*sqrt(i–1)

Fitting asinh(x) in range (0.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	-9.45731234311857e-11	0	-Inf	0
x1	0.494361877127175	0	Inf	0
x2	-4.87826739746734	0	-Inf	0
x3	12.5233063402343	0	Inf	0
x4	-12.4961892328691	0	-Inf	0
x5	4.35882395973314	0	Inf	0
x6	14.4543199176728	0	Inf	0
x7	-101.272695355672	0	-Inf	0
x8	287.612607592072	0	Inf	0
x9	-418.590767194185	0	-Inf	0
x10	336.143742500617	0	Inf	0
x11	-142.156496590626	0	-Inf	0
x12	24.806979685054	0	Inf	0

Number of observations: 1001, Error degrees of freedom: 988
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	923.674148232775	1000	0.923674148232775		
Model	923.674148232775	12	76.9728456860646	Inf	0
Residual	0	988	0		

```
A = 0.000000, B = 0.137965, C = 0.040288
order Q(x) = 5.000000, order D(x) = 7.000000
List of powers for Q(x): [0.137965, 0.235399, 0.295937, 0.345710, 0.389074]
List of powers for D(x): [0.137965, 0.235399, 0.295937, 0.345710, 0.389074, 0.428029,
0.463705]
Fitting asinh(x) in range (0.000000, 100.000000)
MSS of errors squared = 1.103858e-07
Corrected MSS of errors squared = 1.911938e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting asinh(x) in range (0.000000, 100.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9

Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	0	0	NaN	NaN
x1	-4.40706200358708e-14	0	-Inf	0
x2	-9.84190114206013e-14	0	-Inf	0
x3	1.5385728729706e-13	0	Inf	0
x4	1.00000000000628	0	Inf	0
x5	-7.77254918072793e-12	0	-Inf	0
x6	1.70613720680692e-12	0	Inf	0
x7	-2.47172482762075e-13	0	-Inf	0
x8	1.82543065395049e-14	0	Inf	0
x9	-4.90525540281574e-16	0	-Inf	0

```
Number of observations: 1001, Error degrees of freedom: 992
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```

	SumSq	DF	MeanSq	F	pValue


```
    Total       923.674148234165    1000    0.923674148234165
    Model       923.674148234165       8     115.459268529271    Inf       0
    Residual                   0     992                    0

A = 0.000000, B = 3.500000
order Q(x) = 3.000000, order D(x) = 6.000000
List of powers for Q(x): [0.000000, 0.028741, 0.181377]
List of powers for D(x): [0.000000, 0.028741, 0.181377, 0.459862, 0.835415, 1.283288]
Fitting asinh(x) in range (0.000000, 100.000000)
MSS of errors squared = 5.220030e-03
Corrected MSS of errors squared = 7.382237e-03
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the inverse hyperbolic sine function. The results are sorted in ascending order using the MSSE values.

Power	*A*	*B*	*C*	*F*	*AICc*	*MSSE*	*Q/D Orders*
A+B*sqrt(i)+C*(i-1)	0	0.4	0	Inf	-Inf	3.26304300E-09	7, 7
A+B*i	0	0.1		Inf	-Inf	2.98564900E-08	6, 7
A+B*i+C*sqrt(i-1)	0.046	0.1	0.02	Inf	-Inf	3.22132100E-08	6, 7
A+B*sqrt(i)+C*sqrt(i-1)	0	0.137	0.04	Inf	-Inf	1.10385800E-07	5, 7
A+B*i-C*(i-1)	0	0.121	0.107	Inf	-Inf	2.61315600E-07	5, 7
A+B*sqrt(i+C)	0	0.124	1.15	Inf	-Inf	2.68396400E-07	7, 6
A+B/i	0	0.1		Inf	-Inf	6.40646100E-07	7, 5
A+B*i+C*(i-1)	0.001	0.1	0	Inf	-Inf	2.00567700E-06	7, 5
A+B*sqrt(i)	0.019	0.1		Inf	-Inf	8.58816200E-05	3, 7
A+B*log10(i)^4	0	3.5		Inf	-Inf	5.22003000E-03	3, 6

RESULTS FOR INVERSE HYPERBOLIC TANGENT

Using Power A+B* i

```
Fitting atanh(x) in range (0.000000, 0.999000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -2.30085982196643e-12    0     -Inf       0
    x1                -0.158448303603618    0     -Inf       0
    x2                  1.65846676105782    0      Inf       0
    x3                 -7.12988623084183    0     -Inf       0
    x4                  15.9150254770242    0      Inf       0
    x5                 -19.0832209340183    0     -Inf       0
    x6                  11.6002042565199    0      Inf       0
    x7                 -2.80214037028888    0     -Inf       0
    x8                  79.6604003704102    0      Inf       0
    x9                 -324.702376031031    0     -Inf       0
    x10                 585.896150615975    0      Inf       0
    x11                -589.013135755234    0     -Inf       0
    x12                 344.503436327721    0      Inf       0
    x13                -110.456617980199    0     -Inf       0
    x14                 15.1121422911412    0      Inf       0


Number of observations: 1000, Error degrees of freedom: 985
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       334.064284144156    999    0.334398682826983
    Model       334.064284144156     14     23.8617345817255    Inf       0
    Residual                   0    985                     0

A = 0.261003, B = 0.102500
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.363503, 0.466003, 0.568503, 0.671003, 0.773503, 0.876003,
0.978503]
List of powers for D(x): [0.363503, 0.466003, 0.568503, 0.671003, 0.773503, 0.876003,
0.978503]
Fitting atanh(x) in range (0.000000, 0.999000)
MSS of errors squared = 1.051883e-04
Corrected MSS of errors squared = 1.487587e-04
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting atanh(x) in range (0.000000, 0.999000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                         Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     2.3318675031609e-12     0      Inf       0
    x1              -0.00195305300367667    0     -Inf       0
    x2                0.0547478667275775    0      Inf       0
    x3                -0.30925690712084     0     -Inf       0
    x4                 0.782553737840804    0      Inf       0
    x5                 -1.0554462906156     0     -Inf       0
    x6                 0.733510208032959    0      Inf       0
    x7                -0.204155189766783    0     -Inf       0
    x8              0.000647063893415588    0      Inf       0
    x9                -0.16243198315025     0     -Inf       0
    x10                6.16207849660325     0      Inf       0
    x11               -55.1832074491972     0     -Inf       0
    x12                178.259096794142     0      Inf       0
    x13               -232.380321157842     0     -Inf       0
    x14                104.304138143935     0      Inf       0


Number of observations: 1000, Error degrees of freedom: 985
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF          MeanSq            F     pValue
                  _______________    ___    _________________    ___    ______

    Total         334.064284144336    999    0.334398682827163
    Model         334.064284144336     14     23.8617345817383    Inf       0
    Residual                     0    985                     0

A = 0.000000, B = 1.837778
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [1.837778, 0.918889, 0.612593, 0.459445, 0.367556, 0.306296,
0.262540]
List of powers for D(x): [1.837778, 0.918889, 0.612593, 0.459445, 0.367556, 0.306296,
0.262540]
```



```
Fitting atanh(x) in range (0.000000, 0.999000)
MSS of errors squared = 1.078602e-04
Corrected MSS of errors squared = 1.525374e-04
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting atanh(x) in range (0.000000, 0.999000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -5.75214955491607e-13    0     -Inf        0
    x1               0.00729482009831316    0      Inf        0
    x2              -0.0522577148085721     0     -Inf        0
    x3               0.121219654230536      0      Inf        0
    x4              -0.115028231756669      0     -Inf        0
    x5               0.0387714968859214     0      Inf        0
    x6                 39.7703905804479     0      Inf        0
    x7               -183.029317071131      0     -Inf        0
    x8                269.383071450245      0      Inf        0
    x9                                0     0      NaN      NaN
    x10              -361.640752700482      0     -Inf        0
    x11               329.273975484538      0      Inf        0
    x12              -92.7573677503239      0     -Inf        0


Number of observations: 1000, Error degrees of freedom: 988
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq            F     pValue
                ________________    ___    ________________    ___    ______

    Total       334.064284144416    999    0.334398682827243
    Model       334.064284144416     11     30.3694803767651    Inf       0
    Residual                   0    988                    0

A = 0.000000, B = 0.100000
order Q(x) = 5.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
```



```
Fitting atanh(x) in range (0.000000, 0.999000)
MSS of errors squared = 3.549299e-04
Corrected MSS of errors squared = 5.019467e-04
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting atanh(x) in range (0.000000, 0.999000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                         Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    1.03193563015194e-10     0      Inf        0
    x1                -21.5660655122623     0     -Inf        0
    x2                 197.860664960711     0      Inf        0
    x3                -713.133668555098     0     -Inf        0
    x4                 1313.73061610416     0      Inf        0
    x5                -1317.84796223936     0     -Inf        0
    x6                   687.1329360586     0      Inf        0
    x7                -146.176520834498     0     -Inf        0
    x8                 793.378797003805     0      Inf        0
    x9                -3965.77490982119     0     -Inf        0
    x10                7327.17710113582     0      Inf        0
    x11               -5390.15018012404     0     -Inf        0
    x12                               0     0      NaN      NaN
    x13                1939.33899955769     0      Inf        0
    x14               -702.969807747584     0     -Inf        0


Number of observations: 1000, Error degrees of freedom: 986
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF          MeanSq           F     pValue
                   _________________    ___    ___________________    ___    ______

    Total       334.064284142942     999    0.334398682825768
    Model       334.064284142942      13     25.6972526263801     Inf       0
    Residual                   0     986                    0

A = 0.005932, B = 0.100000, C = 2.000000
order Q(x) = 7.000000, order D(x) = 7.000000
```



```
List of powers for Q(x): [0.179137, 0.205932, 0.229538, 0.250881, 0.270507, 0.288774,
0.305932]
List of powers for D(x): [0.179137, 0.205932, 0.229538, 0.250881, 0.270507, 0.288774,
0.305932]
Fitting atanh(x) in range (0.000000, 0.999000)
MSS of errors squared = 1.777659e-04
Corrected MSS of errors squared = 3.078996e-04
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting atanh(x) in range (0.000000, 0.999000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -4.15986018691166e-12    0     -Inf       0
    x1             -0.000617436852522798    0     -Inf       0
    x2                0.0123573877325102    0      Inf       0
    x3                -0.107149888690631    0     -Inf       0
    x4                 0.523698003956056    0      Inf       0
    x5                 -1.00548988223505    0     -Inf       0
    x6                  0.81112097460273    0      Inf       0
    x7                -0.233918185916186    0     -Inf       0
    x8                  2.70005951278169    0      Inf       0
    x9                 -2.96843077881497    0     -Inf       0
    x10                0.961089506332603    0      Inf       0
    x11                 1.54989962578125    0      Inf       0
    x12                -2.08813207803059    0     -Inf       0
    x13                 1.03621508422275    0      Inf       0
    x14                -0.190701110736395    0     -Inf       0


Number of observations: 1000, Error degrees of freedom: 985
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq            F     pValue
                ___________________    ___    ___________________    ___    ______

    Total       334.064284144385    999    0.334398682827212
    Model       334.064284144385     14     23.8617345817418    Inf       0
    Residual                   0    985                    0
```



```
A = 0.018324, B = 0.128119, C = 0.077411
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.146443, 0.351973, 0.557504, 0.763034, 0.968564, 1.174095,
1.379625]
List of powers for D(x): [0.146443, 0.351973, 0.557504, 0.763034, 0.968564, 1.174095,
1.379625]
Fitting atanh(x) in range (0.000000, 0.999000)
MSS of errors squared = 1.009742e-04
Corrected MSS of errors squared = 1.748925e-04
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting atanh(x) in range (0.000000, 0.999000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     2.36930150305595e-07    0      Inf       0
    x1               5.47781089276638e-08   0      Inf       0
    x2              -2.39143412402834e-08   0     -Inf       0
    x3              -2.67847214407635e-07   0     -Inf       0
    x4                0.00916208643192161   0      Inf       0
    x5               -0.0803475588474016    0     -Inf       0
    x6                 0.884355763923178    0      Inf       0
    x7                 0.229696198828427    0      Inf       0
    x8               -0.0489560248359355    0     -Inf       0
    x9                0.00608953451483071   0      Inf       0


Number of observations: 999, Error degrees of freedom: 989
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F      pValue
                  _______________    ___    _________________    ___    ______

    Total         333.586619215435    998    0.334255129474384
    Model         333.586619215435      9     37.0651799128261    Inf       0
    Residual                     0    989                    0

A = 0.000000, B = 0.100000, C = 0.145206
```



```
order Q(x) = 3.000000, order D(x) = 6.000000
List of powers for Q(x): [0.100000, 0.054794, 0.009587]
List of powers for D(x): [0.100000, 0.054794, 0.009587, -0.035619, -0.080826, -
0.126032]
Fitting atanh(x) in range (0.000000, 0.999000)
MSS of errors squared = NaN
Corrected MSS of errors squared = NaN
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting atanh(x) in range (0.000000, 0.999000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                      ____________________    __    _____    ______

    (Intercept)    -3.42618025198238e-12    0     -Inf       0
    x1             -0.000791215035396002    0     -Inf       0
    x2                0.0239370651601764    0      Inf       0
    x3                -0.212657092348578    0     -Inf       0
    x4                 0.932298358897832    0      Inf       0
    x5                 -1.70787673662436    0     -Inf       0
    x6                  1.35435356039914    0      Inf       0
    x7                -0.389262984129981    0     -Inf       0
    x8                  3.09137521413423    0      Inf       0
    x9                 -4.86637937308446    0     -Inf       0
    x10                 3.58138210176815    0      Inf       0
    x11                0.512225640681218    0      Inf       0
    x12                -2.59315120960918    0     -Inf       0
    x13                    1.60643590951    0      Inf       0
    x14               -0.331888517783217    0     -Inf       0


Number of observations: 1000, Error degrees of freedom: 985
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq           F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       334.064284144385    999    0.334398682827212
    Model       334.064284144385     14     23.8617345817418    Inf       0
    Residual                   0    985                    0
```



```
A = 0.000000, B = 0.165127, C = 0.062594
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.165127, 0.392848, 0.583902, 0.768923, 0.950822, 1.130725,
1.309211]
List of powers for D(x): [0.165127, 0.392848, 0.583902, 0.768923, 0.950822, 1.130725,
1.309211]
Fitting atanh(x) in range (0.000000, 0.999000)
MSS of errors squared = 1.003655e-04
Corrected MSS of errors squared = 1.738381e-04
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting atanh(x) in range (0.000000, 0.999000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -3.48370624139838e-10    0     -Inf       0
    x1              6.45047396941165e-05    0      Inf       0
    x2              -0.00439467635564524    0     -Inf       0
    x3                  0.35629682713656    0      Inf       0
    x4                 -1.29208397831106    0     -Inf       0
    x5                  1.77352771691334    0      Inf       0
    x6                 -1.0752109561327     0     -Inf       0
    x7                 0.241799974219377    0      Inf       0
    x8                    2.013399585833    0      Inf       0
    x9                 -1.7567086431427     0     -Inf       0
    x10                0.889926478617581    0      Inf       0
    x11             -0.000587182922377199    0     -Inf       0
    x12                -0.324789617706907    0     -Inf       0
    x13                 0.253108616033955    0      Inf       0
    x14                -0.074349093186511    0     -Inf       0


Number of observations: 1000, Error degrees of freedom: 985
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                  _____________    ___    ____________________    ___    ______
```



```
    Total       334.064284144386    999    0.334398682827214
    Model       334.064284144386     14     23.8617345817419    Inf       0
    Residual                   0    985                    0

A = 0.053503, B = 0.120172, C = 0.317733
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.173676, 0.541186, 0.897114, 1.247048, 1.593150, 1.936531,
2.277849]
List of powers for D(x): [0.173676, 0.541186, 0.897114, 1.247048, 1.593150, 1.936531,
2.277849]
Fitting atanh(x) in range (0.000000, 0.999000)
MSS of errors squared = 1.099334e-04
Corrected MSS of errors squared = 1.904103e-04
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting atanh(x) in range (0.000000, 0.999000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                         Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -1.92697010089688e-12    0     -Inf       0
    x1              -0.00197496244066215    0     -Inf       0
    x2                 0.238405487547916    0      Inf       0
    x3                -2.26502268126409     0     -Inf       0
    x4                 7.80649359618891     0      Inf       0
    x5                -12.320185978307      0     -Inf       0
    x6                 9.09693851931381     0      Inf       0
    x7                -2.55465383616748     0     -Inf       0
    x8                 5.53434637938925     0      Inf       0
    x9                -57.5555507986433     0     -Inf       0
    x10                217.427927895504     0      Inf       0
    x11               -391.976206973507     0     -Inf       0
    x12                376.241978153025     0      Inf       0
    x13               -186.156132676164     0     -Inf       0
    x14                37.4836379846232     0      Inf       0


Number of observations: 1000, Error degrees of freedom: 985
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```


	SumSq	DF	MeanSq	F	pValue
Total	334.06428414434	999	0.334398682827167		
Model	334.06428414434	14	23.8617345817386	Inf	0
Residual	0	985	0		

A = 0.057592, B = 0.100000, C = 0.194943
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.157592, 0.393956, 0.506488, 0.595243, 0.671084, 0.738446, 0.799677]
List of powers for D(x): [0.157592, 0.393956, 0.506488, 0.595243, 0.671084, 0.738446, 0.799677]
Fitting atanh(x) in range (0.000000, 0.999000)
MSS of errors squared = 1.128601e-04
Corrected MSS of errors squared = 1.954795e-04
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*log10(i)^4

Fitting atanh(x) in range (0.000000, 0.999000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	-1.59082886128147e-13	0	-Inf	0
x1	1.29931662335091	0	Inf	0
x2	-1.65292484246198	0	-Inf	0
x3	0.413390338127791	0	Inf	0
x4	-0.0597819990200343	0	-Inf	0
x5	37.8422277926793	0	Inf	0
x6	-41.5101060008679	0	-Inf	0
x7	4.66787816804407	0	Inf	0

Number of observations: 1000, Error degrees of freedom: 992
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	334.064284144382	999	0.334398682827209		


```
    Model       334.064284144382      7     47.7234691634831    Inf       0
    Residual                   0    992                    0

A = 0.009898, B = 0.100000
order Q(x) = 4.000000, order D(x) = 3.000000
List of powers for Q(x): [0.009898, 0.010719, 0.015080, 0.023037]
List of powers for D(x): [0.009898, 0.010719, 0.015080]
Fitting atanh(x) in range (0.000000, 0.999000)
MSS of errors squared = 1.337882e-03
Corrected MSS of errors squared = 1.892052e-03
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the inverse hyperbolic tangent function. The results are sorted in ascending order using the MSSE values.

Power	*A*	*B*	*C*	*F*	*AICc*	*MSSE*	*Q/D Orders*
A+B*i+C*sqrt(i-1)	0	0.165	0.062	Inf	-Inf	1.00365500E-04	7, 7
A+B*i+C*(i-1)	0.018	0.128	0.077	Inf	-Inf	1.00974200E-04	7, 7
A+B*i	0.261	0.102		Inf	-Inf	1.05188300E-04	7, 7
A+B/i	0	1.837		Inf	-Inf	1.07860200E-04	7, 7
A+B*sqrt(i)+C*(i-1)	0.053	0.12	0.317	Inf	-Inf	1.09933400E-04	7, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.057	0.1	0.194	Inf	-Inf	1.12860100E-04	7, 7
A+B*sqrt(i+C)	0.005	0.1	2	Inf	-Inf	1.77765900E-04	7, 7
A+B*sqrt(i)	0	0.1		Inf	-Inf	3.54929900E-04	5, 7
A+B*log10(i)^4	0.009	0.1		Inf	-Inf	1.33788200E-03	4, 3
A+B*i-C*(i-1)	0	0.1	0.145	Inf	-Inf	NaN	3, 6

RESULTS FOR THE COSINE INTEGRAL

Using Power A+B* i

```
Fitting Ci(x) in range (0.500000, 20.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	-10.8577019026503	0.0612016753022261	-177.40857336066	7.90297216898543e-206
x1	63.4357244937007	0.357555142153841	177.415220800843	7.84956843026664e-206
x2	-158.47178042512	0.894592092266134	-177.144177547655	1.03514366427269e-205
x3	219.45792298996	1.24283474955091	176.578521858405	1.84648231534831e-205
x4	-181.97230502112	1.03565418895738	-175.707593288755	4.51747221879484e-205
x5	90.3564313900483	0.517720414368087	174.527464790691	1.52910864381384e-204
x6	-24.8788235098404	0.143774074323662	-173.041096782397	7.18544088357097e-204
x7	2.93056120725084	0.0171119522247877	171.258145695717	4.67868169216433e-203
x8	4.79886732028317	0.00343954653253112	1395.20348827837	0
x9	-9.60524751916249	0.0142407213778302	-674.49164015777	3.5346529440251e-311
x10	10.2634539299239	0.0235953365261139	434.978069440329	1.58107865262715e-276
x11	-6.17437862666663	0.0195534444175407	-315.76935985395	3.02543023918294e-251
x12	1.98271938786574	0.00810324242733895	244.682225127115	3.91612321661173e-231
x13	-0.265501097378095	0.00134327212672726	-197.652502494011	2.51704974507315e-214

```
Number of observations: 196, Error degrees of freedom: 182
Root Mean Squared Error: 4.42e-09
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: 1.58e+16, p-value = 0
```

	SumSq	DF	MeanSq	F	pValue
Total	4.00540843003334	195	0.020540556051453		
Model	4.00540843003334	13	0.308108340771795	1.57839114238442e+16	0


```
    Residual    3.5527136788005e-15    182    1.95204048285742e-17

A = 0.000000, B = 0.100000
order Q(x) = 7.000000, order D(x) = 6.000000
List of powers for Q(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
List of powers for D(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000]
Fitting Ci(x) in range (0.500000, 20.000000)
MSS of errors squared = 3.225425e-04
Corrected MSS of errors squared = 4.561440e-04
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -7.092657e+03
AIC =  -7.122978e+03
AICc =  -7.092657e+03
```

Using Power A+B/i

```
Fitting Ci(x) in range (0.500000, 20.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     1.23382784992146e-06    0      Inf        0
    x1              2.47031196232775e-07    0      Inf        0
    x2             -6.15422578337748e-06    0     -Inf        0
    x3              2.05309199985766e-05    0      Inf        0
    x4             -1.58575551694842e-05    0     -Inf        0
    x5               -0.0132201710694911    0     -Inf        0
    x6                  2.07409241633583    0      Inf        0
    x7                 -29.0393393176227    0     -Inf        0
    x8                  106.058556479983    0      Inf        0
    x9                 -118.024163302899    0     -Inf        0
    x10                                0    0      NaN      NaN
    x11                 39.9440738952802    0      Inf        0


Number of observations: 196, Error degrees of freedom: 185
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       4.00540843003133    195    0.0205405560514427
    Model       4.00540843003133     10     0.400540843003133    Inf       0
    Residual                   0    185                      0
```



```
A = 0.003985, B = 0.100000
order Q(x) = 4.000000, order D(x) = 7.000000
List of powers for Q(x): [0.103985, 0.053985, 0.037318, 0.028985]
List of powers for D(x): [0.103985, 0.053985, 0.037318, 0.028985, 0.023985, 0.020652,
0.018271]
Fitting Ci(x) in range (0.500000, 20.000000)
MSS of errors squared = 1.660945e-02
Corrected MSS of errors squared = 2.348930e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting Ci(x) in range (0.500000, 20.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     -2.7103486713701e-06    0     -Inf       0
    x1               1.77030402569127e-05   0      Inf       0
    x2              -2.63783763917053e-05   0     -Inf       0
    x3               1.13790240153827e-05   0      Inf       0
    x4                   43.8829282335994   0      Inf       0
    x5                  -244.431393483013   0     -Inf       0
    x6                   558.939689168165   0      Inf       0
    x7                  -634.117578984367   0     -Inf       0
    x8                   355.727020087637   0      Inf       0
    x9                   -79.000664982219   0     -Inf       0


Number of observations: 196, Error degrees of freedom: 186
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                  _________________    ___    ________________    ___    ______

    Total       4.00540843002808     195    0.020540556051426
    Model       4.00540843002808       9    0.445045381114231    Inf       0
    Residual                   0     186                    0

A = 0.000000, B = 0.100000
order Q(x) = 3.000000, order D(x) = 6.000000
```



```
List of powers for Q(x): [0.100000, 0.141421, 0.173205]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949]
Fitting Ci(x) in range (0.500000, 20.000000)
MSS of errors squared = 2.036249e-02
Corrected MSS of errors squared = 2.879691e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting Ci(x) in range (0.500000, 20.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    0.00137927629005035     0      Inf        0
    x1             -0.0468383738428659     0     -Inf        0
    x2               0.146569252001989     0      Inf        0
    x3                               0     0      NaN      NaN
    x4              -0.575825600078233     0     -Inf        0
    x5               0.934546865939261     0      Inf        0
    x6              -0.603261837771303     0     -Inf        0
    x7               0.143430414992839     0      Inf        0
    x8                92.5645373110536     0      Inf        0
    x9               -682.83327761725     0     -Inf        0
    x10               2082.8040777113     0      Inf        0
    x11              -3316.1855656124     0     -Inf        0
    x12               2913.41444331197     0      Inf        0
    x13              -1343.58404063428     0     -Inf        0
    x14               254.819825537146     0      Inf        0


Number of observations: 196, Error degrees of freedom: 182
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                 ____________________    ___    ____________________    ___    ______

    Total        4.00540843000466    195    0.0205405560513059
    Model        4.00540843000466     13     0.308108340769589    Inf       0
    Residual                    0    182                      0

A = 0.000000, B = 0.103386, C = 0.125301
```



```
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.109672, 0.150721, 0.182772, 0.209986, 0.234058, 0.255875,
0.275972]
List of powers for D(x): [0.109672, 0.150721, 0.182772, 0.209986, 0.234058, 0.255875,
0.275972]
Fitting Ci(x) in range (0.500000, 20.000000)
MSS of errors squared = 7.856315e-02
Corrected MSS of errors squared = 1.360754e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting Ci(x) in range (0.500000, 20.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     -13.7973978706768    0     -Inf       0
    x1               73.7116064845174    0      Inf       0
    x2              -176.970375958908    0     -Inf       0
    x3               240.132121081904    0      Inf       0
    x4              -195.987953936838    0     -Inf       0
    x5               95.9474188004369    0      Inf       0
    x6              -26.0658152093312    0     -Inf       0
    x7               3.03044323293563    0      Inf       0
    x8               4.00717178045524    0      Inf       0
    x9              -6.48416094649987    0     -Inf       0
    x10              4.84043235292065    0      Inf       0
    x11             -0.919061904788914    0     -Inf       0
    x12             -0.955107471469705    0     -Inf       0
    x13              0.623902991411966    0      Inf       0
    x14             -0.11331498759221    0     -Inf       0


Number of observations: 196, Error degrees of freedom: 181
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       4.00540843002877    195    0.0205405560514296
    Model       4.00540843002877     14     0.286100602144912    Inf       0
```


Residual	0	181	0	

A = 0.000000, B = 0.100000, C = 0.005918
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.205918, 0.311836, 0.417754, 0.523672, 0.629590, 0.735508]
List of powers for D(x): [0.100000, 0.205918, 0.311836, 0.417754, 0.523672, 0.629590, 0.735508]
Fitting Ci(x) in range (0.500000, 20.000000)
MSS of errors squared = 3.570571e-04
Corrected MSS of errors squared = 6.184410e-04
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*i–C*(i–1)

Fitting Ci(x) in range (0.500000, 20.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	2.63533722964036e-08	0	Inf	0
x1	2.28541510191577e-10	0	Inf	0
x2	-1.01957969589824e-09	0	-Inf	0
x3	-2.55614302771504e-08	0	-Inf	0
x4	-0.000600896179569849	0	-Inf	0
x5	0.00802389258244724	0	Inf	0
x6	1.01389868657367	0	Inf	0
x7	-0.0309128679298285	0	-Inf	0
x8	0.0128343852071224	0	Inf	0
x9	-0.00375701835376645	0	-Inf	0
x10	0.000513818097446059	0	Inf	0

Number of observations: 196, Error degrees of freedom: 185
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	4.00540843003174	195	0.0205405560514448		
Model	4.00540843003174	10	0.400540843003174	Inf	0
Residual	0	185	0		


```
A = 0.054527, B = 0.104682, C = 0.185142
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.159208, 0.078748, -0.001712]
List of powers for D(x): [0.159208, 0.078748, -0.001712, -0.082173, -0.162633, -
0.243093, -0.323554]
Fitting Ci(x) in range (0.500000, 20.000000)
MSS of errors squared = 6.473171e-02
Corrected MSS of errors squared = 1.121186e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting Ci(x) in range (0.500000, 20.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______

    (Intercept)     -14.1504781266895     0     -Inf       0
    x1               82.6334177110934     0      Inf       0
    x2              -206.327187853089     0     -Inf       0
    x3               285.583497718018     0      Inf       0
    x4              -236.678890941742     0     -Inf       0
    x5               117.457607597174     0      Inf       0
    x6              -32.3232929126401     0     -Inf       0
    x7               3.80536557983531     0      Inf       0
    x8               4.35340733716146     0      Inf       0
    x9              -7.43221872602512     0     -Inf       0
    x10              5.84236686089748     0      Inf       0
    x11             -1.3730542404655     0     -Inf       0
    x12             -0.95248560546198     0     -Inf       0
    x13             0.692116968856975     0      Inf       0
    x14            -0.130247508454478     0     -Inf       0


Number of observations: 196, Error degrees of freedom: 181
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                ____________________    ___    ____________________    ___    ______

    Total       4.00540843003184     195    0.0205405560514454
```


Model	4.00540843003184	14	0.286100602145132	Inf	0
Residual	0	181	0		

A = 0.000000, B = 0.100279, C = 0.000000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100279, 0.200557, 0.300836, 0.401115, 0.501394, 0.601672, 0.701951]
List of powers for D(x): [0.100279, 0.200557, 0.300836, 0.401115, 0.501394, 0.601672, 0.701951]
Fitting Ci(x) in range (0.500000, 20.000000)
MSS of errors squared = 3.532208e-04
Corrected MSS of errors squared = 6.117964e-04
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*sqrt(i)+C*(i–1)

Fitting Ci(x) in range (0.500000, 20.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	-1.09107858528485e-05	0	-Inf	0
x1	0.000287953909014347	0	Inf	0
x2	-0.000825463113678138	0	-Inf	0
x3	0.00082961710330531	0	Inf	0
x4	-0.000281196801712115	0	-Inf	0
x5	223.616091101931	0	Inf	0
x6	-1824.11461606163	0	-Inf	0
x7	5865.99981680031	0	Inf	0
x8	-9660.72866530842	0	-Inf	0
x9	8692.77363432352	0	Inf	0
x10	-4081.68635672251	0	-Inf	0
x11	785.140095867716	0	Inf	0

Number of observations: 196, Error degrees of freedom: 184
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	4.00540843001664	195	0.0205405560513674		

Model	4.00540843001664	11	0.364128039092422	Inf	0
Residual	0	184	0		

A = 0.048949, B = 0.108820, C = 0.000000
order Q(x) = 4.000000, order D(x) = 7.000000
List of powers for Q(x): [0.157769, 0.202844, 0.237431, 0.266590]
List of powers for D(x): [0.157769, 0.202844, 0.237431, 0.266590, 0.292279, 0.315503, 0.336861]
Fitting Ci(x) in range (0.500000, 20.000000)
MSS of errors squared = 3.648642e-02
Corrected MSS of errors squared = 6.319634e-02
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*sqrt(i)+C*sqrt(i–1)

Fitting Ci(x) in range (0.500000, 20.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	-2.7103486713701e-06	0	-Inf	0
x1	1.77030402569127e-05	0	Inf	0
x2	-2.63783763917053e-05	0	-Inf	0
x3	1.13790240153827e-05	0	Inf	0
x4	43.8829282335994	0	Inf	0
x5	-244.431393483013	0	-Inf	0
x6	558.939689168165	0	Inf	0
x7	-634.117578984367	0	-Inf	0
x8	355.727020087637	0	Inf	0
x9	-79.000664982219	0	-Inf	0

Number of observations: 196, Error degrees of freedom: 186
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	4.00540843002808	195	0.020540556051426		
Model	4.00540843002808	9	0.445045381114231	Inf	0
Residual	0	186	0		


```
A = 0.000000, B = 0.100000, C = 0.000000
order Q(x) = 3.000000, order D(x) = 6.000000
List of powers for Q(x): [0.100000, 0.141421, 0.173205]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949]
Fitting Ci(x) in range (0.500000, 20.000000)
MSS of errors squared = 2.036249e-02
Corrected MSS of errors squared = 3.526887e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting Ci(x) in range (0.500000, 20.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7

Estimated Coefficients:
                          Estimate            SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     4.87616632614704e-12    0      Inf        0
    x1                                 0    0      NaN      NaN
    x2             -6.15049805619635e-12    0     -Inf        0
    x3              1.46979545252498e-12    0      Inf        0
    x4             -1.95481969286238e-13    0     -Inf        0
    x5                 0.999999999999203    0      Inf        0
    x6              9.40315595404661e-13    0      Inf        0
    x7             -1.42964704389682e-13    0     -Inf        0


Number of observations: 196, Error degrees of freedom: 189
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                 _____________    ___    ____________________    ___    ______

    Total        4.00540843003173    195    0.0205405560514448
    Model        4.00540843003173      6     0.667568071671955    Inf       0
    Residual                    0    189                     0

A = 0.000000, B = 0.584434
order Q(x) = 4.000000, order D(x) = 3.000000
List of powers for Q(x): [0.000000, 0.004799, 0.030287, 0.076788]
List of powers for D(x): [0.000000, 0.004799, 0.030287]
Fitting Ci(x) in range (0.500000, 20.000000)
MSS of errors squared = 1.032852e-02
```



```
Corrected MSS of errors squared = 1.460674e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the cosine integral. The results are sorted in ascending order using the MSSE values.

Power	*A*	*B*	*C*	*F*	*AICc*	*MSSE*	*Q/D Orders*
A+B*i	0	0.1		1.5E16	-7.09E+03	3.22542500E-04	7, 6
A+B*i+C*sqrt(i-1)	0	0.1	0	Inf	-Inf	3.53220800E-04	7, 7
A+B*i+C*(i-1)	0	0.1	0.005	Inf	-Inf	3.57057100E-04	7, 7
A+B*log10(i)^4	0	0.584		Inf	-Inf	1.03285200E-02	4, 3
A+B/i	0.003	0.1		Inf	-Inf	1.66094500E-02	4, 7
A+B*sqrt(i)	0	0.1		Inf	-Inf	2.03624900E-02	3, 6
A+B*sqrt(i)+C*sqrt(i-1)	0	0.1	0	Inf	-Inf	2.03624900E-02	3, 6
A+B*sqrt(i)+C*(i-1)	0.048	0.108	0	Inf	-Inf	3.64864200E-02	4, 7
A+B*i-C*(i-1)	0.054	0.104	0.185	Inf	-Inf	6.47317100E-02	3, 7
A+B*sqrt(i+C)	0	0.103	0.125	Inf	-Inf	7.85631500E-02	7, 7

RESULTS FOR THE COSINE FUNCTION

Using Power A+B* i

```
Fitting cos(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______

    (Intercept)      1.00000000006398       0      Inf        0
    x1              -2.83764391147102       0     -Inf        0
    x2               6.21321791127825       0      Inf        0
    x3              -7.60467155968501       0     -Inf        0
    x4               5.84734252676208       0      Inf        0
    x5              -3.21599309055961       0     -Inf        0
    x6              0.886715117833606       0      Inf        0
    x7               2.83754959217052       0      Inf        0
    x8              -6.21117223686066       0     -Inf        0
    x9               7.58544205410103       0      Inf        0
    x10             -5.74235593483609       0     -Inf        0
    x11              2.48657843151995       0      Inf        0
    x12             -0.516864607214192      0     -Inf        0
    x13             0.0259980465678655      0      Inf        0


Number of observations: 501, Error degrees of freedom: 487
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```


	SumSq	DF	MeanSq	F	pValue
Total	47.6265936125824	500	0.0952531872251649		
Model	47.6265936125824	13	3.6635841240448	Inf	0
Residual	0	487	0		

```
A = 0.500000, B = 0.293947
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.793947, 1.087894, 1.381841, 1.675789, 1.969736, 2.263683]
List of powers for D(x): [0.793947, 1.087894, 1.381841, 1.675789, 1.969736, 2.263683,
2.557630]
Fitting cos(x) in range (0.000000, 1.570796)
MSS of errors squared = 1.021816e-11
Corrected MSS of errors squared = 1.445066e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting cos(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	0.999999999530184	0	Inf	0
x1	-0.0507538956493821	0	-Inf	0
x2	1.15895533982233	0	Inf	0
x3	11.8358423564934	0	Inf	0
x4	-79.3512137125645	0	-Inf	0
x5	106.525269162186	0	Inf	0
x6	0	0	NaN	NaN
x7	-40.6288894615876	0	-Inf	0
x8	-0.0108001555506238	0	-Inf	0
x9	2.69333374248133	0	Inf	0
x10	-55.8576987720605	0	-Inf	0
x11	290.111985192389	0	Inf	0
x12	-568.777049602484	0	-Inf	0
x13	462.984690952333	0	Inf	0
x14	-131.049898522441	0	-Inf	0

```
Number of observations: 501, Error degrees of freedom: 487
```



```
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       47.6265936125819    500    0.0952531872251639
    Model       47.6265936125819     13      3.66358412404476    Inf       0
    Residual                   0    487                     0

A = 2.000000, B = 1.569551
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [3.569551, 2.784776, 2.523184, 2.392388, 2.313910, 2.261592,
2.224222]
List of powers for D(x): [3.569551, 2.784776, 2.523184, 2.392388, 2.313910, 2.261592,
2.224222]
Fitting cos(x) in range (0.000000, 1.570796)
MSS of errors squared = 2.309491e-12
Corrected MSS of errors squared = 3.266113e-12
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting cos(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      1.00000000012039    0      Inf       0
    x1               -1.06857662734986    0     -Inf       0
    x2                3.20922821916932    0      Inf       0
    x3               -6.48154228209613    0     -Inf       0
    x4                6.89986969018347    0      Inf       0
    x5               -4.52307712354533    0     -Inf       0
    x6                1.42181971489968    0      Inf       0
    x7                1.0684853051289     0      Inf       0
    x8               -3.20437651731659    0     -Inf       0
    x9                5.97768900600931    0      Inf       0
    x10              -6.79448434870351    0     -Inf       0
    x11               4.20835921283277    0      Inf       0
    x12              -1.16682372343738    0     -Inf       0
    x13             0.0639927498961305    0      Inf       0
```



```
Number of observations: 501, Error degrees of freedom: 487
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq            F      pValue
                ________________    ___    ________________    ___    ______

    Total       47.6265936125825    500    0.095253187225165
    Model       47.6265936125825     13     3.66358412404481    Inf        0
    Residual                   0    487                    0

A = 0.404976, B = 0.915218
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [1.320194, 1.699290, 1.990181, 2.235413, 2.451466, 2.646794]
List of powers for D(x): [1.320194, 1.699290, 1.990181, 2.235413, 2.451466, 2.646794,
2.826416]
Fitting cos(x) in range (0.000000, 1.570796)
MSS of errors squared = 2.308064e-12
Corrected MSS of errors squared = 3.264095e-12
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting cos(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    0.999999996254675    0      Inf       0
    x1              2.91666183874307    0      Inf       0
    x2             -13.0717242476773    0     -Inf       0
    x3              17.984803022493     0      Inf       0
    x4             -7.72388652629697    0     -Inf       0
    x5             -3.16781488309747    0     -Inf       0
    x6              2.64595455667386    0      Inf       0
    x7             -3.95995499368001    0     -Inf       0
    x8              16.0265980001766    0      Inf       0
    x9             -26.7411568358824    0     -Inf       0
    x10             22.2010350161842    0      Inf       0
    x11            -8.97694782504876    0     -Inf       0
    x12             1.36956181393598    0      Inf       0
```



```
Number of observations: 501, Error degrees of freedom: 488
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       47.6265936125804    500    0.0952531872251608
    Model       47.6265936125804     12      3.96888280104837    Inf       0
    Residual                   0    488                     0

A = 1.000000, B = 0.751445, C = 0.981473
order Q(x) = 6.000000, order D(x) = 6.000000
List of powers for Q(x): [2.057770, 2.297516, 2.499405, 2.677166, 2.837813, 2.985504]
List of powers for D(x): [2.057770, 2.297516, 2.499405, 2.677166, 2.837813, 2.985504]
Fitting cos(x) in range (0.000000, 1.570796)
MSS of errors squared = 1.450776e-11
Corrected MSS of errors squared = 2.512818e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting cos(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    0.999999999999535    0      Inf       0
    x1             -10.1954796025179    0     -Inf       0
    x2              26.7688352505184    0      Inf       0
    x3             -34.1292122809161    0     -Inf       0
    x4               24.995224558908    0      Inf       0
    x5             -10.2959490365398    0     -Inf       0
    x6              1.87702532168987    0      Inf       0
    x7              10.1962443230062    0      Inf       0
    x8             -26.7803439814613    0     -Inf       0
    x9              34.2026703428384    0      Inf       0
    x10            -25.2500544889046    0     -Inf       0
    x11             10.8002581186732    0      Inf       0
    x12            -2.40256154349193    0     -Inf       0
    x13            0.195948762048893    0      Inf       0
```



```
Number of observations: 501, Error degrees of freedom: 487
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ________________    ___    ________________    ___    ______

    Total       47.6265936125822    500    0.0952531872251643
    Model       47.6265936125822     13      3.66358412404478    Inf       0
    Residual                   0    487                     0

A = 0.195822, B = 0.172239, C = 0.011933
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.368061, 0.552233, 0.736406, 0.920578, 1.104750, 1.288922]
List of powers for D(x): [0.368061, 0.552233, 0.736406, 0.920578, 1.104750, 1.288922,
1.473094]
Fitting cos(x) in range (0.000000, 1.570796)
MSS of errors squared = 3.999109e-11
Corrected MSS of errors squared = 6.926660e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting cos(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ________________    ___    _____    ______

    (Intercept)     0.99999999998236    0       Inf       0
    x1               -16.120310345245    0      -Inf       0
    x2                38.6135901777788    0       Inf       0
    x3               -36.4368825838819    0      -Inf       0
    x4                12.8594226126403    0       Inf       0
    x5                 3.8558042167698    0       Inf       0
    x6               -5.30766163657254    0      -Inf       0
    x7                1.57066249631245    0       Inf       0
    x8                16.1590681061509    0       Inf       0
    x9               -39.0479456535516    0      -Inf       0
    x10               38.4713183031115    0       Inf       0
    x11              -17.9525104323899    0      -Inf       0
    x12               3.30598529376893    0       Inf       0
```



```
Number of observations: 501, Error degrees of freedom: 488
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq            F      pValue
                ________________    ___    ________________    ___    ______

    Total       47.6265936125825    500    0.095253187225165
    Model       47.6265936125825     12     3.96888280104854    Inf       0
    Residual                   0    488                    0

A = 0.328389, B = 0.141152, C = 0.000000
order Q(x) = 7.000000, order D(x) = 5.000000
List of powers for Q(x): [0.469541, 0.610693, 0.751845, 0.892997, 1.034149, 1.175301,
1.316453]
List of powers for D(x): [0.469541, 0.610693, 0.751845, 0.892997, 1.034149]
Fitting cos(x) in range (0.000000, 1.570796)
MSS of errors squared = 1.668555e-09
Corrected MSS of errors squared = 2.890022e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting cos(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ________________    ___    _____    ______

    (Intercept)    0.999999999992061    0      Inf       0
    x1             -74.0205874614701    0     -Inf       0
    x2              297.852403373871    0      Inf       0
    x3             -531.499282643359    0     -Inf       0
    x4              528.759665604945    0      Inf       0
    x5             -305.708915901567    0     -Inf       0
    x6              96.5054749106032    0      Inf       0
    x7             -12.884318589707     0     -Inf       0
    x8              74.0324749800789    0      Inf       0
    x9             -297.979079953762    0     -Inf       0
    x10             532.063451450809    0      Inf       0
    x11            -530.10113941903     0     -Inf       0
    x12             307.49756943738     0      Inf       0
    x13            -97.7623639008035    0     -Inf       0
```


x14	13.2408710915133	0	Inf	0

Number of observations: 501, Error degrees of freedom: 486
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	47.6265936125532	500	0.0952531872251064		
Model	47.6265936125532	14	3.4018995437538	Inf	0
Residual	0	486	0		

A = 0.294525, B = 0.111659, C = 0.000000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.406184, 0.517842, 0.629501, 0.741160, 0.852819, 0.964477, 1.076136]
List of powers for D(x): [0.406184, 0.517842, 0.629501, 0.741160, 0.852819, 0.964477, 1.076136]
Fitting cos(x) in range (0.000000, 1.570796)
MSS of errors squared = 8.178935e-11
Corrected MSS of errors squared = 1.416633e-10
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*sqrt(i)+C*(i–1)

Fitting cos(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	1.00000000028507	0	Inf	0
x1	-3.00910448115923	0	-Inf	0
x2	1.56610768860418	0	Inf	0
x3	-0.692510753646427	0	-Inf	0
x4	0.0326755151426206	0	Inf	0
x5	1.29072476143831	0	Inf	0
x6	-0.585453557502998	0	-Inf	0
x7	3.00909910633481	0	Inf	0
x8	-1.56564794502628	0	-Inf	0
x9	0.210248260695334	0	Inf	0
x10	-0.0951779016728096	0	-Inf	0

x11	0.341568738517691	0	Inf	0
x12	-0.211928623435554	0	-Inf	0
x13	0.0476501917073728	0	Inf	0

Number of observations: 501, Error degrees of freedom: 487
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	47.6265936125815	500	0.095253187225163		
Model	47.6265936125815	13	3.66358412404473	Inf	0
Residual	0	487	0		

A = 0.550000, B = 0.400000, C = 0.374918
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.950000, 1.490604, 1.992657, 2.474754, 2.944100, 3.404386]
List of powers for D(x): [0.950000, 1.490604, 1.992657, 2.474754, 2.944100, 3.404386, 3.857809]
Fitting cos(x) in range (0.000000, 1.570796)
MSS of errors squared = 7.877539e-11
Corrected MSS of errors squared = 1.364430e-10
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*sqrt(i)+C*sqrt(i–1)

Fitting cos(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	0.999999999965469	0	Inf	0
x1	-2.84825537040488	0	-Inf	0
x2	8.63077752837764	0	Inf	0
x3	-11.6385203301734	0	-Inf	0
x4	-1.81689029918978	0	-Inf	0
x5	20.5202829473184	0	Inf	0
x6	-20.6223074874591	0	-Inf	0
x7	6.82855918040392	0	Inf	0
x8	2.84943609348871	0	Inf	0
x9	-8.87452286402508	0	-Inf	0


```
    x10              14.5392811930098     0       Inf        0
    x11             -10.3639145176721     0      -Inf        0
    x12               2.7504309194102     0       Inf        0


Number of observations: 501, Error degrees of freedom: 488
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq           DF            MeanSq            F      pValue
                 ________________    ___    ____________________    ___    ______

    Total        47.6265936125794    500    0.0952531872251587
    Model        47.6265936125794     12      3.96888280104828    Inf        0
    Residual                    0    488                     0

A = 0.268952, B = 0.100000, C = 0.341449
order Q(x) = 7.000000, order D(x) = 5.000000
List of powers for Q(x): [0.368952, 0.751822, 0.925039, 1.060359, 1.175456, 1.277404,
1.369902]
List of powers for D(x): [0.368952, 0.751822, 0.925039, 1.060359, 1.175456]
Fitting cos(x) in range (0.000000, 1.570796)
MSS of errors squared = 1.117161e-09
Corrected MSS of errors squared = 1.934979e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting cos(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                         Estimate           SE    tStat    pValue
                     ____________________    __    _____    ______

    (Intercept)     6.67087006615928e-09     0      Inf        0
    x1                                 0     0      NaN      NaN
    x2             -8.42207134940915e-09     0     -Inf        0
    x3              2.03999871751251e-09     0      Inf        0
    x4             -2.88008038769335e-10     0     -Inf        0
    x5                 0.999999993329123     0      Inf        0
    x6              8.43055448464781e-09     0      Inf        0
    x7             -2.06629067932774e-09     0     -Inf        0
    x8              3.30412399304244e-10     0      Inf        0
    x9             -4.71088357994288e-11     0     -Inf        0
```



```
    x10             3.35524767533505e-11    0      Inf        0
    x11             -1.1704745699842e-11    0     -Inf        0


Number of observations: 501, Error degrees of freedom: 490
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       47.6265936125817    500    0.0952531872251634
    Model       47.6265936125817     10      4.76265936125817    Inf       0
    Residual                   0    490                     0

A = 0.000000, B = 1.318864
order Q(x) = 4.000000, order D(x) = 7.000000
List of powers for Q(x): [0.000000, 0.010830, 0.068346, 0.173284]
List of powers for D(x): [0.000000, 0.010830, 0.068346, 0.173284, 0.314800, 0.483566,
0.672711]
Fitting cos(x) in range (0.000000, 1.570796)
MSS of errors squared = 4.323874e-06
Corrected MSS of errors squared = 6.114881e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the cosine function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i)	0.404	0.915		Inf	-Inf	2.30806400E-12	6, 7
A+B/i	2	1.569		Inf	-Inf	2.30949100E-12	7, 7
A+B*i	0.5	0.293		Inf	-Inf	1.02181600E-11	6, 7
A+B*sqrt(i+C)	1	0.751	0.981	Inf	-Inf	1.45077600E-11	6, 6
A+B*i+C*(i-1)	0.195	0.172	0.011	Inf	-Inf	3.99910900E-11	6, 7
A+B*sqrt(i)+C*(i-1)	0.55	0.4	0.374	Inf	-Inf	7.87753900E-11	6, 7
A+B*i+C*sqrt(i-1)	0.294	0.111	0	Inf	-Inf	8.17893500E-11	7, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.268	0.1	0.341	Inf	-Inf	1.11716100E-09	7, 5
A+B*i-C*(i-1)	0.328	0.141	0	Inf	-Inf	1.66855500E-09	7, 5
A+B*log10(i)^4	0	1.318		Inf	-Inf	4.32387400E-06	4, 7

RESULTS FOR THE HYPERBOLIC COSINE FUNCTION

Using Power A+B* i

```
Fitting cosh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE     tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      1.00000003761644    0       Inf       0
    x1              -5.19150343165304    0      -Inf       0
    x2               9.42867299469357    0       Inf       0
    x3              -9.55413864523507    0      -Inf       0
    x4               6.55942077944298    0       Inf       0
    x5              -3.38454151697485    0      -Inf       0
    x6               1.16675676203538    0       Inf       0
    x7              -0.206189159477844    0      -Inf       0
    x8               5.19030041213919    0       Inf       0
    x9              -9.4068440948231     0      -Inf       0
    x10              9.38802661477074    0       Inf       0
    x11             -5.85725495371894    0      -Inf       0
    x12              2.23047638431803    0       Inf       0
    x13             -0.468406730819815    0      -Inf       0
    x14             0.0413385842730231    0       Inf       0


Number of observations: 501, Error degrees of freedom: 486
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF          MeanSq          F      pValue
                    _____________    ___    ____________________    ___    ______
```



```
    Total       167322.933901256    500    334.645867802511
    Model       167322.933901256     14     11951.638135804    Inf       0
    Residual                   0    486                   0

A = 0.113375, B = 0.383545
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.496920, 0.880465, 1.264010, 1.647556, 2.031101, 2.414646,
2.798191]
List of powers for D(x): [0.496920, 0.880465, 1.264010, 1.647556, 2.031101, 2.414646,
2.798191]
Fitting cosh(x) in range (0.000000, 5.000000)
MSS of errors squared = 5.970871e-08
Corrected MSS of errors squared = 8.444087e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting cosh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)       0.99999999999912    0      Inf        0
    x1               -0.0418649513064423    0     -Inf        0
    x2                  2.64428633441156    0      Inf        0
    x3                 -20.0853498791941    0     -Inf        0
    x4                  42.3650030043131    0      Inf        0
    x5                 -25.8820742499361    0     -Inf        0
    x6                0.0169737696546466    0      Inf        0
    x7                -0.953513112425266    0     -Inf        0
    x8                   4.6879424308728    0      Inf        0
    x9                                 0    0      NaN      NaN
    x10                -12.4215731823203    0     -Inf        0
    x11                                0    0      NaN      NaN
    x12                 9.67016992677498    0      Inf        0


Number of observations: 501, Error degrees of freedom: 490
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq          F     pValue
                 _____________    ___    _____________    ___    ______
```



```
    Total       167322.933901257    500    334.645867802513
    Model       167322.933901257     10    16732.2933901257    Inf       0
    Residual                   0    490                   0

A = 0.000000, B = 0.100000
order Q(x) = 5.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.050000, 0.033333, 0.025000, 0.020000]
List of powers for D(x): [0.100000, 0.050000, 0.033333, 0.025000, 0.020000, 0.016667,
0.014286]
Fitting cosh(x) in range (0.000000, 5.000000)
MSS of errors squared = 1.844386e-01
Corrected MSS of errors squared = 2.608355e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting cosh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ________________    ___    _____    ______

    (Intercept)    0.999999542490982    0      Inf       0
    x1             -5.93211653241034    0     -Inf       0
    x2              27.2853233054993    0      Inf       0
    x3             -52.0259185004678    0     -Inf       0
    x4              53.1163480364365    0      Inf       0
    x5             -28.8921295213015    0     -Inf       0
    x6              6.65749230194046    0      Inf       0
    x7              5.99676594836134    0      Inf       0
    x8             -26.0514227213365    0     -Inf       0
    x9                48.69073561645    0      Inf       0
    x10            -46.3031338348608    0     -Inf       0
    x11             22.102657353553    0      Inf       0
    x12            -4.21909903841467    0     -Inf       0


Number of observations: 501, Error degrees of freedom: 488
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq          F     pValue
                  _____________    ___    _____________    ___    ______
```



```
    Total       167322.933901201    500    334.645867802403
    Model       167322.933901201     12    13943.5778251001    Inf       0
    Residual                   0    488                   0

A = 1.122875, B = 0.714893
order Q(x) = 6.000000, order D(x) = 6.000000
List of powers for Q(x): [1.837767, 2.133885, 2.361105, 2.552660, 2.721423, 2.873997]
List of powers for D(x): [1.837767, 2.133885, 2.361105, 2.552660, 2.721423, 2.873997]
Fitting cosh(x) in range (0.000000, 5.000000)
MSS of errors squared = 5.110217e-09
Corrected MSS of errors squared = 7.226938e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting cosh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      1.00000030148222    0      Inf       0
    x1              -9.66609567874522    0     -Inf       0
    x2               37.1565322453097    0      Inf       0
    x3              -64.3273370123518    0     -Inf       0
    x4               64.4072982310385    0      Inf       0
    x5              -38.056186550206     0     -Inf       0
    x6               11.8047867472123    0      Inf       0
    x7              -1.28331241478765    0     -Inf       0
    x8               9.56125779469517    0      Inf       0
    x9              -35.5765448599645    0     -Inf       0
    x10                 57.99322838507    0      Inf       0
    x11             -49.3028085389875    0     -Inf       0
    x12              21.4258730377836    0      Inf       0
    x13             -3.77218644360277    0     -Inf       0


Number of observations: 501, Error degrees of freedom: 487
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF          MeanSq          F     pValue
                    _________________    ___    _________________    ___    ______
```



```
    Total       167322.933901276    500    334.645867802552
    Model       167322.933901276     13    12870.9949154828    Inf       0
    Residual                   0    487                   0

A = 0.426332, B = 0.849609, C = 0.980844
order Q(x) = 7.000000, order D(x) = 6.000000
List of powers for Q(x): [1.622092, 1.893192, 2.121476, 2.322472, 2.504115, 2.671107,
2.826509]
List of powers for D(x): [1.622092, 1.893192, 2.121476, 2.322472, 2.504115, 2.671107]
Fitting cosh(x) in range (0.000000, 5.000000)
MSS of errors squared = 3.185721e-09
Corrected MSS of errors squared = 5.517831e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting cosh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ________________    ___    _____    ______

    (Intercept)    0.999999999460146    0      Inf       0
    x1             -139.222106533179    0     -Inf       0
    x2              492.733897203201    0      Inf       0
    x3             -699.967352618705    0     -Inf       0
    x4              477.941770067106    0      Inf       0
    x5             -132.420257965267    0     -Inf       0
    x6             -8.78486346679109    0     -Inf       0
    x7              8.75870410366847    0      Inf       0
    x8              141.181641122106    0      Inf       0
    x9             -508.212068430337    0     -Inf       0
    x10             750.543796320775    0      Inf       0
    x11            -565.274915996607    0     -Inf       0
    x12             216.25541892029     0      Inf       0
    x13            -33.5196585051332    0     -Inf       0


Number of observations: 501, Error degrees of freedom: 487
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq          F     pValue
```


Total	167322.933901196	500	334.645867802392		
Model	167322.933901196	13	12870.9949154766	Inf	0
Residual	0	487	0		

A = 0.500000, B = 0.100000, C = 0.000000
order Q(x) = 7.000000, order D(x) = 6.000000
List of powers for Q(x): [0.600000, 0.700000, 0.800000, 0.900000, 1.000000, 1.100000, 1.200000]
List of powers for D(x): [0.600000, 0.700000, 0.800000, 0.900000, 1.000000, 1.100000]
Fitting cosh(x) in range (0.000000, 5.000000)
MSS of errors squared = 3.449671e-07
Corrected MSS of errors squared = 5.975005e-07
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*i–C*(i–1)

Fitting cosh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	1.00000000000859	0	Inf	0
x1	274.942305444366	0	Inf	0
x2	-1505.35437490459	0	-Inf	0
x3	3436.51418869217	0	Inf	0
x4	-4194.29713842935	0	-Inf	0
x5	2892.4442936555	0	Inf	0
x6	-1070.00566379786	0	-Inf	0
x7	164.756023006919	0	Inf	0
x8	-14.3409489746016	0	-Inf	0
x9	52.5215106542517	0	Inf	0
x10	-65.9581899805477	0	-Inf	0
x11	28.7778656220461	0	Inf	0

Number of observations: 501, Error degrees of freedom: 489
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue


```
    Total       167322.933901299    500    334.645867802599
    Model       167322.933901299     11     15211.175809209    Inf       0
    Residual                   0    489                   0

A = 0.092200, B = 0.156876, C = 0.192786
order Q(x) = 7.000000, order D(x) = 4.000000
List of powers for Q(x): [0.249076, 0.213166, 0.177256, 0.141345, 0.105435, 0.069525,
0.033615]
List of powers for D(x): [0.249076, 0.213166, 0.177256, 0.141345]
Fitting cosh(x) in range (0.000000, 5.000000)
MSS of errors squared = 2.727809e-04
Corrected MSS of errors squared = 4.724703e-04
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting cosh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)      0.999999999988582     0      Inf       0
    x1               -6.49566073919565     0     -Inf       0
    x2                28.3907973853386     0      Inf       0
    x3               -56.3308191035069     0     -Inf       0
    x4                57.1105137143194     0      Inf       0
    x5               -31.5874078915506     0     -Inf       0
    x6                8.85772680523187     0      Inf       0
    x7               -0.928286222628174    0     -Inf       0
    x8                6.49918440816551     0      Inf       0
    x9               -28.5171835569756     0     -Inf       0
    x10               57.1204006694083     0      Inf       0
    x11              -59.2265384818577     0     -Inf       0
    x12               34.4152373433091     0      Inf       0
    x13              -10.6923053618845     0     -Inf       0
    x14               1.39027622646702     0      Inf       0


Number of observations: 501, Error degrees of freedom: 486
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```


	SumSq	DF	MeanSq	F	pValue
Total	167322.93390126	500	334.64586780252		
Model	167322.93390126	14	11951.6381358043	Inf	0
Residual	0	486	0		

```
A = 0.212820, B = 0.100000, C = 0.177532
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.312820, 0.590352, 0.763888, 0.920314, 1.067884, 1.209794,
1.347683]
List of powers for D(x): [0.312820, 0.590352, 0.763888, 0.920314, 1.067884, 1.209794,
1.347683]
Fitting cosh(x) in range (0.000000, 5.000000)
MSS of errors squared = 1.863380e-07
Corrected MSS of errors squared = 3.227468e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting cosh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	1.00000006863123	0	Inf	0
x1	-2.93484536692877	0	-Inf	0
x2	5.1384567095687	0	Inf	0
x3	-5.24437874034525	0	-Inf	0
x4	4.2839530267997	0	Inf	0
x5	-2.67414352287349	0	-Inf	0
x6	1.03134225122127	0	Inf	0
x7	-0.175653992698149	0	-Inf	0
x8	2.93406489107087	0	Inf	0
x9	-5.12157196424284	0	-Inf	0
x10	5.11265188683958	0	Inf	0
x11	-3.14359739738701	0	-Inf	0
x12	1.15423028868768	0	Inf	0
x13	-0.230066858269749	0	-Inf	0
x14	0.0190407936371533	0	Inf	0


```
Number of observations: 501, Error degrees of freedom: 486
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq          F     pValue
                ___________________    ___    ________________    ___    ______

    Total       167322.933901251    500    334.645867802503
    Model       167322.933901251     14    11951.6381358037    Inf       0
    Residual                   0    486                   0

A = 0.477002, B = 0.336892, C = 0.292704
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.813895, 1.246144, 1.645925, 2.028899, 2.401132, 2.765736,
3.124559]
List of powers for D(x): [0.813895, 1.246144, 1.645925, 2.028899, 2.401132, 2.765736,
3.124559]
Fitting cosh(x) in range (0.000000, 5.000000)
MSS of errors squared = 5.461935e-08
Corrected MSS of errors squared = 9.460349e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting cosh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ____________________    __    _____    ______

    (Intercept)     1.00000021635585     0      Inf       0
    x1               -2.6342789856004    0     -Inf       0
    x2                9.1338022169512    0      Inf       0
    x3              -19.0410566790929    0     -Inf       0
    x4               24.9867486203873    0      Inf       0
    x5              -20.6103046665438    0     -Inf       0
    x6               9.61211923887416    0      Inf       0
    x7              -1.91051059594204    0     -Inf       0
    x8               2.63304721142352    0      Inf       0
    x9              -9.03749967642746    0     -Inf       0
    x10              18.2442815621177    0      Inf       0
    x11             -20.3861112550331    0     -Inf       0
    x12              13.0357211687451    0      Inf       0
    x13             -4.48183678463891    0     -Inf       0
```


x14	0.644704140849949	0	Inf	0

Number of observations: 501, Error degrees of freedom: 486
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	167322.933901271	500	334.645867802542		
Model	167322.933901271	14	11951.6381358051	Inf	0
Residual	0	486	0		

A = 0.253753, B = 0.700000, C = 0.313988
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.953753, 1.557691, 1.910235, 2.197597, 2.446977, 2.670495, 2.874890]
List of powers for D(x): [0.953753, 1.557691, 1.910235, 2.197597, 2.446977, 2.670495, 2.874890]
Fitting cosh(x) in range (0.000000, 5.000000)
MSS of errors squared = 1.013756e-08
Corrected MSS of errors squared = 1.755878e-08
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*log10(i)^4

Fitting cosh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	0	0	NaN	NaN
x1	-1.36168987508264e-10	0	-Inf	0
x2	1.71179308581942e-10	0	Inf	0
x3	-5.70978413327695e-11	0	-Inf	0
x4	4.31104598292336e-11	0	Inf	0
x5	-3.32259341582777e-11	0	-Inf	0
x6	1.52605910799632e-11	0	Inf	0
x7	-3.37976369583935e-12	0	-Inf	0
x8	1.00000000013624	0	Inf	0
x9	-1.58300162281077e-10	0	-Inf	0
x10	2.22646680239719e-11	0	Inf	0


```
Number of observations: 501, Error degrees of freedom: 491
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq          F     pValue
                ___________________    ___    ________________    ___    ______

    Total       167322.933901258    500    334.645867802515
    Model       167322.933901258      9    18591.4371001397    Inf       0
    Residual                   0    491                   0

A = 0.000000, B = 1.412746
order Q(x) = 7.000000, order D(x) = 3.000000
List of powers for Q(x): [0.000000, 0.011601, 0.073211, 0.185619, 0.337208, 0.517989,
0.720597]
List of powers for D(x): [0.000000, 0.011601, 0.073211]
Fitting cosh(x) in range (0.000000, 5.000000)
MSS of errors squared = 4.610947e-02
Corrected MSS of errors squared = 6.520864e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the hyperbolic cosine function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i+C)	0.426	0.849	0.98	Inf	-Inf	3.18572100E-09	7, 6
A+B*sqrt(i)	1.122	0.714		Inf	-Inf	5.11021700E-09	6, 6
A+B*sqrt(i)+C*sqrt(i-1)	0.253	0.7	0.313	Inf	-Inf	1.01375600E-08	7, 7
A+B*sqrt(i)+C*(i-1)	0.477	0.336	0.292	Inf	-Inf	5.46193500E-08	7, 7
A+B*i	0.113	0.383		Inf	-Inf	5.97087100E-08	7, 7
A+B*i+C*sqrt(i-1)	0.212	0.1	0.177	Inf	-Inf	1.86338000E-07	7, 7
A+B*i+C*(i-1)	0.5	0.1	0	Inf	-Inf	3.44967100E-07	7, 6
A+B*i-C*(i-1)	0.092	0.156	0.192	Inf	-Inf	2.72780900E-04	7, 4
A+B*log10(i)^4	0	1.412		Inf	-Inf	4.61094700E-02	7, 3
A+B/i	0	0.1		Inf	-Inf	1.84438600E-01	5, 7

RESULTS FOR THE DIGAMMA FUNCTION

Using Power A+B* i

```
Fitting digamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______

    (Intercept)    -5.28623963656016     0      -Inf         0
    x1              42.3784834374483     0       Inf         0
    x2                             0     0       NaN       NaN
    x3             -160.063351804008     0      -Inf         0
    x4                182.9404806626     0       Inf         0
    x5             -60.040896784895     0      -Inf         0
    x6              7.00921976256799     0       Inf         0
    x7             -11.7999302328169     0      -Inf         0
    x8              14.8949347674795     0       Inf         0
    x9             -17.0355259638384     0      -Inf         0
    x10              9.1671589844638     0       Inf         0
    x11            -1.48974684970119     0      -Inf         0
    x12            0.129917375485397     0       Inf         0


Number of observations: 981, Error degrees of freedom: 969
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq            F      pValue
                ___________________    ___    ___________________    ___    ______

    Total       716.522284336722     980    0.731145188098696
```


Model	716.522284336722	11	65.1383894851565	Inf	0
Residual	0	969	0		

A = 0.148594, B = 0.100000
order Q(x) = 5.000000, order D(x) = 7.000000
List of powers for Q(x): [0.248594, 0.348594, 0.448594, 0.548594, 0.648594]
List of powers for D(x): [0.248594, 0.348594, 0.448594, 0.548594, 0.648594, 0.748594, 0.848594]
Fitting digamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 6.501735e-10
Corrected MSS of errors squared = 9.194841e-10
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B/i

Fitting digamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	-2.01693828382219	0	-Inf	0
x1	12.9692331781861	0	Inf	0
x2	-60.2941286298228	0	-Inf	0
x3	49.2727426135126	0	Inf	0
x4	-0.515138948084874	0	-Inf	0
x5	15.9773464098578	0	Inf	0
x6	-402.626668953048	0	-Inf	0
x7	1808.26188159657	0	Inf	0
x8	-2687.77889956352	0	-Inf	0
x9	1267.5616888822	0	Inf	0

Number of observations: 981, Error degrees of freedom: 971
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	716.522284330928	980	0.731145188092783		
Model	716.522284330928	9	79.6135871478808	Inf	0
Residual	0	971	0		


```
A = 0.179165, B = 0.311893
order Q(x) = 3.000000, order D(x) = 6.000000
List of powers for Q(x): [0.491058, 0.335111, 0.283129]
List of powers for D(x): [0.491058, 0.335111, 0.283129, 0.257138, 0.241544, 0.231147]
Fitting digamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 1.930917e-10
Corrected MSS of errors squared = 2.730729e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting digamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    _________________    __    _____    ______

    (Intercept)    -1.77142438628825     0     -Inf        0
    x1              433.959029899299     0      Inf        0
    x2                             0     0      NaN      NaN
    x3             -889.16640211061     0     -Inf        0
    x4                             0     0      NaN      NaN
    x5                             0     0      NaN      NaN
    x6              455.838931016782     0      Inf        0
    x7              257.328130604393     0      Inf        0
    x8             -1528.99434204548     0     -Inf        0
    x9              3609.98459386437     0      Inf        0
    x10            -4068.46358444006     0     -Inf        0
    x11             2212.11307111189     0      Inf        0
    x12            -482.944721410203     0     -Inf        0


Number of observations: 981, Error degrees of freedom: 971
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F      pValue
                  _______________    ___    _________________    ___    ______

    Total        716.52228423779    980    0.731145187997745
    Model        716.52228423779      9     79.6135871375322    Inf       0
    Residual                   0    971                    0

A = 0.595524, B = 0.100000
```



```
order Q(x) = 6.000000, order D(x) = 6.000000
List of powers for Q(x): [0.695524, 0.736946, 0.768730, 0.795524, 0.819131, 0.840473]
List of powers for D(x): [0.695524, 0.736946, 0.768730, 0.795524, 0.819131, 0.840473]
Fitting digamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 3.340423e-10
Corrected MSS of errors squared = 4.724072e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting digamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ________________    ___    _____    ______

    (Intercept)    -17.3767071785233    0      -Inf        0
    x1              1587.71534099751    0       Inf        0
    x2             -6117.76721785554    0      -Inf        0
    x3              7442.2733515424     0       Inf        0
    x4                            0    0       NaN       NaN
    x5             -6772.72993966551    0      -Inf        0
    x6              4972.48410306524    0       Inf        0
    x7             -1094.60570190915    0      -Inf        0
    x8              40.8085376511807    0       Inf        0
    x9             -119.934156488532    0      -Inf        0
    x10             123.071555660662    0       Inf        0
    x11            -42.957674294736     0      -Inf        0


Number of observations: 99, Error degrees of freedom: 88
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq           F      pValue
                  _____________    ___    _________________    ___    ______

    Total        77.1013803282087    98    0.786748778859273
    Model        77.1013803282087    10     7.71013803282087    Inf       0
    Residual                    0    88                     0

A = 0.000000, B = 0.100000, C = 0.825989
order Q(x) = 7.000000, order D(x) = 4.000000
```



```
List of powers for Q(x): [0.135129, 0.168107, 0.195601, 0.219681, 0.241371, 0.261266,
0.279750]
List of powers for D(x): [0.135129, 0.168107, 0.195601, 0.219681]
Fitting digamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 3.932838e-11
Corrected MSS of errors squared = 6.811876e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting digamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -4.27747293243368     0     -Inf        0
    x1              33.7230066935861     0      Inf        0
    x2                             0     0      NaN      NaN
    x3             -125.671003462376     0     -Inf        0
    x4              131.601521445285     0      Inf        0
    x5             -28.2804299693213     0     -Inf        0
    x6             -7.67387381684377     0     -Inf        0
    x7             0.330674990968134     0      Inf        0
    x8              2.96359822807054     0      Inf        0
    x9              2.37803181139254     0      Inf        0
    x10            -3.14974252246048     0     -Inf        0
    x11            -6.32948550686111     0     -Inf        0
    x12             4.70843951786422     0      Inf        0


Number of observations: 981, Error degrees of freedom: 969
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F      pValue
                  _______________    ___    _________________    ___    ______

    Total       716.52228433554     980    0.73114518809749
    Model       716.52228433554      11    65.1383894850491    Inf        0
    Residual                  0     969                   0

A = 0.234885, B = 0.100000, C = 0.044701
order Q(x) = 7.000000, order D(x) = 5.000000
```



```
List of powers for Q(x): [0.334885, 0.479585, 0.624286, 0.768987, 0.913687, 1.058388,
1.203089]
List of powers for D(x): [0.334885, 0.479585, 0.624286, 0.768987, 0.913687]
Fitting digamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 1.821049e-10
Corrected MSS of errors squared = 3.154150e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting digamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate          SE     tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)       -3.47656358826854     0     -Inf       0
    x1                 100.893504251306     0      Inf       0
    x2                -187.65768088061      0     -Inf       0
    x3                 82.332710361909      0      Inf       0
    x4                 6.56331139679358     0      Inf       0
    x5                0.663875786979556     0      Inf       0
    x6               -0.0965704081570141    0     -Inf       0
    x7               -0.00228111948210221   0     -Inf       0
    x8                 5.66202878109956     0      Inf       0
    x9                 1.23303303822742     0      Inf       0
    x10               -7.24676224305973     0     -Inf       0


Number of observations: 981, Error degrees of freedom: 970
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                 _______________    ___    _________________    ___    ______

    Total        716.522284336425    980    0.731145188098393
    Model        716.522284336425     10     71.6522284336425    Inf       0
    Residual                    0    970                    0

A = 0.297059, B = 0.184350, C = 0.057748
order Q(x) = 7.000000, order D(x) = 3.000000
List of powers for Q(x): [0.481409, 0.608010, 0.734611, 0.861213, 0.987814, 1.114415,
1.241016]
```



```
List of powers for D(x): [0.481409, 0.608010, 0.734611]
Fitting digamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 1.555065e-10
Corrected MSS of errors squared = 2.693451e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting digamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)     -1.12849202141196     0     -Inf        0
    x1               12.5123132849532     0      Inf        0
    x2              -26.972103111015      0     -Inf        0
    x3                             0      0      NaN       NaN
    x4               24.7489487559323     0      Inf        0
    x5              -8.74023728588752     0     -Inf        0
    x6               -1.0574338237795     0     -Inf        0
    x7               2.77235815355909     0      Inf        0
    x8               1.91902783629064     0      Inf        0
    x9              -6.81719789552356     0     -Inf        0
    x10             0.684347033950835     0      Inf        0
    x11              1.28013067971888     0      Inf        0
    x12             0.0563301689053562    0      Inf        0


Number of observations: 981, Error degrees of freedom: 969
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______

    Total        716.522284335408    980    0.731145188097355
    Model        716.522284335408     11     65.1383894850371    Inf       0
    Residual                    0    969                    0

A = 0.216284, B = 0.203409, C = 0.201586
order Q(x) = 6.000000, order D(x) = 6.000000
List of powers for Q(x): [0.419694, 0.824688, 1.111597, 1.379078, 1.636502, 1.887499]
List of powers for D(x): [0.419694, 0.824688, 1.111597, 1.379078, 1.636502, 1.887499]
```



```
Fitting digamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 1.554940e-09
Corrected MSS of errors squared = 2.693235e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting digamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      -1.56067994977082     0     -Inf        0
    x1                75.5755314111177     0      Inf        0
    x2               -152.106665928995     0     -Inf        0
    x3                             0     0      NaN       NaN
    x4                129.496717775274     0      Inf        0
    x5               -52.7415629901084     0     -Inf        0
    x6                7.00605115326526     0      Inf        0
    x7                5.3963305605513      0      Inf        0
    x8               -18.5908981943842     0     -Inf        0
    x9                2.24698352383432     0      Inf        0
    x10               2.69811936052171     0      Inf        0
    x11              -0.0743906252178552    0     -Inf        0


Number of observations: 981, Error degrees of freedom: 970
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                  _____________    ___    ___________________    ___    ______

    Total        716.522284336281    980    0.731145188098246
    Model        716.522284336281     10     71.6522284336281    Inf        0
    Residual                    0    970                     0

A = 0.286541, B = 0.367442, C = 0.042313
order Q(x) = 5.000000, order D(x) = 6.000000
List of powers for Q(x): [0.653983, 0.848496, 1.007596, 1.148365, 1.277419]
List of powers for D(x): [0.653983, 0.848496, 1.007596, 1.148365, 1.277419, 1.398152]
Fitting digamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 2.203713e-10
```



```
Corrected MSS of errors squared = 3.816943e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting digamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ________________    __    _____    ______

    (Intercept)     -2.9815917484297    0     -Inf       0
    x1               30.0097517682567   0      Inf       0
    x2              -94.2972686538877   0     -Inf       0
    x3               55.9790819528192   0      Inf       0
    x4               54.2590114640021   0      Inf       0
    x5              -43.2324347223404   0     -Inf       0
    x6               3.93686445638749   0      Inf       0
    x7               1.2071956647313    0      Inf       0
    x8              -11.8702870647582   0     -Inf       0
    x9               5.79917380148489   0      Inf       0
    x10              1.4702413092081    0      Inf       0


Number of observations: 981, Error degrees of freedom: 970
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F      pValue
                  _____________    ___    _________________    ___    ______

    Total         716.522284336169    980    0.731145188098131
    Model         716.522284336169     10     71.6522284336169    Inf       0
    Residual                     0    970                    0

A = 0.101148, B = 0.293316, C = 0.179718
order Q(x) = 5.000000, order D(x) = 5.000000
List of powers for Q(x): [0.394464, 0.695678, 0.863346, 0.999061, 1.116459]
List of powers for D(x): [0.394464, 0.695678, 0.863346, 0.999061, 1.116459]
Fitting digamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 1.612427e-09
Corrected MSS of errors squared = 2.792806e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
```



```
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting digamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)                        0    0      NaN      NaN
    x1              2.78559846608508e-08    0      Inf        0
    x2             -3.47579425660411e-08    0     -Inf        0
    x3              6.81335947668265e-09    0      Inf        0
    x4                  1.00000000257542    0      Inf        0
    x5             -2.60764230108039e-09    0     -Inf        0
    x6              -1.0771725393225e-10    0     -Inf        0
    x7             -1.13440794513948e-11    0     -Inf        0
    x8              3.16891928135001e-13    0      Inf        0


Number of observations: 981, Error degrees of freedom: 973
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                 _____________    ____    _________________    ____    ______

    Total        716.52228433692    980    0.731145188098898
    Model        716.52228433692      7     102.360326333846    Inf        0
    Residual                   0    973                    0

A = 0.000000, B = 3.021862
order Q(x) = 3.000000, order D(x) = 5.000000
List of powers for Q(x): [0.000000, 0.024815, 0.156599]
List of powers for D(x): [0.000000, 0.024815, 0.156599, 0.397039, 0.721288]
Fitting digamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 2.273145e-07
Corrected MSS of errors squared = 3.214713e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the digamma function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i+C)	0	0.1	0.825	Inf	-Inf	3.93283800E-11	7, 4
A+B*i-C*(i-1)	0.297	0.184	0.057	Inf	-Inf	1.55506500E-10	7, 3
A+B*i+C*(i-1)	0.234	0.1	0.044	Inf	-Inf	1.82104900E-10	7, 5
A+B/i	0.179	0.311		Inf	-Inf	1.93091700E-10	3, 6
A+B*sqrt(i)+C*(i-1)	0.286	0.367	0.042	Inf	-Inf	2.20371300E-10	5, 6
A+B*sqrt(i)	0.595	0.1		Inf	-Inf	3.34042300E-10	6, 6
A+B*i	0.148	0.1		Inf	-Inf	6.50173500E-10	5, 7
A+B*i+C*sqrt(i-1)	0.216	0.203	0.201	Inf	-Inf	1.55494000E-09	6, 6
A+B*sqrt(i)+C*sqrt(i-1)	0.101	0.293	0.179	Inf	-Inf	1.61242700E-09	5, 5
A+B*log10(i)^4	0	3.021		Inf	-Inf	2.27314500E-07	3, 5

RESULTS FOR THE ERROR FUNCTION

Using Power A+B* i

```
Fitting erf(x) in range (0.000000, 2.100000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                      ____________________    __    _____    ______

    (Intercept)    -1.87740924935337e-11     0     -Inf       0
    x1                  1.03530122771787     0      Inf       0
    x2                 -7.50054706650226     0     -Inf       0
    x3                  20.1053906183722     0      Inf       0
    x4                 -22.2403541282385     0     -Inf       0
    x5                  6.36409483592401     0      Inf       0
    x6                  4.94384562378568     0      Inf       0
    x7                  -2.7062385721464     0     -Inf       0
    x8                  493.040927127705     0      Inf       0
    x9                 -2423.08000841371     0     -Inf       0
    x10                 5143.94142792366     0      Inf       0
    x11                -5999.68144901832     0     -Inf       0
    x12                 4035.07405282292     0      Inf       0
    x13                -1477.82055575748     0     -Inf       0
    x14                 229.523834177133     0      Inf       0


Number of observations: 211, Error degrees of freedom: 196
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq          F      pValue
                ___________________    ___    ____________________    ___    ______
```



```
    Total       18.2280044424114    210    0.08680002115434
    Model       18.2280044424114     14     1.3020003173151    Inf       0
    Residual                   0    196                   0

A = 0.500000, B = 0.100000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.600000, 0.700000, 0.800000, 0.900000, 1.000000, 1.100000,
1.200000]
List of powers for D(x): [0.600000, 0.700000, 0.800000, 0.900000, 1.000000, 1.100000,
1.200000]
Fitting erf(x) in range (0.000000, 2.100000)
MSS of errors squared = 1.058465e-06
Corrected MSS of errors squared = 1.496895e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting erf(x) in range (0.000000, 2.100000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    3.37306817848118e-14    0      Inf        0
    x1             0.000307560185404361    0      Inf        0
    x2              0.00928065951955667    0      Inf        0
    x3               -0.173862393377827    0     -Inf        0
    x4                0.837148250941183    0      Inf        0
    x5                -1.69645607096286    0     -Inf        0
    x6                 1.54273919251598    0      Inf        0
    x7               -0.519074787378861    0     -Inf        0
    x8               -0.0062919644874074    0     -Inf        0
    x9                0.879783094772137    0      Inf        0
    x10               -12.8308047412923    0     -Inf        0
    x11                49.321338376758    0      Inf        0
    x12               -57.6103532108872    0     -Inf        0
    x13                               0    0      NaN      NaN
    x14                21.2462306507044    0      Inf        0


Number of observations: 211, Error degrees of freedom: 197
R-squared: 1,  Adjusted R-Squared: 1
```



```
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       18.2280044423161    210    0.0868000211538862
    Model       18.2280044423161     13      1.40215418787047    Inf       0
    Residual                   0    197                     0

A = 0.000000, B = 0.858846
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.858846, 0.429423, 0.286282, 0.214711, 0.171769, 0.143141,
0.122692]
List of powers for D(x): [0.858846, 0.429423, 0.286282, 0.214711, 0.171769, 0.143141,
0.122692]
Fitting erf(x) in range (0.000000, 2.100000)
MSS of errors squared = 1.314502e-08
Corrected MSS of errors squared = 1.858986e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting erf(x) in range (0.000000, 2.100000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    7.77936447933404e-12    0      Inf       0
    x1               0.0826105675116671    0      Inf       0
    x2                 -2.0035158445681    0     -Inf       0
    x3                 16.6653736834764    0      Inf       0
    x4                -41.9192366422803    0     -Inf       0
    x5                 44.9633159708813    0      Inf       0
    x6                -21.1381977487601    0     -Inf       0
    x7                 3.37282033619422    0      Inf       0
    x8                 47.4082938375172    0      Inf       0
    x9                -322.398413730969    0     -Inf       0
    x10                974.185937230295    0      Inf       0
    x11               -1586.71400974492    0     -Inf       0
    x12                1449.33935505891    0      Inf       0
    x13               -700.811542486742    0     -Inf       0
    x14                139.962884521746    0      Inf       0
```



```
Number of observations: 211, Error degrees of freedom: 196
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       18.2280044423284    210    0.0868000211539448
    Model       18.2280044423284     14      1.30200031730917    Inf       0
    Residual                   0    196                     0

A = 0.106211, B = 0.555514
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.661725, 0.891826, 1.068389, 1.217239, 1.348378, 1.466936,
1.575962]
List of powers for D(x): [0.661725, 0.891826, 1.068389, 1.217239, 1.348378, 1.466936,
1.575962]
Fitting erf(x) in range (0.000000, 2.100000)
MSS of errors squared = 3.007229e-09
Corrected MSS of errors squared = 4.252863e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting erf(x) in range (0.000000, 2.100000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    7.52731680046938e-11     0      Inf        0
    x1                -3.17271004696044     0     -Inf        0
    x2                 31.8060746502786     0      Inf        0
    x3                -111.249131362506     0     -Inf        0
    x4                 178.046729578834     0      Inf        0
    x5                -133.723181337169     0     -Inf        0
    x6                 38.2943638192458     0      Inf        0
    x7                 1819.39356789207     0      Inf        0
    x8                 -9688.5934306253     0     -Inf        0
    x9                 18763.3966318403     0      Inf        0
    x10               -14322.7732727706     0     -Inf        0
    x11                             0     0      NaN      NaN
    x12                5448.5757477895     0      Inf        0
```



```
    x13                -2019.0017898714    0      -Inf        0


Number of observations: 211, Error degrees of freedom: 198
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                 ________________    ___    ____________________    ___    ______

    Total        18.2280044424008    210    0.0868000211542896
    Model        18.2280044424008     12      1.51900037020007    Inf       0
    Residual                    0    198                     0

A = 0.168138, B = 0.148292, C = 1.340222
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.394991, 0.439159, 0.477076, 0.510823, 0.541533, 0.569902]
List of powers for D(x): [0.394991, 0.439159, 0.477076, 0.510823, 0.541533, 0.569902,
0.596395]
Fitting erf(x) in range (0.000000, 2.100000)
MSS of errors squared = 1.146963e-08
Corrected MSS of errors squared = 1.986599e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting erf(x) in range (0.000000, 2.100000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                    ____________________    ___    _____    ______

    (Intercept)    -9.79109539666852e-12     0     -Inf       0
    x1               0.00408583982358763     0      Inf       0
    x2               -0.0485267735990758     0     -Inf       0
    x3                 0.216981703515167     0      Inf       0
    x4                -0.266793576045706     0     -Inf       0
    x5                 0.100877689898994     0      Inf       0
    x6                  4.10777431097042     0      Inf       0
    x7                 -8.15156677212599     0     -Inf       0
    x8                  9.77454355644478     0      Inf       0
    x9                 -7.19688196071673     0     -Inf       0
    x10                 3.00321572369814     0      Inf       0
    x11               -0.544946354909321     0     -Inf       0
```



```
Number of observations: 211, Error degrees of freedom: 199
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       18.2280044423162    210    0.0868000211538868
    Model       18.2280044423162     11      1.65709131293784    Inf       0
    Residual                   0    199                     0

A = 0.074053, B = 0.123117, C = 0.087317
order Q(x) = 5.000000, order D(x) = 6.000000
List of powers for Q(x): [0.197170, 0.407605, 0.618039, 0.828474, 1.038909]
List of powers for D(x): [0.197170, 0.407605, 0.618039, 0.828474, 1.038909, 1.249343]
Fitting erf(x) in range (0.000000, 2.100000)
MSS of errors squared = 4.595840e-08
Corrected MSS of errors squared = 7.960229e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting erf(x) in range (0.000000, 2.100000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -2.66824058136634e-14    0     -Inf       0
    x1               -0.0612591531994926    0     -Inf       0
    x2                 0.514547277057595    0      Inf       0
    x3                -1.78007091037132     0     -Inf       0
    x4                 3.22692703746542     0      Inf       0
    x5                -3.21024449165791     0     -Inf       0
    x6                 1.6576870686594      0      Inf       0
    x7               -0.347426656825618     0     -Inf       0
    x8                 4.35299872131725     0      Inf       0
    x9                -7.63449663902528     0     -Inf       0
    x10                6.74817612120377     0      Inf       0
    x11               -3.0088273851366      0     -Inf       0
    x12                0.541959111958549    0      Inf       0
```



```
Number of observations: 211, Error degrees of freedom: 198
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ________________    ___    ____________________    ___    ______

    Total       18.228004442316    210    0.0868000211538859
    Model       18.228004442316     12        1.519000370193    Inf       0
    Residual                  0    198                     0

A = 0.000000, B = 0.100000, C = 0.000000
order Q(x) = 7.000000, order D(x) = 5.000000
List of powers for Q(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
List of powers for D(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000]
Fitting erf(x) in range (0.000000, 2.100000)
MSS of errors squared = 4.568448e-07
Corrected MSS of errors squared = 7.912783e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting erf(x) in range (0.000000, 2.100000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -2.53538212918212e-14    0     -Inf       0
    x1              -0.00157172319904631    0     -Inf       0
    x2                0.0166758666191414    0      Inf       0
    x3               -0.0683848874294313    0     -Inf       0
    x4                 0.136183896602023    0      Inf       0
    x5               -0.133546164274213     0     -Inf       0
    x6                0.0595358364273103    0      Inf       0
    x7               -0.00876770861173876    0     -Inf       0
    x8                  5.52634081197987     0      Inf       0
    x9                 -13.0217879688067     0     -Inf       0
    x10                 16.7147153066417     0      Inf       0
    x11                -12.2990305513788     0     -Inf       0
    x12                 4.90711328549641     0      Inf       0
    x13               -0.827499354349039     0     -Inf       0
```



```
Number of observations: 211, Error degrees of freedom: 197
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq            F     pValue
                ___________________    ___    ___________________    ___    ______

    Total       18.2280044423161    210    0.086800021153886
    Model       18.2280044423161     13     1.40215418787047    Inf       0
    Residual                   0    197                    0

A = 0.000000, B = 0.100000, C = 0.000000
order Q(x) = 7.000000, order D(x) = 6.000000
List of powers for Q(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
List of powers for D(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000]
Fitting erf(x) in range (0.000000, 2.100000)
MSS of errors squared = 1.346369e-08
Corrected MSS of errors squared = 2.331979e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting erf(x) in range (0.000000, 2.100000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -5.16678287310899e-13    0     -Inf       0
    x1                0.0121654918978234    0      Inf       0
    x2                -0.282622837420879    0     -Inf       0
    x3                  2.10837845785753    0      Inf       0
    x4                 -4.47440027235066    0     -Inf       0
    x5                  3.74752079078277    0      Inf       0
    x6                 -1.10413833078495    0     -Inf       0
    x7               -0.00208304747547659    0     -Inf       0
    x8                  12.1989597934527    0      Inf       0
    x9                 -51.8815306455194    0     -Inf       0
    x10                 115.661480507803    0      Inf       0
    x11                -151.073990972946    0     -Inf       0
    x12                 116.533348155784    0      Inf       0
```



```
    x13                -49.2579010377729    0      -Inf       0
    x14                 8.81391419382531    0       Inf       0


Number of observations: 211, Error degrees of freedom: 196
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF            MeanSq            F      pValue
                ________________    ___    ____________________    ___    ______

    Total       18.228004442315    210    0.0868000211538811
    Model       18.228004442315     14      1.30200031730822    Inf       0
    Residual                  0    196                     0

A = 0.000000, B = 0.400000, C = 0.082636
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.400000, 0.648321, 0.858092, 1.047908, 1.224971, 1.392975,
1.554116]
List of powers for D(x): [0.400000, 0.648321, 0.858092, 1.047908, 1.224971, 1.392975,
1.554116]
Fitting erf(x) in range (0.000000, 2.100000)
MSS of errors squared = 2.245008e-06
Corrected MSS of errors squared = 3.888469e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting erf(x) in range (0.000000, 2.100000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    1.34311460818705e-10     0      Inf        0
    x1                  3.93752111582157    0      Inf        0
    x2                 -16.8756345715308    0     -Inf        0
    x3                  -18.868431417492    0     -Inf        0
    x4                  143.387189927751    0      Inf        0
    x5                 -182.656409913133    0     -Inf        0
    x6                  71.1043501512367    0      Inf        0
    x7                  2092.70667915025    0      Inf        0
    x8                 -12133.8720398329    0     -Inf        0
    x9                  20217.5798028106    0      Inf        0
```


x10	0	0	NaN	NaN
x11	-31409.9773089595	0	-Inf	0
x12	30062.2189470999	0	Inf	0
x13	-8827.69000131443	0	-Inf	0

Number of observations: 211, Error degrees of freedom: 198
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	18.2280044430375	210	0.0868000211573214		
Model	18.2280044430375	12	1.51900037025313	Inf	0
Residual	0	198	0		

A = 0.500000, B = 0.158971, C = 0.000000
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.658971, 0.724819, 0.775346, 0.817943, 0.855471, 0.889398]
List of powers for D(x): [0.658971, 0.724819, 0.775346, 0.817943, 0.855471, 0.889398, 0.920598]
Fitting erf(x) in range (0.000000, 2.100000)
MSS of errors squared = 1.246308e-08
Corrected MSS of errors squared = 2.158669e-08
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*log10(i)^4

Fitting erf(x) in range (0.000000, 2.100000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	0	0	NaN	NaN
x1	-7.35839509515437e-16	0	-Inf	0
x2	-1.15692664222352e-14	0	-Inf	0
x3	5.20858098705128e-14	0	Inf	0
x4	-1.20426633153809e-13	0	-Inf	0
x5	8.68050067117744e-14	0	Inf	0
x6	-2.29710988088434e-14	0	-Inf	0
x7	1.00000000000431	0	Inf	0
x8	-5.05101091344547e-12	0	-Inf	0


```
    x9              7.60337787132167e-13    0      Inf        0


Number of observations: 211, Error degrees of freedom: 202
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                ____________________    ___    ____________________    ___    ______

    Total       18.2280044423161    210    0.086800021153886
    Model       18.2280044423161      8     2.27850055528951    Inf       0
    Residual                   0    202                     0

A = 0.000000, B = 3.183182
order Q(x) = 6.000000, order D(x) = 3.000000
List of powers for Q(x): [0.000000, 0.026140, 0.164959, 0.418235, 0.759794, 1.167125]
List of powers for D(x): [0.000000, 0.026140, 0.164959]
Fitting erf(x) in range (0.000000, 2.100000)
MSS of errors squared = 3.419280e-04
Corrected MSS of errors squared = 4.835593e-04
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the error function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i)	0.106	0.555		Inf	-Inf	3.00722900E-09	7, 7
A+B*sqrt(i+C)	0.168	0.148	1.34	Inf	-Inf	1.14696300E-08	6, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.5	0.158	0	Inf	-Inf	1.24630800E-08	6, 7
A+B/i	0	0.858		Inf	-Inf	1.31450200E-08	7, 7
A+B*i+C*sqrt(i-1)	0	0.1	0	Inf	-Inf	1.34636900E-08	7, 6
A+B*i+C*(i-1)	0.074	0.123	0.087	Inf	-Inf	4.59584000E-08	5, 6
A+B*i-C*(i-1)	0	0.1	0	Inf	-Inf	4.56844800E-07	7, 5
A+B*i	0.5	0.1		Inf	-Inf	1.05846500E-06	7, 7
A+B*sqrt(i)+C*(i-1)	0	0.4	0.082	Inf	-Inf	2.24500800E-06	7, 7
A+B*log10(i)^4	0	3.183		Inf	-Inf	3.41928000E-04	6, 3

RESULTS FOR THE EXPONENTIAL FUNCTION

Using Power A+B* i

```
Fitting exp(x) in range (0.000000, 2.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)     0.999999993916318      0      Inf        0
    x1              -1.20527712687302      0     -Inf        0
    x2               2.68922344593565      0      Inf        0
    x3              -2.4168028368735       0     -Inf        0
    x4               1.80675203027805      0      Inf        0
    x5              -0.728090191452736     0     -Inf        0
    x6               0.19603144441784      0      Inf        0
    x7               1.20861778349971      0      Inf        0
    x8              -1.5042461548019       0     -Inf        0
    x9               1.18429881199309      0      Inf        0
    x10             -0.441935997396052     0     -Inf        0
    x11              0.0596313968142522    0      Inf        0


Number of observations: 201, Error degrees of freedom: 189
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF          MeanSq          F      pValue
                   ____________________    ___    ____________________    ___    ______

    Total          650.160904122341      200    3.25080452061171
    Model          650.160904122341       11    59.1055367383947        Inf       0
```



```
    Residual                   0    189                   0

A = 0.163844, B = 0.431727
order Q(x) = 6.000000, order D(x) = 5.000000
List of powers for Q(x): [0.595571, 1.027299, 1.459026, 1.890753, 2.322481, 2.754208]
List of powers for D(x): [0.595571, 1.027299, 1.459026, 1.890753, 2.322481]
Fitting exp(x) in range (0.000000, 2.000000)
MSS of errors squared = 3.295222e-09
Corrected MSS of errors squared = 4.660148e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting exp(x) in range (0.000000, 2.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      0.999999999946151     0      Inf       0
    x1              -0.0238465805613957    0     -Inf       0
    x2                1.37103804964088     0      Inf       0
    x3               -11.6085810354062     0     -Inf       0
    x4                26.7890616775246     0      Inf       0
    x5               -17.4949912279066     0     -Inf       0
    x6               0.0199083483920018    0      Inf       0
    x7                -1.4209143706928     0     -Inf       0
    x8                11.8211714156381     0      Inf       0
    x9               -27.0090807828413     0     -Inf       0
    x10               17.5768927697567     0      Inf       0


Number of observations: 201, Error degrees of freedom: 190
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq          F     pValue
                  _________________    ___    _________________    ___    ______

    Total        650.160904122351     200    3.25080452061175
    Model        650.160904122351      10    65.0160904122351     Inf       0
    Residual                   0     190                   0

A = 0.000000, B = 1.275872
```



```
order Q(x) = 5.000000, order D(x) = 5.000000
List of powers for Q(x): [1.275872, 0.637936, 0.425291, 0.318968, 0.255174]
List of powers for D(x): [1.275872, 0.637936, 0.425291, 0.318968, 0.255174]
Fitting exp(x) in range (0.000000, 2.000000)
MSS of errors squared = 2.871380e-07
Corrected MSS of errors squared = 4.060745e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting exp(x) in range (0.000000, 2.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    0.999999999980808    0      Inf       0
    x1              -46.962527983001    0     -Inf       0
    x2              129.874740403595    0      Inf       0
    x3              -95.433575178441    0     -Inf       0
    x4             -15.4994012623929    0     -Inf       0
    x5              27.0283267517359    0      Inf       0
    x6              52.4136067516097    0      Inf       0
    x7             -167.606701623701    0     -Inf       0
    x8              181.296183408899    0      Inf       0
    x9             -65.1058706781908    0     -Inf       0


Number of observations: 201, Error degrees of freedom: 191
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq          F     pValue
                  _________________    ___    _________________    ___    ______

    Total        650.160904122377    200    3.25080452061189
    Model        650.160904122377      9    72.2401004580419    Inf       0
    Residual                    0    191                   0

A = 0.083921, B = 0.100000
order Q(x) = 5.000000, order D(x) = 4.000000
List of powers for Q(x): [0.183921, 0.225343, 0.257127, 0.283921, 0.307528]
List of powers for D(x): [0.183921, 0.225343, 0.257127, 0.283921]
Fitting exp(x) in range (0.000000, 2.000000)
```



```
MSS of errors squared = 2.745695e-06
Corrected MSS of errors squared = 3.883000e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting exp(x) in range (0.000000, 2.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      1.00000000002397     0      Inf       0
    x1              -169.704225217909     0     -Inf       0
    x2               754.903060466914     0      Inf       0
    x3              -1365.28325965497     0     -Inf       0
    x4               1247.29907261989     0      Inf       0
    x5              -576.101386228965     0     -Inf       0
    x6               108.080989606663     0      Inf       0
    x7               159.055990045051     0      Inf       0
    x8              -654.922265792614     0     -Inf       0
    x9               1028.72624493289     0      Inf       0
    x10             -722.729551809769     0     -Inf       0
    x11              190.798121460866     0      Inf       0


Number of observations: 201, Error degrees of freedom: 189
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq          F     pValue
                  _______________    ___    _______________    ___    ______

    Total        650.160904122465    200    3.25080452061233
    Model        650.160904122465     11    59.1055367384059    Inf       0
    Residual                    0    189                   0

A = 0.215295, B = 0.234212, C = 1.103622
order Q(x) = 6.000000, order D(x) = 5.000000
List of powers for Q(x): [0.554993, 0.627909, 0.689748, 0.744409, 0.793928, 0.839532]
List of powers for D(x): [0.554993, 0.627909, 0.689748, 0.744409, 0.793928]
Fitting exp(x) in range (0.000000, 2.000000)
MSS of errors squared = 1.195179e-08
Corrected MSS of errors squared = 2.070110e-08
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting exp(x) in range (0.000000, 2.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)     0.999999999999946    0      Inf       0
    x1              -5.26459886324561    0     -Inf       0
    x2               6.62351933883474    0      Inf       0
    x3              -4.39500588682177    0     -Inf       0
    x4              0.805712699812568    0      Inf       0
    x5              0.622091253953599    0      Inf       0
    x6             -0.482691904820773    0     -Inf       0
    x7             0.0960599415091427    0      Inf       0
    x8               5.26495827249125    0      Inf       0
    x9               -6.6363462619358    0     -Inf       0
    x10              5.20763545559433    0      Inf       0
    x11             -5.27417782774082    0     -Inf       0
    x12              4.25591513699347    0      Inf       0
    x13             -1.71114693391594    0     -Inf       0
    x14             0.259170351727046    0      Inf       0


Number of observations: 201, Error degrees of freedom: 186
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq          F     pValue
                ____________________    ___    ____________________    ___    ______

    Total       650.160904122344    200    3.25080452061172
    Model       650.160904122344     14    46.4400645801674    Inf       0
    Residual                   0    186                   0

A = 0.123382, B = 0.176224, C = 0.152633
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.299606, 0.628462, 0.957319, 1.286176, 1.615033, 1.943889,
2.272746]
List of powers for D(x): [0.299606, 0.628462, 0.957319, 1.286176, 1.615033, 1.943889,
2.272746]
```



```
Fitting exp(x) in range (0.000000, 2.000000)
MSS of errors squared = 8.063728e-11
Corrected MSS of errors squared = 1.396679e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting exp(x) in range (0.000000, 2.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate          SE     tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     0.999999999946293    0       Inf       0
    x1              -2.48948739113973    0      -Inf       0
    x2               2.98939601886377    0       Inf       0
    x3              -1.52805748419834    0      -Inf       0
    x4              0.329459751458094    0       Inf       0
    x5               2.49281565283461    0       Inf       0
    x6              -3.04382413747108    0      -Inf       0
    x7               2.07486675231353    0       Inf       0
    x8             0.0574532437423314    0       Inf       0
    x9              -1.50723523291177    0      -Inf       0
    x10              1.03264217794966    0       Inf       0
    x11             -0.217564542021843    0      -Inf       0


Number of observations: 201, Error degrees of freedom: 189
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq          F     pValue
                 ___________________    ___    ___________________    ___    ______

    Total        650.160904122341    200     3.2508045206117
    Model        650.160904122341     11    59.1055367383946    Inf       0
    Residual                    0    189                   0

A = 0.139485, B = 0.250000, C = 0.005583
order Q(x) = 4.000000, order D(x) = 7.000000
List of powers for Q(x): [0.389485, 0.633903, 0.878320, 1.122737]
List of powers for D(x): [0.389485, 0.633903, 0.878320, 1.122737, 1.367155, 1.611572,
1.855989]
Fitting exp(x) in range (0.000000, 2.000000)
```



```
MSS of errors squared = 4.573696e-09
Corrected MSS of errors squared = 7.921874e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting exp(x) in range (0.000000, 2.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      1.00000000551173     0      Inf       0
    x1               1.65239522610172     0      Inf       0
    x2              -1.53735764283304     0     -Inf       0
    x3               2.58950753244784     0      Inf       0
    x4              -1.37714138514757     0     -Inf       0
    x5              0.548827656799103     0      Inf       0
    x6               -1.6540339516731     0     -Inf       0
    x7               2.19552220912594     0      Inf       0
    x8              -0.591867350585311     0     -Inf       0
    x9               0.180349098675421     0      Inf       0
    x10             -0.274367883660966     0     -Inf       0
    x11             0.0862914854339068     0      Inf       0


Number of observations: 201, Error degrees of freedom: 189
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F      pValue
                  _______________    ___    _________________    ___    ______

    Total        650.16090412234    200     3.2508045206117
    Model        650.16090412234     11    59.1055367383945    Inf       0
    Residual                   0    189                   0

A = 0.109739, B = 0.332559, C = 0.169398
order Q(x) = 5.000000, order D(x) = 6.000000
List of powers for Q(x): [0.442298, 0.944255, 1.346981, 1.733380, 2.111329]
List of powers for D(x): [0.442298, 0.944255, 1.346981, 1.733380, 2.111329, 2.483877]
Fitting exp(x) in range (0.000000, 2.000000)
MSS of errors squared = 3.560869e-09
Corrected MSS of errors squared = 6.167605e-09
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting exp(x) in range (0.000000, 2.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      0.99999999999655     0      Inf        0
    x1              -23.0479206528905     0     -Inf        0
    x2               92.9861251967369     0      Inf        0
    x3              -120.758654103576     0     -Inf        0
    x4                             0     0      NaN      NaN
    x5               134.458383536135     0      Inf        0
    x6              -114.701975649472     0     -Inf        0
    x7               30.8101768805748     0      Inf        0
    x8               23.1564467106774     0      Inf        0
    x9              -85.3598908809494     0     -Inf        0
    x10              79.8632460474089     0      Inf        0
    x11              112.555298377728     0      Inf        0
    x12             -290.430504050506     0     -Inf        0
    x13              217.229951350559     0      Inf        0
    x14             -56.289035358098      0     -Inf        0


Number of observations: 201, Error degrees of freedom: 187
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq           DF           MeanSq           F      pValue
                  _________________    ___    _________________    ___    ______

    Total        650.160904122271     200    3.25080452061136
    Model        650.160904122271      13    50.0123772401747     Inf       0
    Residual                    0     187                   0

A = 0.550000, B = 0.400000, C = 0.000000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.950000, 1.115685, 1.242820, 1.350000, 1.444427, 1.529796,
1.608301]
List of powers for D(x): [0.950000, 1.115685, 1.242820, 1.350000, 1.444427, 1.529796,
1.608301]
```



```
Fitting exp(x) in range (0.000000, 2.000000)
MSS of errors squared = 9.260218e-12
Corrected MSS of errors squared = 1.603917e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting exp(x) in range (0.000000, 2.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9

Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)      0.999999992882471     0       Inf        0
    x1              -0.296839509428743     0      -Inf        0
    x2              -0.453694731023085     0      -Inf        0
    x3               1.83608419797343      0       Inf        0
    x4              -1.8925313555347       0      -Inf        0
    x5               0.75485269511939      0       Inf        0
    x6               0.296635778529638     0       Inf        0
    x7               1.1042519666183       0       Inf        0
    x8              -1.06743541785815      0      -Inf        0
    x9               0.317845291308441     0       Inf        0


Number of observations: 201, Error degrees of freedom: 191
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq          F     pValue
                  _____________    ___    _________________    ___    ______

    Total        650.160904122341    200    3.25080452061171
    Model        650.160904122341      9    72.2401004580379    Inf       0
    Residual                   0    191                   0

A = 0.046439, B = 0.210203, C = 0.609702
order Q(x) = 5.000000, order D(x) = 4.000000
List of powers for Q(x): [0.256642, 0.953412, 1.272769, 1.522879, 1.735870]
List of powers for D(x): [0.256642, 0.953412, 1.272769, 1.522879]
Fitting exp(x) in range (0.000000, 2.000000)
MSS of errors squared = 1.955620e-08
Corrected MSS of errors squared = 3.387233e-08
R-Squared =   1.00000000
```



```
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting exp(x) in range (0.000000, 2.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                      ____________________    __    _____    ______

    (Intercept)                          0    0      NaN      NaN
    x1              -1.0077223533868e-10    0     -Inf        0
    x2               1.24883667733323e-10    0      Inf        0
    x3              -2.73282489415117e-11    0     -Inf        0
    x4               3.14633172202497e-12    0      Inf        0
    x5                   1.00000000010077    0      Inf        0
    x6              -1.25080069259733e-10    0     -Inf        0
    x7               2.78811182790211e-11    0      Inf        0
    x8              -3.76781399054828e-12    0     -Inf        0
    x9               2.19763564037908e-13    0      Inf        0


Number of observations: 201, Error degrees of freedom: 192
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq          DF           MeanSq          F     pValue
                   _______________    ___    _______________    ___    ______

    Total       650.160904122341    200    3.25080452061171
    Model       650.160904122341      8    81.2701130152927    Inf       0
    Residual                   0    192                   0

A = 0.000000, B = 1.595510
order Q(x) = 4.000000, order D(x) = 5.000000
List of powers for Q(x): [0.000000, 0.013102, 0.082683, 0.209633]
List of powers for D(x): [0.000000, 0.013102, 0.082683, 0.209633, 0.380832]
Fitting exp(x) in range (0.000000, 2.000000)
MSS of errors squared = 3.020454e-03
Corrected MSS of errors squared = 4.271567e-03
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```


Comments

The following table shows the summary results for the ten models that fit the exponential function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i)+C*(i-1)	0.55	0.4	0	Inf	-Inf	9.26021800E-12	7, 7
A+B*i+C*(i-1)	0.123	0.176	0.152	Inf	-Inf	8.06372800E-11	7, 7
A+B*i	0.163	0.431		Inf	-Inf	3.29522200E-09	6, 5
A+B*i+C*sqrt(i-1)	0.109	0.332	0.169	Inf	-Inf	3.56086900E-09	5, 6
A+B*i-C*(i-1)	0.139	0.25	0.005	Inf	-Inf	4.57369600E-09	4, 7
A+B*sqrt(i+C)	0.215	0.234	1.103	Inf	-Inf	1.19517900E-08	6, 5
A+B*sqrt(i)+C*sqrt(i-1)	0.046	0.21	0.609	Inf	-Inf	1.95562000E-08	5, 4
A+B/i	0	1.275		Inf	-Inf	2.87138000E-07	5, 5
A+B*sqrt(i)	0.083	0.1		Inf	-Inf	2.74569500E-06	5, 4
A+B*log10(i)^4	0	1.595		Inf	-Inf	3.02045400E-03	4, 5

RESULTS FOR THE FRESNEL COSINE

Using Power A+B* i

```
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -7.82498321437591e-05    0     -Inf       0
    x1              0.000210470338019939    0      Inf       0
    x2             -0.000188518805767774    0     -Inf       0
    x3              5.62308980613566e-05    0      Inf       0
    x4                  6.69092536053876    0      Inf       0
    x5                  -19.156653474278    0     -Inf       0
    x6                  30.4249166050997    0      Inf       0
    x7                 -28.9507976648984    0     -Inf       0
    x8                  16.5056104302928    0      Inf       0
    x9                  -5.2207759269688    0     -Inf       0
    x10                0.706774743466249    0      Inf       0


Number of observations: 451, Error degrees of freedom: 440
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F      pValue
                  _____________    ___    ____________________    ___    ______

    Total         10.8248115391142    450    0.0240551367535871
    Model         10.8248115391142     10     1.08248115391142    Inf       0
    Residual                     0    440                     0
```



```
A = 0.000000, B = 0.100000
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.200000, 0.300000]
List of powers for D(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
MSS of errors squared = 4.707377e-02
Corrected MSS of errors squared = 6.657236e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     0.000567568402155187    0      Inf       0
    x1              -0.000255294530847131   0     -Inf       0
    x2               0.00231376701209572    0      Inf       0
    x3              -0.00262530433083697    0     -Inf       0
    x4               0.0650205904377489     0      Inf       0
    x5                -4.51340459877061     0     -Inf       0
    x6                 34.0905639528398     0      Inf       0
    x7                -70.0245289900059     0     -Inf       0
    x8                 41.3823482279123     0      Inf       0


Number of observations: 451, Error degrees of freedom: 442
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______

    Total       10.8248115391142    450    0.0240551367535871
    Model       10.8248115391142      8      1.35310144238928    Inf       0
    Residual                   0    442                     0

A = 0.019705, B = 0.340734
order Q(x) = 3.000000, order D(x) = 5.000000
List of powers for Q(x): [0.360440, 0.190072, 0.133283]
```



```
List of powers for D(x): [0.360440, 0.190072, 0.133283, 0.104889, 0.087852]
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
MSS of errors squared = 2.638361e-02
Corrected MSS of errors squared = 3.731207e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE     tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -0.208085343982628    0     -Inf        0
    x1               7.87274749481302    0      Inf        0
    x2              -35.2987190836814    0     -Inf        0
    x3               50.8218773830584    0      Inf        0
    x4                             0    0      NaN      NaN
    x5              -65.6661772163842    0     -Inf        0
    x6               58.7584687926542    0      Inf        0
    x7              -16.2801119492744    0     -Inf        0
    x8               35.9371932617265    0      Inf        0
    x9              -158.062175730233    0     -Inf        0
    x10              224.412416227318    0      Inf        0
    x11                            0    0      NaN      NaN
    x12             -283.969911851004    0     -Inf        0
    x13              251.978027240745    0      Inf        0
    x14             -69.2955492339086    0     -Inf        0


Number of observations: 451, Error degrees of freedom: 438
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                ____________________    ___    ____________________    ___    ______

    Total       10.8248115391145    450    0.0240551367535877
    Model       10.8248115391145     12     0.902067628259538    Inf       0
    Residual                   0    438                     0

A = 0.000000, B = 0.100000
order Q(x) = 7.000000, order D(x) = 7.000000
```



```
List of powers for Q(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
MSS of errors squared = 2.803037e-02
Corrected MSS of errors squared = 3.964093e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                       Estimate          SE     tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -0.204075294861438     0     -Inf        0
    x1               7.65174744459708     0      Inf        0
    x2              -34.1426071440201     0     -Inf        0
    x3               48.9569948793088     0      Inf        0
    x4                             0     0      NaN      NaN
    x5              -62.8093465102123     0     -Inf        0
    x6               56.0216102614902     0      Inf        0
    x7              -15.4743233746778     0     -Inf        0
    x8               35.1061963800855     0      Inf        0
    x9              -152.928574004723     0     -Inf        0
    x10              215.53223106784     0      Inf        0
    x11                            0     0      NaN      NaN
    x12             -269.581460534899     0     -Inf        0
    x13              238.040014175124     0      Inf        0
    x14             -65.168407372681     0     -Inf        0


Number of observations: 451, Error degrees of freedom: 438
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F      pValue
                  ________________    ___    ____________________    ___    ______

    Total       10.8248115391159     450    0.0240551367535909
    Model       10.8248115391159      12      0.90206762825966    Inf       0
    Residual                   0     438                     0
```



```
A = 0.000000, B = 0.123303, C = 0.000000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.123303, 0.174377, 0.213568, 0.246607, 0.275715, 0.302030,
0.326230]
List of powers for D(x): [0.123303, 0.174377, 0.213568, 0.246607, 0.275715, 0.302030,
0.326230]
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
MSS of errors squared = 2.317256e-02
Corrected MSS of errors squared = 4.013605e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                       Estimate          SE     tStat    pValue
                   ____________________    __    _____    ______

    (Intercept)    -0.213665115150878     0     -Inf       0
    x1               8.66921517463282     0      Inf       0
    x2              -34.6566884089561     0     -Inf       0
    x3               62.8068578845535     0      Inf       0
    x4              -63.7019190330805     0     -Inf       0
    x5               37.478993633528      0      Inf       0
    x6              -12.0112756149984     0     -Inf       0
    x7               1.62848455922045     0      Inf       0
    x8               35.0237580685407     0      Inf       0
    x9              -133.150926758473     0     -Inf       0
    x10              230.624585565832     0      Inf       0
    x11             -224.552370768981     0     -Inf       0
    x12              127.333729284775     0      Inf       0
    x13             -39.4720223168711     0     -Inf       0
    x14              5.19324352039676     0      Inf       0


Number of observations: 451, Error degrees of freedom: 436
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                 _____________    ___    ____________________    ___    ______
```


Total	10.8248115391158	450	0.0240551367535906		
Model	10.8248115391158	14	0.773200824222555	Inf	0
Residual	0	436	0		

A = 0.146369, B = 0.102732, C = 0.000000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.249101, 0.351832, 0.454564, 0.557295, 0.660027, 0.762758, 0.865490]
List of powers for D(x): [0.249101, 0.351832, 0.454564, 0.557295, 0.660027, 0.762758, 0.865490]
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
MSS of errors squared = 9.604513e-01
Corrected MSS of errors squared = 1.663551e+00
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*i–C*(i–1)

Fitting FresnelCosine(x) in range (0.500000, 5.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	0.000658422130012715	0	Inf	0
x1	-1.05986786243563e-05	0	-Inf	0
x2	7.94386712018448e-05	0	Inf	0
x3	-0.000593787388773735	0	-Inf	0
x4	-0.000150101289399633	0	-Inf	0
x5	1.66265563863331e-05	0	Inf	0
x6	0.00557359681233615	0	Inf	0
x7	-0.0621809544584865	0	-Inf	0
x8	0.768789388761456	0	Inf	0
x9	0.385785700424757	0	Inf	0
x10	-0.127260825380808	0	-Inf	0
x11	0.0335471864631476	0	Inf	0
x12	-0.00425409262329064	0	-Inf	0

Number of observations: 451, Error degrees of freedom: 438
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue


```
    Total       10.8248115391137    450    0.024055136753586
    Model       10.8248115391137     12    0.902067628259474    Inf       0
    Residual                   0    438                    0

A = 0.000000, B = 0.104357, C = 0.150412
order Q(x) = 5.000000, order D(x) = 7.000000
List of powers for Q(x): [0.104357, 0.058302, 0.012247, -0.033808, -0.079863]
List of powers for D(x): [0.104357, 0.058302, 0.012247, -0.033808, -0.079863, -
0.125918, -0.171973]
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
MSS of errors squared = 1.077030e-01
Corrected MSS of errors squared = 1.865471e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE     tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -0.179433014195043    0     -Inf       0
    x1               1.74574141068874    0      Inf       0
    x2              -12.5017461171825    0     -Inf       0
    x3               32.9048544817159    0      Inf       0
    x4              -42.4369660227377    0     -Inf       0
    x5               29.6982584381246    0      Inf       0
    x6              -10.866703526622     0     -Inf       0
    x7               1.6360968094549     0      Inf       0
    x8               7.46278160545254    0      Inf       0
    x9              -44.1658844875562    0     -Inf       0
    x10               106.64659659331    0      Inf       0
    x11             -128.997994872584    0     -Inf       0
    x12              85.7667361459313    0      Inf       0
    x13             -30.0842194153693    0     -Inf       0
    x14              4.37187116004808    0      Inf       0


Number of observations: 451, Error degrees of freedom: 436
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```


	SumSq	DF	MeanSq	F	pValue
Total	10.8248115391183	450	0.0240551367535961		
Model	10.8248115391183	14	0.773200824222733	Inf	0
Residual	0	436	0		

```
A = 0.226051, B = 0.100000, C = 0.236449
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.326051, 0.662500, 0.860441, 1.035593, 1.198949, 1.354767,
1.505231]
List of powers for D(x): [0.326051, 0.662500, 0.860441, 1.035593, 1.198949, 1.354767,
1.505231]
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
MSS of errors squared = 5.958619e-02
Corrected MSS of errors squared = 1.032063e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	-0.198084167834828	0	-Inf	0
x1	0.866536446675748	0	Inf	0
x2	-2.22671642713347	0	-Inf	0
x3	3.4845995766605	0	Inf	0
x4	-3.22878882037357	0	-Inf	0
x5	1.76124514288857	0	Inf	0
x6	-0.524881565845925	0	-Inf	0
x7	0.0661107401844604	0	Inf	0
x8	3.82662205872364	0	Inf	0
x9	-8.47961937872162	0	-Inf	0
x10	11.9023430810626	0	Inf	0
x11	-10.1343453824008	0	-Inf	0
x12	5.16782952216567	0	Inf	0
x13	-1.45868957749395	0	-Inf	0
x14	0.17583654329335	0	Inf	0


```
Number of observations: 451, Error degrees of freedom: 436
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ____________________    ___    ______

    Total       10.8248115391138    450    0.0240551367535861
    Model       10.8248115391138     14     0.773200824222412    Inf       0
    Residual                   0    436                     0

A = 0.001921, B = 0.173228, C = 0.171772
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.175149, 0.418674, 0.645505, 0.863693, 1.076359, 1.285102,
1.490872]
List of powers for D(x): [0.175149, 0.418674, 0.645505, 0.863693, 1.076359, 1.285102,
1.490872]
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
MSS of errors squared = 1.155420e+00
Corrected MSS of errors squared = 2.001246e+00
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -0.208085343982628    0      -Inf        0
    x1               7.87274749481302    0       Inf        0
    x2              -35.2987190836814    0      -Inf        0
    x3               50.8218773830584    0       Inf        0
    x4                              0    0       NaN      NaN
    x5              -65.6661772163842    0      -Inf        0
    x6               58.7584687926542    0       Inf        0
    x7              -16.2801119492744    0      -Inf        0
    x8               35.9371932617265    0       Inf        0
    x9              -158.062175730233    0      -Inf        0
    x10              224.412416227318    0       Inf        0
    x11                             0    0       NaN      NaN
    x12             -283.969911851004    0      -Inf        0
    x13              251.978027240745    0       Inf        0
```


x14	-69.2955492339086	0	-Inf	0

Number of observations: 451, Error degrees of freedom: 438
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	10.8248115391145	450	0.0240551367535877		
Model	10.8248115391145	12	0.902067628259538	Inf	0
Residual	0	438	0		

A = 0.000000, B = 0.100000, C = 0.000000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949, 0.264575]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949, 0.264575]
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
MSS of errors squared = 2.803037e-02
Corrected MSS of errors squared = 4.855002e-02
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*log10(i)^4

Fitting FresnelCosine(x) in range (0.500000, 5.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	0.00141373994447756	0	Inf	0
x1	-0.32827987358211	0	-Inf	0
x2	0.378688789786403	0	Inf	0
x3	-0.0518223892662673	0	-Inf	0
x4	1773.03586172418	0	Inf	0
x5	-2260.7922571258	0	-Inf	0
x6	592.702998148852	0	Inf	0
x7	-115.183579256126	0	-Inf	0
x8	11.2369762131893	0	Inf	0


```
Number of observations: 451, Error degrees of freedom: 442
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       10.8248115391193    450    0.0240551367535985
    Model       10.8248115391193      8      1.35310144238991    Inf       0
    Residual                   0    442                     0

A = 0.099312, B = 0.329071
order Q(x) = 3.000000, order D(x) = 5.000000
List of powers for Q(x): [0.099312, 0.102014, 0.116365]
List of powers for D(x): [0.099312, 0.102014, 0.116365, 0.142548, 0.177858]
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
MSS of errors squared = 3.862884e-02
Corrected MSS of errors squared = 5.462944e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the Fresnel cosine. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i+C)	0	0.123	0	Inf	-Inf	2.31725600E-02	7, 7
A+B/i	0.019	0.34		Inf	-Inf	2.63836100E-02	3, 5
A+B*sqrt(i)	0	0.1		Inf	-Inf	2.80303700E-02	7, 7
A+B*sqrt(i)+C*sqrt(i-1)	0	0.1	0	Inf	-Inf	2.80303700E-02	7, 7
A+B*log10(i)^4	0.099	0.329		Inf	-Inf	3.86288400E-02	3, 5
A+B*i	0	0.1		Inf	-Inf	4.70737700E-02	3, 7
A+B*i+C*sqrt(i-1)	0.226	0.1	0.236	Inf	-Inf	5.95861900E-02	7, 7
A+B*i-C*(i-1)	0	0.104	0.15	Inf	-Inf	1.07703000E-01	5, 7
A+B*i+C*(i-1)	0.146	0.102	0	Inf	-Inf	9.60451300E-01	7, 7
A+B*sqrt(i)+C*(i-1)	0.001	0.173	0.171	Inf	-Inf	1.15542000E+00	7, 7

RESULTS FOR THE FRESNEL SINE

Using Power A+B* i

```
Fitting FresnelSine(x) in range (1.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    0.00199499373335929     0      Inf       0
    x1               -0.11887993237616     0     -Inf       0
    x2                0.338738006359876    0      Inf       0
    x3               -0.371435327673556    0     -Inf       0
    x4                0.1843718477742      0      Inf       0
    x5               -0.0347841332174057   0     -Inf       0
    x6                284.799701350957     0      Inf       0
    x7               -1286.27379853018     0     -Inf       0
    x8                2488.02563924193     0      Inf       0
    x9               -2618.79346597405     0     -Inf       0
    x10               1574.40838374873     0      Inf       0
    x11              -510.92387957874      0     -Inf       0
    x12               69.7574022630659     0      Inf       0


Number of observations: 401, Error degrees of freedom: 388
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       7.36458791571622     400    0.0184114697892905
```



```
    Model       7.36458791571622     12     0.613715659643018    Inf       0
    Residual                   0    388                     0

A = 0.404551, B = 0.100000
order Q(x) = 5.000000, order D(x) = 7.000000
List of powers for Q(x): [0.504551, 0.604551, 0.704551, 0.804551, 0.904551]
List of powers for D(x): [0.504551, 0.604551, 0.704551, 0.804551, 0.904551, 1.004551,
1.104551]
Fitting FresnelSine(x) in range (1.000000, 5.000000)
MSS of errors squared = 4.465937e-02
Corrected MSS of errors squared = 6.315789e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting FresnelSine(x) in range (1.000000, 5.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     -0.00035859768394241    0     -Inf        0
    x1              -4.24712401218693e-05   0     -Inf        0
    x2               0.000931380537326324   0      Inf        0
    x3              -0.00237869372533758    0     -Inf        0
    x4                                      0      NaN      NaN
    x5               0.00184838224556664    0      Inf        0
    x6                                      0      NaN      NaN
    x7                                      0      NaN      NaN
    x8                0.0188651469481039    0      Inf        0
    x9                 -1.01504690196963    0     -Inf        0
    x10                  4.8978414587383    0      Inf        0
    x11                                     0      NaN      NaN
    x12                -12.7650719318947    0     -Inf        0
    x13                                     0      NaN      NaN
    x14                 9.86341222774978    0      Inf        0


Number of observations: 401, Error degrees of freedom: 391
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ___________________    ___    ____________________    ___    ______
```



```
    Total       7.36458791572862    400    0.0184114697893215
    Model       7.36458791572862      9     0.818287546192068    Inf       0
    Residual                   0    391                     0

A = 0.000000, B = 0.100000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.050000, 0.033333, 0.025000, 0.020000, 0.016667,
0.014286]
List of powers for D(x): [0.100000, 0.050000, 0.033333, 0.025000, 0.020000, 0.016667,
0.014286]
Fitting FresnelSine(x) in range (1.000000, 5.000000)
MSS of errors squared = 2.435858e-01
Corrected MSS of errors squared = 3.444823e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting FresnelSine(x) in range (1.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -0.0979231148015773     0     -Inf        0
    x1                22.8927899628106     0      Inf        0
    x2               -130.599206117056     0     -Inf        0
    x3                264.126543012223     0      Inf        0
    x4               -206.907296481654     0     -Inf        0
    x5                               0     0      NaN      NaN
    x6                80.4556317107849     0      Inf        0
    x7               -29.8705385905962     0     -Inf        0
    x8                215.392885400098     0      Inf        0
    x9               -1097.34841700918     0     -Inf        0
    x10               1673.55407131011     0      Inf        0
    x11                              0     0      NaN      NaN
    x12              -2283.84661904429     0     -Inf        0
    x13               2073.99764631814     0      Inf        0
    x14              -580.749568205431     0     -Inf        0


Number of observations: 401, Error degrees of freedom: 388
R-squared: 1,  Adjusted R-Squared: 1
```



```
F-statistic vs. constant model: Inf, p-value = 0
```

	SumSq	DF	MeanSq	F	pValue
Total	7.36458791571126	400	0.0184114697892781		
Model	7.36458791571126	12	0.613715659642605	Inf	0
Residual	0	388	0		

```
A = 0.101500, B = 0.109050
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.210549, 0.255719, 0.290379, 0.319599, 0.345342, 0.368616,
0.390018]
List of powers for D(x): [0.210549, 0.255719, 0.290379, 0.319599, 0.345342, 0.368616,
0.390018]
Fitting FresnelSine(x) in range (1.000000, 5.000000)
MSS of errors squared = 2.893078e-02
Corrected MSS of errors squared = 4.091431e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting FresnelSine(x) in range (1.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	3.49158598554652e-05	0	Inf	0
x1	-0.000442843601534044	0	-Inf	0
x2	0.000744705506273715	0	Inf	0
x3	-0.000336745656562684	0	-Inf	0
x4	120.963900415796	0	Inf	0
x5	-595.243330566703	0	-Inf	0
x6	1097.04503792399	0	Inf	0
x7	-806.254623501242	0	-Inf	0
x8	0	0	NaN	NaN
x9	288.696101434267	0	Inf	0
x10	-104.207085809115	0	-Inf	0

```
Number of observations: 401, Error degrees of freedom: 391
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```


	SumSq	DF	MeanSq	F	pValue
Total	7.36458791572549	400	0.0184114697893137		
Model	7.36458791572549	9	0.818287546191721	Inf	0
Residual	0	391	0		

```
A = 0.000000, B = 0.100000, C = 0.678897
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.129572, 0.163673, 0.191805]
List of powers for D(x): [0.129572, 0.163673, 0.191805, 0.216308, 0.238304, 0.258436,
0.277108]
Fitting FresnelSine(x) in range (1.000000, 5.000000)
MSS of errors squared = 2.795289e-01
Corrected MSS of errors squared = 4.841582e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting FresnelSine(x) in range (1.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	-0.0887369040121108	0	-Inf	0
x1	38.6220239122778	0	Inf	0
x2	-173.32073713485	0	-Inf	0
x3	329.727329817249	0	Inf	0
x4	-338.664334201549	0	-Inf	0
x5	197.370457816354	0	Inf	0
x6	-61.7200618646809	0	-Inf	0
x7	8.07408038368541	0	Inf	0
x8	484.31938939023	0	Inf	0
x9	-2236.54824417451	0	-Inf	0
x10	4392.97324487932	0	Inf	0
x11	-4674.81042705441	0	-Inf	0
x12	2833.15836760946	0	Inf	0
x13	-924.958002081186	0	-Inf	0
x14	126.865601069231	0	Inf	0

```
Number of observations: 401, Error degrees of freedom: 386
```



```
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       7.36458791568046    400    0.0184114697892012
    Model       7.36458791568046     14     0.526041993977176    Inf       0
    Residual                   0    386                     0

A = 0.500000, B = 0.100000, C = 0.000000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.600000, 0.700000, 0.800000, 0.900000, 1.000000, 1.100000,
1.200000]
List of powers for D(x): [0.600000, 0.700000, 0.800000, 0.900000, 1.000000, 1.100000,
1.200000]
Fitting FresnelSine(x) in range (1.000000, 5.000000)
MSS of errors squared = 6.288563e-02
Corrected MSS of errors squared = 1.089211e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting FresnelSine(x) in range (1.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)    -0.101851224693886     0     -Inf       0
    x1               1.34808278420268     0      Inf       0
    x2              -3.69902465136695     0     -Inf       0
    x3               4.60062998697109     0      Inf       0
    x4              -3.04757849146647     0     -Inf       0
    x5               1.04676147166677     0      Inf       0
    x6             -0.147010091686941     0     -Inf       0
    x7               13.6963104942697     0      Inf       0
    x8              -38.6304935328993     0     -Inf       0
    x9               49.7082717677606     0      Inf       0
    x10             -34.296831478157     0     -Inf       0
    x11              12.3572036492653     0      Inf       0
    x12             -1.83449243882706     0     -Inf       0
```



```
Number of observations: 401, Error degrees of freedom: 388
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       7.36458791572757    400    0.0184114697893189
    Model       7.36458791572757     12     0.613715659643964    Inf       0
    Residual                   0    388                     0

A = 0.030000, B = 0.221616, C = 0.085551
order Q(x) = 6.000000, order D(x) = 6.000000
List of powers for Q(x): [0.251616, 0.387681, 0.523745, 0.659810, 0.795875, 0.931940]
List of powers for D(x): [0.251616, 0.387681, 0.523745, 0.659810, 0.795875, 0.931940]
Fitting FresnelSine(x) in range (1.000000, 5.000000)
MSS of errors squared = 3.074406e-02
Corrected MSS of errors squared = 5.325028e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting FresnelSine(x) in range (1.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -0.0579945361640116     0     -Inf       0
    x1               0.349220781214121     0      Inf       0
    x2              -0.896051497912748     0     -Inf       0
    x3               1.26921688827317      0      Inf       0
    x4              -1.07104137468937      0     -Inf       0
    x5               0.537925730194566     0      Inf       0
    x6              -0.14869883533214      0     -Inf       0
    x7               0.017422886801178     0      Inf       0
    x8               6.29192421506499      0      Inf       0
    x9              -16.9664420006874      0     -Inf       0
    x10              25.4168981136274      0      Inf       0
    x11             -22.8454144377617      0     -Inf       0
    x12              12.3201741456938      0      Inf       0
    x13             -3.69104127501231      0     -Inf       0
    x14              0.473901102439784     0      Inf       0
```



```
Number of observations: 401, Error degrees of freedom: 386
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ________________    ___    ________________    ___    ______

    Total       7.36458791572831    400    0.0184114697893208
    Model       7.36458791572831     14     0.526041993980594    Inf       0
    Residual                   0    386                     0

A = 0.000000, B = 0.100000, C = 0.000000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
List of powers for D(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
Fitting FresnelSine(x) in range (1.000000, 5.000000)
MSS of errors squared = 3.876296e-02
Corrected MSS of errors squared = 6.713941e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting FresnelSine(x) in range (1.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -0.0124695339762302     0     -Inf        0
    x1                0.98065893416895     0      Inf        0
    x2               -3.71319729818831     0     -Inf        0
    x3                4.19931382335418     0      Inf        0
    x4                               0     0      NaN      NaN
    x5               -2.57993993090229     0     -Inf        0
    x6                1.12563412989723     0      Inf        0
    x7                152.905342948801     0      Inf        0
    x8               -765.501584735289     0     -Inf        0
    x9                1157.63278754769     0      Inf        0
    x10                              0     0      NaN      NaN
    x11              -1566.95432501426     0     -Inf        0
    x12               1419.94952514857     0      Inf        0
```



```
    x13              -397.031746296024    0      -Inf        0


Number of observations: 401, Error degrees of freedom: 389
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       7.36458791573667    400    0.0184114697893417
    Model       7.36458791573667     11     0.669507992339697    Inf       0
    Residual                   0    389                      0

A = 0.071045, B = 0.100000, C = 0.000000
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.171045, 0.212466, 0.244250, 0.271045, 0.294652, 0.315994]
List of powers for D(x): [0.171045, 0.212466, 0.244250, 0.271045, 0.294652, 0.315994,
0.335620]
Fitting FresnelSine(x) in range (1.000000, 5.000000)
MSS of errors squared = 9.070531e-02
Corrected MSS of errors squared = 1.571062e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting FresnelSine(x) in range (1.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)    -0.080855685748405    0      -Inf       0
    x1               3.53231997351224    0       Inf       0
    x2              -22.4265452733347    0      -Inf       0
    x3               60.4128265230683    0       Inf       0
    x4              -85.8597339752778    0      -Inf       0
    x5               67.8426976078292    0       Inf       0
    x6              -28.292559433108     0      -Inf       0
    x7               4.87185662893488    0       Inf       0
    x8               47.6089032464505    0       Inf       0
    x9              -318.876002406532    0      -Inf       0
    x10              900.740713154109    0       Inf       0
    x11             -1339.21545830862    0      -Inf       0
```


x12	1105.7370912458	0	Inf	0
x13	-481.592223968966	0	-Inf	0
x14	86.5969565148014	0	Inf	0

Number of observations: 401, Error degrees of freedom: 386
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	7.36458791576563	400	0.0184114697894141		
Model	7.36458791576563	14	0.526041993983259	Inf	0
Residual	0	386	0		

A = 0.000000, B = 0.388152, C = 0.006714
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.388152, 0.555644, 0.681794, 0.787933, 0.881362, 0.965787, 1.043399]
List of powers for D(x): [0.388152, 0.555644, 0.681794, 0.787933, 0.881362, 0.965787, 1.043399]
Fitting FresnelSine(x) in range (1.000000, 5.000000)
MSS of errors squared = 3.714196e-02
Corrected MSS of errors squared = 6.433176e-02
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*log10(i)^4

Fitting FresnelSine(x) in range (1.000000, 5.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	-6.14454988943739e-10	0	-Inf	0
x1	0	0	NaN	NaN
x2	7.99695443593263e-10	0	Inf	0
x3	-2.31059966682614e-10	0	-Inf	0
x4	5.15417348892737e-11	0	Inf	0
x5	-5.72046212932854e-12	0	-Inf	0
x6	1.00000000081081	0	Inf	0
x7	-1.05826565575761e-09	0	-Inf	0
x8	3.10258908095811e-10	0	Inf	0


```
    x9              -7.09044302804523e-11    0      -Inf         0
    x10              8.09768419964426e-12    0       Inf         0


Number of observations: 401, Error degrees of freedom: 391
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F      pValue
                  ________________    ___    ____________________    ___    ______

    Total       7.36458791572851    400    0.0184114697893213
    Model       7.36458791572851      9     0.818287546192056    Inf       0
    Residual                   0    391                      0

A = 0.000000, B = 1.640122
order Q(x) = 5.000000, order D(x) = 5.000000
List of powers for Q(x): [0.000000, 0.013468, 0.084995, 0.215494, 0.391481]
List of powers for D(x): [0.000000, 0.013468, 0.084995, 0.215494, 0.391481]
Fitting FresnelSine(x) in range (1.000000, 5.000000)
MSS of errors squared = 5.377593e-01
Corrected MSS of errors squared = 7.605066e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the Fresnel sine function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i)	0.101	0.109		Inf	-Inf	2.89307800E-02	7, 7
A+B*i-C*(i-1)	0.03	0.221	0.085	Inf	-Inf	3.07440600E-02	6, 6
A+B*sqrt(i)+C*sqrt(i-1)	0	0.388	0.006	Inf	-Inf	3.71419600E-02	7, 7
A+B*i+C*sqrt(i-1)	0	0.1	0	Inf	-Inf	3.87629600E-02	7, 7
A+B*i	0.404	0.1		Inf	-Inf	4.46593700E-02	5, 7
A+B*i+C*(i-1)	0.5	0.1	0	Inf	-Inf	6.28856300E-02	7, 7
A+B*sqrt(i)+C*(i-1)	0.071	0.1	0	Inf	-Inf	9.07053100E-02	6, 7
A+B/i	0	0.1		Inf	-Inf	2.43585800E-01	7, 7
A+B*sqrt(i+C)	0	0.1	0.678	Inf	-Inf	2.79528900E-01	3, 7
A+B*log10(i)^4	0	1.64		Inf	-Inf	5.37759300E-01	5, 5

RESULTS FOR THE BESSEL J0(X)

Using Power A+B* i

```
Fitting J0(x) in range (2.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      1.04516654041521e-06    0      Inf       0
    x1              -2.43417694456647e-06    0     -Inf       0
    x2               1.87951487414986e-06    0      Inf       0
    x3              -4.81277384114398e-07    0     -Inf       0
    x4                   5.68138955718568    0      Inf       0
    x5                  -13.8144505323762    0     -Inf       0
    x6                   18.6355902043765    0      Inf       0
    x7                  -15.0629146159393    0     -Inf       0
    x8                   7.29514011172701    0      Inf       0
    x9                  -1.96018043599893    0     -Inf       0
    x10                 0.225421378873146    0      Inf       0


Number of observations: 281, Error degrees of freedom: 270
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F      pValue
                  _________________    ___    ____________________    ___    ______

    Total       7.94294139670076     280    0.0283676478453599
    Model       7.94294139670076      10     0.794294139670076     Inf       0
    Residual                   0     270                     0
```



```
A = 0.000000, B = 0.100000
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.200000, 0.300000]
List of powers for D(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
Fitting J0(x) in range (2.000000, 30.000000)
MSS of errors squared = 4.357535e-02
Corrected MSS of errors squared = 6.162485e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting J0(x) in range (2.000000, 30.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     0.00330917382956018    0      Inf        0
    x1              -0.00956141906123751   0     -Inf        0
    x2                 0.495001336387036   0      Inf        0
    x3                 -2.75485561648368   0     -Inf        0
    x4                  3.66170048488443   0      Inf        0
    x5                                 0   0      NaN      NaN
    x6                 -1.39559363373393   0     -Inf        0
    x7                 -3.43369129595611   0     -Inf        0
    x8                  448.027661252887   0      Inf        0
    x9                 -5232.36208274088   0     -Inf        0
    x10                 16442.6919675133   0      Inf        0
    x11                -16153.1478822938   0     -Inf        0
    x12                                0   0      NaN      NaN
    x13                 4499.2240240517    0      Inf        0


Number of observations: 281, Error degrees of freedom: 269
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F      pValue
                  _____________    ___    ____________________    ___    ______

    Total        7.9429413970904    280    0.0283676478467514
    Model        7.9429413970904     11     0.722085581553673    Inf        0
```



```
    Residual                  0    269                     0

A = 0.136448, B = 0.181709
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.318157, 0.227302, 0.197018, 0.181875, 0.172790, 0.166733]
List of powers for D(x): [0.318157, 0.227302, 0.197018, 0.181875, 0.172790, 0.166733,
0.162406]
Fitting J0(x) in range (2.000000, 30.000000)
MSS of errors squared = 2.078523e-02
Corrected MSS of errors squared = 2.939476e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting J0(x) in range (2.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    0.00411543403778359     0      Inf        0
    x1              -0.110599333222122     0     -Inf        0
    x2               0.342530502946113     0      Inf        0
    x3                               0     0      NaN      NaN
    x4              -1.34581081549289      0     -Inf        0
    x5               2.18547327909025      0      Inf        0
    x6              -1.41138751727953      0     -Inf        0
    x7              0.335678462364494      0      Inf        0
    x8               64.5789331410118      0      Inf        0
    x9              -463.625602693724      0     -Inf        0
    x10              1390.11391580667      0      Inf        0
    x11             -2180.50405530353      0     -Inf        0
    x12              1889.3768787947       0      Inf        0
    x13             -860.042733039531      0     -Inf        0
    x14              161.102663206701      0      Inf        0


Number of observations: 281, Error degrees of freedom: 267
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF          MeanSq            F     pValue
                  ____________    ___    ____________________    ___    ______
```



```
    Total       7.94294139668597    280    0.028367647845307
    Model       7.94294139668597     13    0.610995492052767    Inf       0
    Residual                   0    267                    0

A = 0.000000, B = 0.100000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
Fitting J0(x) in range (2.000000, 30.000000)
MSS of errors squared = 1.448260e-01
Corrected MSS of errors squared = 2.048149e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting J0(x) in range (2.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -0.0591678772506568     0     -Inf        0
    x1                23.8603240733894     0      Inf        0
    x2               -113.051849583973     0     -Inf        0
    x3                199.412650675127     0      Inf        0
    x4               -140.642751927779     0     -Inf        0
    x5                               0     0      NaN      NaN
    x6                46.8882230340302     0      Inf        0
    x7               -16.4074285249095     0     -Inf        0
    x8                396.048533868575     0      Inf        0
    x9               -1871.24541385415     0     -Inf        0
    x10               3292.76524726205     0      Inf        0
    x11              -2317.3963433184      0     -Inf        0
    x12                              0     0      NaN      NaN
    x13               769.740182804549     0      Inf        0
    x14              -268.912213299745     0     -Inf        0


Number of observations: 281, Error degrees of freedom: 268
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```


	SumSq	DF	MeanSq	F	pValue
Total	7.94294139671335	280	0.0283676478454048		
Model	7.94294139671335	12	0.661911783059446	Inf	0
Residual	0	268	0		

```
A = 0.006906, B = 0.100000, C = 1.675297
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.170469, 0.198617, 0.223130, 0.245135, 0.265272, 0.283949,
0.301444]
List of powers for D(x): [0.170469, 0.198617, 0.223130, 0.245135, 0.265272, 0.283949,
0.301444]
Fitting J0(x) in range (2.000000, 30.000000)
MSS of errors squared = 3.656333e-02
Corrected MSS of errors squared = 6.332954e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting J0(x) in range (2.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	-0.00101761092894995	0	-Inf	0
x1	0.00347938367623816	0	Inf	0
x2	-0.00500986866387603	0	-Inf	0
x3	0.00374391073032639	0	Inf	0
x4	-0.0014082196586	0	-Inf	0
x5	0.000212050794586773	0	Inf	0
x6	4.65775800585941	0	Inf	0
x7	-10.1471238751472	0	-Inf	0
x8	12.7609113180759	0	Inf	0
x9	-9.70187455438361	0	-Inf	0
x10	4.43324091899871	0	Inf	0
x11	-1.12519621667696	0	-Inf	0
x12	0.122271864922209	0	Inf	0

```
Number of observations: 281, Error degrees of freedom: 268
R-squared: 1,  Adjusted R-Squared: 1
```



```
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       7.94294139670021    280    0.0283676478453579
    Model       7.94294139670021     12     0.661911783058351    Inf       0
    Residual                   0    268                     0

A = 0.001816, B = 0.109502, C = 0.015475
order Q(x) = 5.000000, order D(x) = 7.000000
List of powers for Q(x): [0.111318, 0.236295, 0.361271, 0.486248, 0.611225]
List of powers for D(x): [0.111318, 0.236295, 0.361271, 0.486248, 0.611225, 0.736201,
0.861178]
Fitting J0(x) in range (2.000000, 30.000000)
MSS of errors squared = 1.163867e-01
Corrected MSS of errors squared = 2.015876e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting J0(x) in range (2.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    0.00159409033900336    0      Inf        0
    x1              -0.411765675166239    0     -Inf        0
    x2                1.85296510423389    0      Inf        0
    x3               -3.15099940106463    0     -Inf        0
    x4                2.40356918029008    0      Inf        0
    x5               -0.69536326563267    0     -Inf        0
    x6               -200.904560003592    0     -Inf        0
    x7                1014.45537448284    0      Inf        0
    x8               -1911.81479299676    0     -Inf        0
    x9                1454.55202606522    0      Inf        0
    x10                              0    0      NaN      NaN
    x11              -589.657993355093    0     -Inf        0
    x12               234.369945449556    0      Inf        0


Number of observations: 281, Error degrees of freedom: 269
R-squared: 1,  Adjusted R-Squared: 1
```



```
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ________________    ___    ____________________    ___    ______

    Total       7.94294139670545    280    0.0283676478453766
    Model       7.94294139670545     11     0.722085581518678    Inf       0
    Residual                   0    269                     0

A = 0.082804, B = 0.100000, C = 0.114728
order Q(x) = 5.000000, order D(x) = 7.000000
List of powers for Q(x): [0.182804, 0.168076, 0.153347, 0.138619, 0.123890]
List of powers for D(x): [0.182804, 0.168076, 0.153347, 0.138619, 0.123890, 0.109162,
0.094434]
Fitting J0(x) in range (2.000000, 30.000000)
MSS of errors squared = 1.990995e-02
Corrected MSS of errors squared = 3.448504e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting J0(x) in range (2.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate            SE    tStat    pValue
                    ____________________    ___    _____    ______

    (Intercept)      2.09900116585693e-06    0      Inf       0
    x1              -4.68977422456532e-06    0     -Inf       0
    x2               3.46802347047324e-06    0      Inf       0
    x3              -8.49188660111527e-07    0     -Inf       0
    x4                   5.48012452598476    0      Inf       0
    x5                  -12.8464061533139    0     -Inf       0
    x6                   16.6986331183066    0      Inf       0
    x7                  -12.9991702475114    0     -Inf       0
    x8                   6.06024317013542    0      Inf       0
    x9                  -1.56669820286688    0     -Inf       0
    x10                 0.173261784806812    0      Inf       0


Number of observations: 281, Error degrees of freedom: 270
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
```


Total	7.94294139670023	280	0.028367647845358		
Model	7.94294139670023	10	0.794294139670023	Inf	0
Residual	0	270	0		

A = 0.000000, B = 0.117216, C = 0.000000
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.117216, 0.234433, 0.351649]
List of powers for D(x): [0.117216, 0.234433, 0.351649, 0.468866, 0.586082, 0.703298, 0.820515]
Fitting J0(x) in range (2.000000, 30.000000)
MSS of errors squared = 1.728190e-01
Corrected MSS of errors squared = 2.993314e-01
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*sqrt(i)+C*(i–1)

Fitting J0(x) in range (2.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	-0.000256385150614329	0	-Inf	0
x1	0.00390338492099405	0	Inf	0
x2	-0.0126936194583862	0	-Inf	0
x3	0.0172897275379227	0	Inf	0
x4	-0.0108147398516984	0	-Inf	0
x5	0.00257161183205898	0	Inf	0
x6	35.6424689382841	0	Inf	0
x7	-208.570609950533	0	-Inf	0
x8	536.834746469554	0	Inf	0
x9	-740.657748800791	0	-Inf	0
x10	572.930793970429	0	Inf	0
x11	-235.226478428876	0	-Inf	0
x12	40.0468272256238	0	Inf	0

Number of observations: 281, Error degrees of freedom: 268
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue

Total	7.94294139668666	280	0.0283676478453095		
Model	7.94294139668666	12	0.661911783057222	Inf	0
Residual	0	268	0		

A = 0.000000, B = 0.124642, C = 0.008693
order Q(x) = 5.000000, order D(x) = 7.000000
List of powers for Q(x): [0.124642, 0.184963, 0.233271, 0.275361, 0.313478]
List of powers for D(x): [0.124642, 0.184963, 0.233271, 0.275361, 0.313478, 0.348772, 0.381927]
Fitting J0(x) in range (2.000000, 30.000000)
MSS of errors squared = 5.602324e-02
Corrected MSS of errors squared = 9.703510e-02
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*sqrt(i)+C*sqrt(i–1)

Fitting J0(x) in range (2.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	0.00662744824103848	0	Inf	0
x1	-4.52320506057333	0	-Inf	0
x2	26.6021626053376	0	Inf	0
x3	-54.4583327945364	0	-Inf	0
x4	42.8774542835066	0	Inf	0
x5	0	0	NaN	NaN
x6	-16.6911150230791	0	-Inf	0
x7	6.18640947474629	0	Inf	0
x8	2366.62223105527	0	Inf	0
x9	-21041.9152941333	0	-Inf	0
x10	70502.2846369557	0	Inf	0
x11	-118886.723800065	0	-Inf	0
x12	108560.575377353	0	Inf	0
x13	-51458.0998144956	0	-Inf	0
x14	9958.25665309183	0	Inf	0

Number of observations: 281, Error degrees of freedom: 267
R-squared: 1, Adjusted R-Squared: 1


```
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq            F      pValue
                ___________________    ___    ________________    ___    ______

    Total       7.94294139795548    280    0.028367647849841
    Model       7.94294139795548     13    0.610995492150421    Inf       0
    Residual                   0    267                    0

A = 0.191058, B = 0.100000, C = 0.000083
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.291058, 0.332563, 0.364381, 0.391203, 0.414832, 0.436194,
0.455838]
List of powers for D(x): [0.291058, 0.332563, 0.364381, 0.391203, 0.414832, 0.436194,
0.455838]
Fitting J0(x) in range (2.000000, 30.000000)
MSS of errors squared = 4.250955e-01
Corrected MSS of errors squared = 7.362869e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting J0(x) in range (2.000000, 30.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)       2.99688531505215     0      Inf       0
    x1               -11.9946573886114     0     -Inf       0
    x2                9.81111162457279     0      Inf       0
    x3               -0.889766423417033    0     -Inf       0
    x4                0.0823558157397766   0      Inf       0
    x5               -0.00617399065954897  0     -Inf       0
    x6                0.000293607105718371 0      Inf       0
    x7               -6.3082653463866e-06  0     -Inf       0
    x8                3.88784155068136     0      Inf       0
    x9               -3.12797020508908     0     -Inf       0
    x10               0.258890436009218    0      Inf       0
    x11              -0.0198616091910341   0     -Inf       0
    x12               0.00107820945213439  0      Inf       0
    x13              -2.76201222483588e-05 0     -Inf       0
```



```
Number of observations: 281, Error degrees of freedom: 267
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       7.94294139670038    280    0.0283676478453585
    Model       7.94294139670038     13     0.610995492053876    Inf       0
    Residual                   0    267                     0

A = 0.051323, B = 3.500000
order Q(x) = 7.000000, order D(x) = 6.000000
List of powers for Q(x): [0.051323, 0.080065, 0.232701, 0.511185, 0.886738, 1.334611,
1.836563]
List of powers for D(x): [0.051323, 0.080065, 0.232701, 0.511185, 0.886738, 1.334611]
Fitting J0(x) in range (2.000000, 30.000000)
MSS of errors squared = 1.844570e-01
Corrected MSS of errors squared = 2.608616e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the Bessel $J_0(x)$ function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i-C*(i-1)	0.082	0.1	0.114	Inf	-Inf	1.99099500E-02	5, 7
A+B/i	0.136	0.181		Inf	-Inf	2.07852300E-02	6, 7
A+B*sqrt(i+C)	0.006	0.1	1.675	Inf	-Inf	3.65633300E-02	7, 7
A+B*i	0	0.1		Inf	-Inf	4.35753500E-02	3, 7
A+B*sqrt(i)+C*(i-1)	0	0.124	0.008	Inf	-Inf	5.60232400E-02	5, 7
A+B*i+C*(i-1)	0.001	0.109	0.015	Inf	-Inf	1.16386700E-01	5, 7
A+B*sqrt(i)	0	0.1		Inf	-Inf	1.44826000E-01	7, 7
A+B*i+C*sqrt(i-1)	0	0.117	0	Inf	-Inf	1.72819000E-01	3, 7
A+B*log10(i)^4	0.051	3.5		Inf	-Inf	1.84457000E-01	7, 6
A+B*sqrt(i)+C*sqrt(i-1)	0.191	0.1	0	Inf	-Inf	4.25095500E-01	7, 7

RESULTS FOR THE BESSEL J1(X)

Using Power A+B* i

```
Fitting J1(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                    Estimate                SE                tStat              pValue
                ____________________    ____________________    ____________    ____________________

    (Intercept)    -3.38023825616514e-13    1.2209214126979e-08    -2.768596095547e-05       0.999977929100345
    x1               -0.0209474946857677    0.00054886113840632      -38.1653814051967     3.0283871065861e-114
    x2                 0.115002211362598    0.00294336565162512       39.0716699772254    1.08069172934225e-116
    x3                -0.257334361208171    0.00648126413255296      -39.704347168275    2.22865362329464e-118
    x4                 0.301887159056333    0.00752304222953422       40.1283350332893    1.69353270972961e-119
    x5                -0.1964837351927      0.00486507113960385      -40.3866109157654     3.5566291743298e-120
    x6                0.0674312718771317    0.00166458192369043       40.5094341813075    1.69745912969672e-120
    x7               -0.00955015916125282   0.000235693949337575      -40.5193225710453    1.59943457714813e-120
    x8                 5.76390624995364     0.00748223452601495       770.345573894153                    0
    x9                -14.2429238407241      0.03717613666259        -383.120063550247                   0
    x10                19.5552702750191      0.0769056368772311       254.276163218521                    0
    x11               -16.1084154677452      0.0847732025946358      -190.017776546341    5.13198330770138e-303
    x12                7.95957254353815      0.0525088755111721       151.585278984781    3.16418146694421e-275
    x13               -2.1841373951198       0.0173263612441347      -126.058631950731    1.16413715865071e-252
    x14               0.256716536589612      0.00237919337587932      107.900660447465    9.7029777733081e-234


Number of observations: 301, Error degrees of freedom: 286
Root Mean Squared Error: 1.22e-08
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: 5.98e+15, p-value = 0
                    SumSq          DF          MeanSq                F            pValue
                ____________    ____    ____________________    ____________    ______

    Total          12.4852228770457    300      0.0416174095901524
    Model          12.4852228770457     14       0.89180163407469    5.98263961966381e+15       0
    Residual    4.2632564145606e-14    286    1.49064909600021e-16

A = 0.000000, B = 0.100000
```



```
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
List of powers for D(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
Fitting J1(x) in range (0.000000, 30.000000)
MSS of errors squared = 8.049462e-02
Corrected MSS of errors squared = 1.138366e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.012045e+04
AIC =  -1.015213e+04
AICc =  -1.012045e+04
```

Using Power A+B/i

```
Fitting J1(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    3.16161968215517e-14     0      Inf        0
    x1             2.72643140450986e-05     0      Inf        0
    x2             -0.00221568798820221     0     -Inf        0
    x3               0.0214095171510719     0      Inf        0
    x4              -0.0606715703295075     0     -Inf        0
    x5               0.0558332079122937     0      Inf        0
    x6                                0     0      NaN      NaN
    x7              -0.0143827310069892     0     -Inf        0
    x8             -0.00577313312708963     0     -Inf        0
    x9                0.972605705115855     0      Inf        0
    x10              -14.2811248878197     0     -Inf        0
    x11               54.3232798480489     0      Inf        0
    x12              -62.7445887013746     0     -Inf        0
    x13                               0     0      NaN      NaN
    x14               22.7356011690373     0      Inf        0


Number of observations: 301, Error degrees of freedom: 288
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ____________________    ___    ______

    Total       12.4852228770435    300    0.0416174095901451
    Model       12.4852228770435     12      1.04043523975363    Inf       0
```



```
    Residual                  0    288                    0

A = 0.000000, B = 0.100000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.050000, 0.033333, 0.025000, 0.020000, 0.016667,
0.014286]
List of powers for D(x): [0.100000, 0.050000, 0.033333, 0.025000, 0.020000, 0.016667,
0.014286]
Fitting J1(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.928379e-01
Corrected MSS of errors squared = 2.727140e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting J1(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      7.7686219422259e-12    0      Inf       0
    x1               8.82000514705104e-05   0      Inf       0
    x2              -0.000314274026098757   0     -Inf       0
    x3               0.000360350713007178   0      Inf       0
    x4              -0.000134231861217747   0     -Inf       0
    x5                    70.7714698407228  0      Inf       0
    x6                   -527.628298999445  0     -Inf       0
    x7                    1628.34677598811  0      Inf       0
    x8                   -2616.90697951694  0     -Inf       0
    x9                    2316.34740760438  0      Inf       0
    x10                  -1074.84826806061  0     -Inf       0
    x11                   204.917893037115  0      Inf       0


Number of observations: 301, Error degrees of freedom: 289
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______

    Total        12.4852228771261    300    0.0416174095904203
    Model        12.4852228771261     11      1.13502026155692    Inf       0
```



```
    Residual                  0    289                    0

A = 0.000000, B = 0.100000
order Q(x) = 4.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.141421, 0.173205, 0.200000]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
Fitting J1(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.004593e-01
Corrected MSS of errors squared = 1.420709e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting J1(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    3.64996081668451e-13     0      Inf       0
    x1                -3.92266700022375     0     -Inf       0
    x2                 32.2720287036765     0      Inf       0
    x3                -104.80944645705     0     -Inf       0
    x4                 174.894637154774     0      Inf       0
    x5                -159.71059968095     0     -Inf       0
    x6                 76.1508702072661     0      Inf       0
    x7                -14.8748226127619     0     -Inf       0
    x8                 688.942477175452     0      Inf       0
    x9                -4668.85295775682     0     -Inf       0
    x10                13118.5563572754     0      Inf       0
    x11               -19497.5780617789     0     -Inf       0
    x12                16166.7986962779     0      Inf       0
    x13                -7096.078581312     0     -Inf       0
    x14                1289.21206940439     0      Inf       0


Number of observations: 301, Error degrees of freedom: 286
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  ____________    ___    ____________________    ___    ______
```

Total	12.4852228770842	300	0.0416174095902806		
Model	12.4852228770842	14	0.891801634077442	Inf	0
Residual	0	286	0		

A = 0.000000, B = 0.100000, C = 1.399721
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.154910, 0.184383, 0.209755, 0.232373, 0.252977, 0.272024, 0.289823]
List of powers for D(x): [0.154910, 0.184383, 0.209755, 0.232373, 0.252977, 0.272024, 0.289823]
Fitting J1(x) in range (0.000000, 30.000000)
MSS of errors squared = 5.623638e-02
Corrected MSS of errors squared = 9.740426e-02
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*i+C*(i–1)

Fitting J1(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	-3.38023825616514e-13	1.2209214126979e-08	-2.768596095547e-05	0.999977929100345
x1	-0.0209474946857677	0.00054886113840632	-38.1653814051967	3.0283871065861e-114
x2	0.115002211362598	0.00294336565162512	39.0716699772254	1.08069172934225e-116
x3	-0.257334361208171	0.00648126413255296	-39.704347168275	2.22865362329464e-118
x4	0.301887159056333	0.00752304222953422	40.1283350332893	1.69353270972961e-119
x5	-0.1964837351927	0.00486507113960385	-40.3866109157654	3.5566291743298e-120
x6	0.0674312718771317	0.00166458192369043	40.5094341813075	1.69745912969672e-120
x7	-0.00955015916125282	0.000235693949337575	-40.5193225710453	1.59943457714813e-120
x8	5.76390624995364	0.00748223452601495	770.345573894153	0
x9	-14.2429238407241	0.03717613666259	-383.120063550247	0
x10	19.5552702750191	0.0769056368772311	254.276163218521	0
x11	-16.1084154677452	0.0847732025946358	-190.017776546341	5.13198330770138e-303
x12	7.95957254353815	0.0525088755111721	151.585278984781	3.16418146694421e-275
x13	-2.1841373951198	0.0173263612441347	-126.058631950731	1.16413715865071e-252
x14	0.256716536589612	0.00237919337587932	107.900660447465	9.7029777733081e-234

Number of observations: 301, Error degrees of freedom: 286
Root Mean Squared Error: 1.22e-08
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: 5.98e+15, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	12.4852228770457	300	0.0416174095901524		
Model	12.4852228770457	14	0.89180163407469	5.98263961966381e+15	0
Residual	4.2632564145606e-14	286	1.49064909600021e-16		


```
A = 0.000000, B = 0.100000, C = 0.000000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
List of powers for D(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
Fitting J1(x) in range (0.000000, 30.000000)
MSS of errors squared = 8.049462e-02
Corrected MSS of errors squared = 1.394208e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.012045e+04
AIC =  -1.015213e+04
AICc =  -1.012045e+04
```

Using Power A+B*i–C*(i–1)

```
Fitting J1(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -9.16018129644335e-06    0     -Inf       0
    x1               -1.730200353928e-06    0     -Inf       0
    x2              8.86855508466071e-06    0      Inf       0
    x3             -1.78030834347004e-05    0     -Inf       0
    x4              1.98216170682694e-05    0      Inf       0
    x5             -0.00301213626380203     0     -Inf       0
    x6                0.028487389676674     0      Inf       0
    x7               -0.131927312133106     0     -Inf       0
    x8                0.542411683088947     0      Inf       0
    x9                0.640108014930607     0      Inf       0
    x10              -0.085529256220343     0     -Inf       0
    x11               0.00946162456165389    0      Inf       0


Number of observations: 300, Error degrees of freedom: 288
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _____________    ___    ____________________    ___    ______

    Total         12.4839313141885    299    0.0417522786427708
    Model         12.4839313141885     11       1.1349028467444    Inf       0
    Residual                     0    288                     0
```



```
A = 0.142566, B = 0.164313, C = 0.250000
order Q(x) = 4.000000, order D(x) = 7.000000
List of powers for Q(x): [0.306880, 0.221193, 0.135507, 0.049820]
List of powers for D(x): [0.306880, 0.221193, 0.135507, 0.049820, -0.035866, -
0.121553, -0.207239]
Fitting J1(x) in range (0.000000, 30.000000)
MSS of errors squared = NaN
Corrected MSS of errors squared = NaN
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting J1(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	-3.38023825616514e-13	1.2209214126979e-08	-2.768596095547e-05	0.999977929100345
x1	-0.0209474946857677	0.00054886113840632	-38.1653814051967	3.0283871065861e-114
x2	0.115002211362598	0.00294336565162512	39.0716699772254	1.08069172934225e-116
x3	-0.257334361208171	0.00648126413255296	-39.704347168275	2.22865362329464e-118
x4	0.301887159056333	0.00752304222953422	40.1283350332893	1.69353270972961e-119
x5	-0.1964837351927	0.00486507113960385	-40.3866109157654	3.5566291743298e-120
x6	0.0674312718771317	0.00166458192369043	40.5094341813075	1.69745912969672e-120
x7	-0.00955015916125282	0.000235693949337575	-40.5193225710453	1.59943457714813e-120
x8	5.76390624995364	0.00748223452601495	770.345573894153	0
x9	-14.2429238407241	0.03717613666259	-383.120063550247	0
x10	19.5552702750191	0.0769056368772311	254.276163218521	0
x11	-16.1084154677452	0.0847732025946358	-190.017776546341	5.13198330770138e-303
x12	7.95957254353815	0.0525088755111721	151.585278984781	3.16418146694421e-275
x13	-2.1841373951198	0.0173263612441347	-126.058631950731	1.16413715865071e-252
x14	0.256716536589612	0.00237919337587932	107.900660447465	9.7029777733081e-234

```
Number of observations: 301, Error degrees of freedom: 286
Root Mean Squared Error: 1.22e-08
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: 5.98e+15, p-value = 0
```

	SumSq	DF	MeanSq	F	pValue
Total	12.4852228770457	300	0.0416174095901524		
Model	12.4852228770457	14	0.89180163407469	5.98263961966381e+15	0
Residual	4.2632564145606e-14	286	1.49064909600021e-16		

```
A = 0.000000, B = 0.100000, C = 0.000000
order Q(x) = 7.000000, order D(x) = 7.000000
```



```
List of powers for Q(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
List of powers for D(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
Fitting J1(x) in range (0.000000, 30.000000)
MSS of errors squared = 8.049462e-02
Corrected MSS of errors squared = 1.394208e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.012045e+04
AIC =  -1.015213e+04
AICc =  -1.012045e+04
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting J1(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                         Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    4.16779993686073e-14     0      Inf       0
    x1               -0.231122988487284     0     -Inf       0
    x2                 2.43275844472221     0      Inf       0
    x3                -9.16838064406484     0     -Inf       0
    x4                 16.9549032069839     0      Inf       0
    x5                -16.7159189480753     0     -Inf       0
    x6                 8.46357898666374     0      Inf       0
    x7                -1.7358179747193      0     -Inf       0
    x8                 65.7731396031835     0      Inf       0
    x9                -475.962740303627     0     -Inf       0
    x10                1436.00629466761     0      Inf       0
    x11               -2264.5103213503      0     -Inf       0
    x12                1971.5038847276      0      Inf       0
    x13               -901.327008725975     0     -Inf       0
    x14                169.516751192986     0      Inf       0


Number of observations: 301, Error degrees of freedom: 286
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                ____________________    ___    ____________________    ___    ______

    Total       12.4852228771752     300    0.041617409590584
    Model       12.4852228771752      14    0.891801634083943     Inf       0
    Residual                   0     286                    0
```



```
A = 0.000000, B = 0.100000, C = 0.000000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
Fitting J1(x) in range (0.000000, 30.000000)
MSS of errors squared = 5.611945e-02
Corrected MSS of errors squared = 9.720173e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting J1(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE     tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    4.55437568568894e-12     0      Inf       0
    x1                -1.49308157476752     0     -Inf       0
    x2                 15.7169877866518     0      Inf       0
    x3                -59.2355520859034     0     -Inf       0
    x4                 109.546744817736     0      Inf       0
    x5                -108.005447577344     0     -Inf       0
    x6                 54.6863053729183     0      Inf       0
    x7                -11.2159561986887     0     -Inf       0
    x8                 259.116582844721     0      Inf       0
    x9                -2093.56452875217     0     -Inf       0
    x10                6659.07011498771     0      Inf       0
    x11               -10850.857003227     0     -Inf       0
    x12                9665.93931265067     0      Inf       0
    x13               -4495.77634598328     0     -Inf       0
    x14                857.071866252072     0      Inf       0


Number of observations: 301, Error degrees of freedom: 286
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______
```



```
    Total       12.4852228767456    300    0.0416174095891522
    Model       12.4852228767456     14     0.891801634053261    Inf       0
    Residual                   0    286                      0

A = 0.055870, B = 0.100000, C = 0.000000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.155870, 0.197291, 0.229075, 0.255870, 0.279477, 0.300819,
0.320445]
List of powers for D(x): [0.155870, 0.197291, 0.229075, 0.255870, 0.279477, 0.300819,
0.320445]
Fitting J1(x) in range (0.000000, 30.000000)
MSS of errors squared = 2.946177e-01
Corrected MSS of errors squared = 5.102928e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting J1(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -1.54379741130639e-13    0     -Inf        0
    x1                 -47.2096607996728    0     -Inf        0
    x2                  63.3256777451639    0      Inf        0
    x3                 -21.6946419687327    0     -Inf        0
    x4                  6.98930228570547    0      Inf        0
    x5                 -1.58760434322143    0     -Inf        0
    x6                 0.176927083013585    0      Inf        0
    x7                  25.8607253716613    0      Inf        0
    x8                                 0    0      NaN      NaN
    x9                 -48.6625434344838    0     -Inf        0
    x10                 32.7013278537112    0      Inf        0
    x11                -10.2481014442616    0     -Inf        0
    x12                 1.34859164835435    0      Inf        0


Number of observations: 301, Error degrees of freedom: 289
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F     pValue
                  _____________    ___    ____________________    ___    ______
```



```
    Total       12.4852228770439    300    0.0416174095901464
    Model       12.4852228770439     11      1.13502026154945    Inf       0
    Residual                   0    289                     0

A = 0.017685, B = 0.100000
order Q(x) = 6.000000, order D(x) = 6.000000
List of powers for Q(x): [0.017685, 0.018506, 0.022868, 0.030824, 0.041554, 0.054351]
List of powers for D(x): [0.017685, 0.018506, 0.022868, 0.030824, 0.041554, 0.054351]
Fitting J1(x) in range (0.000000, 30.000000)
MSS of errors squared = 9.417841e-01
Corrected MSS of errors squared = 1.331884e+00
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the Bessel $J_1(x)$ function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i)+C*(i-1)	0	0.1	0	Inf	-Inf	5.61194500E-02	7, 7
A+B*sqrt(i+C)	0	0.1	1.399	Inf	-Inf	5.62363800E-02	7, 7
A+B*i	0	0.1		5.90E+15	-1.01E+04	8.04946200E-02	7, 7
A+B*i+C*(i-1)	0	0.1	0	5.90E+15	-1.01E+04	8.04946200E-02	7, 7
A+B*i+C*sqrt(i-1)	0	0.1	0	5.90E+19	-1.01E+04	8.04946200E-02	7, 7
A+B*sqrt(i)	0	0.1		Inf	-Inf	1.00459300E-01	4, 7
A+B/i	0	0.1		Inf	-Inf	1.92837900E-01	7, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.055	0.1	0	Inf	-Inf	2.94617700E-01	7, 7
A+B*log10(i)^4	0.017	0.1		Inf	-Inf	9.41784100E-01	6, 6
A+B*i-C*(i-1)	0.142	0.164	0.25	Inf	-Inf	NaN	4, 7

RESULTS FOR THE BESSEL J2(X)

Using Power A+B* i

Fitting J2(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	7.59306227962375e-14	6.58223188124914e-09	1.15356955157629e-05	0.999990803860698
x1	-0.00052901216428357	1.77278675124421e-05	-29.8407106163383	5.72343964667716e-90
x2	0.00234398211008186	7.93115124650395e-05	29.5541219329929	4.63649835175916e-89
x3	-0.00408371064449921	0.000140476941380456	-29.0703271609483	1.62207607173096e-87
x4	0.00350833307597933	0.000123365288542111	28.4385755299535	1.75932397388015e-85
x5	-0.00148974749723866	5.37827434797724e-05	-27.6993585832778	4.51277504719217e-83
x6	0.000250575633800697	9.31982191808002e-06	26.8863113483523	2.17949032283642e-80
x7	5.83169133638146	0.00553921409202514	1052.8012168328	0
x8	-14.5567066062165	0.027530519135932	-528.747988163346	0
x9	20.1593422016116	0.0569331332931487	354.08805093883	0
x10	-16.7274001895898	0.0627018617886895	-266.776770456394	0
x11	8.31560312856906	0.0387845952589246	214.404793270483	8.46907567475231e-319
x12	-2.29311829262665	0.0127748863803126	-179.502049909467	8.12851448553961e-297
x13	0.270584882405149	0.00175043874765303	154.581177300804	2.23479236901774e-278

Number of observations: 301, Error degrees of freedom: 287
Root Mean Squared Error: 6.58e-09
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: 1.85e+16, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	10.4218514727574	300	0.0347395049091915		
Model	10.4218514727574	13	0.801680882519803	1.85035548344044e+16	0
Residual	1.24344978758018e-14	287	4.33257765707378e-17		

A = 0.000000, B = 0.100000
order Q(x) = 6.000000, order D(x) = 7.000000


```
List of powers for Q(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000]
List of powers for D(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
Fitting J2(x) in range (0.000000, 30.000000)
MSS of errors squared = 4.126671e-01
Corrected MSS of errors squared = 5.835994e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.039855e+04
AIC =  -1.042802e+04
AICc =  -1.039855e+04
```

Using Power A+B/i

```
Fitting J2(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -5.88139809891648e-13    0     -Inf       0
    x1             -5.80312080424327e-09    0     -Inf       0
    x2              1.19389821383093e-07    0      Inf       0
    x3             -3.20172061773455e-07    0     -Inf       0
    x4              2.06962046629735e-07    0      Inf       0
    x5              0.000904495090255909    0      Inf       0
    x6                -0.504470032749013    0     -Inf       0
    x7                  16.0836676494639    0      Inf       0
    x8                 -126.642198556212    0     -Inf       0
    x9                   371.45561822546    0      Inf       0
    x10                 -449.282077388603    0     -Inf       0
    x11                  189.888555605128    0      Inf       0


Number of observations: 301, Error degrees of freedom: 289
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______

    Total       10.4218514727639    300    0.0347395049092128
    Model       10.4218514727639     11     0.947441042978532    Inf       0
    Residual                   0    289                     0

A = 0.006158, B = 0.475762
order Q(x) = 4.000000, order D(x) = 7.000000
List of powers for Q(x): [0.481920, 0.244039, 0.164745, 0.125099]
```



```
List of powers for D(x): [0.481920, 0.244039, 0.164745, 0.125099, 0.101310, 0.085452,
0.074124]
Fitting J2(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.083663e-01
Corrected MSS of errors squared = 1.532531e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting J2(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate           SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)    5.59908566110094e-13     0      Inf        0
    x1             -0.00150161453973932     0     -Inf        0
    x2               0.0117287069866154     0      Inf        0
    x3              -0.0325893534157251     0     -Inf        0
    x4               0.0423727109036417     0      Inf        0
    x5              -0.0263844029314398     0     -Inf        0
    x6              0.00637395990816999     0      Inf        0
    x7                 66.4393608920963     0      Inf        0
    x8                 -482.60108272921     0     -Inf        0
    x9                 1460.03411942579     0      Inf        0
    x10               -2307.50228618876     0     -Inf        0
    x11                2012.70613709862     0      Inf        0
    x12               -921.672591031689     0     -Inf        0
    x13                173.596342475355     0      Inf        0


Number of observations: 301, Error degrees of freedom: 287
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F      pValue
                 _______________    ___    ____________________    ___    ______

    Total        10.4218514727971    300    0.0347395049093237
    Model        10.4218514727971     13     0.801680882522854    Inf       0
    Residual                    0    287                     0

A = 0.000000, B = 0.100000
order Q(x) = 6.000000, order D(x) = 7.000000
```



```
List of powers for Q(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
Fitting J2(x) in range (0.000000, 30.000000)
MSS of errors squared = 3.522285e-02
Corrected MSS of errors squared = 4.981264e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting J2(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -3.17879865487033e-12    0     -Inf       0
    x1               -0.0054805464110776    0     -Inf       0
    x2                 0.041390723065588    0      Inf       0
    x3                -0.112822289148261    0     -Inf       0
    x4                 0.144806418602205    0      Inf       0
    x5                -0.0893070281790776    0     -Inf       0
    x6                 0.0214127473106266    0      Inf       0
    x7                   166.933208285091    0      Inf       0
    x8                  -1264.32518674895    0     -Inf       0
    x9                   3876.1125734658    0      Inf       0
    x10                 -6162.56168891324    0     -Inf       0
    x11                  5391.66429603207    0      Inf       0
    x12                 -2472.99787367996    0     -Inf       0
    x13                  466.174671347867    0      Inf       0


Number of observations: 301, Error degrees of freedom: 287
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ________________    ___    ____________________    ___    ______

    Total       10.421851473029    300    0.0347395049100968
    Model       10.421851473029     13     0.801680882540696    Inf       0
    Residual                  0    287                     0

A = 0.025789, B = 0.103500, C = 0.131166
```



```
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.135868, 0.176883, 0.208933, 0.236155, 0.260237, 0.282067]
List of powers for D(x): [0.135868, 0.176883, 0.208933, 0.236155, 0.260237, 0.282067,
0.302177]
Fitting J2(x) in range (0.000000, 30.000000)
MSS of errors squared = 4.928877e-02
Corrected MSS of errors squared = 8.537066e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting J2(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     7.97404915585691e-14    0      Inf       0
    x1             -0.000529018011990477    0     -Inf       0
    x2               0.00234400683903811    0      Inf       0
    x3              -0.00408375083485452    0     -Inf       0
    x4               0.00350836460105805    0      Inf       0
    x5              -0.00148975944324468    0     -Inf       0
    x6              0.000250577379113001    0      Inf       0
    x7                  5.83168528488302    0      Inf       0
    x8                 -14.5566762197012    0     -Inf       0
    x9                  20.159278688157     0      Inf       0
    x10                -16.7273294645402    0     -Inf       0
    x11                 8.31555888204158    0      Inf       0
    x12                -2.29310354856041    0     -Inf       0
    x13                0.270582838125632    0      Inf       0


Number of observations: 301, Error degrees of freedom: 287
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       10.4218514727575    300    0.0347395049091918
    Model       10.4218514727575     13    0.801680882519811     Inf       0
    Residual                   0    287                    0
```



```
A = 0.000000, B = 0.100001, C = 0.000000
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100001, 0.200001, 0.300002, 0.400002, 0.500003, 0.600003]
List of powers for D(x): [0.100001, 0.200001, 0.300002, 0.400002, 0.500003, 0.600003,
0.700004]
Fitting J2(x) in range (0.000000, 30.000000)
MSS of errors squared = 4.153023e-01
Corrected MSS of errors squared = 7.193247e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting J2(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -0.000137570758536226    0     -Inf       0
    x1              4.12348359114004e-06    0      Inf       0
    x2             -2.77526581565323e-05    0     -Inf       0
    x3              7.90708678110265e-05    0      Inf       0
    x4             -0.000128835063798885    0     -Inf       0
    x5              0.000171376483218569    0      Inf       0
    x6              3.95911577228762e-05    0      Inf       0
    x7               0.00299868426808512    0      Inf       0
    x8              -0.0278310887150271     0     -Inf       0
    x9                0.116672947074024     0      Inf       0
    x10              -0.307405071773208     0     -Inf       0
    x11               0.772502968200166     0      Inf       0
    x12               0.471901050307489     0      Inf       0
    x13              -0.0288395191562172    0     -Inf       0


Number of observations: 300, Error degrees of freedom: 286
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ____________________    ___    ______

    Total       10.4204484729586    299    0.0348509982373198
    Model       10.4204484729586     13     0.801572959458355    Inf       0
    Residual                   0    286                     0
```



```
A = 0.317494, B = 0.144719, C = 0.246086
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.462213, 0.360845, 0.259478, 0.158111, 0.056743, -0.044624]
List of powers for D(x): [0.462213, 0.360845, 0.259478, 0.158111, 0.056743, -0.044624,
-0.145991]
Fitting J2(x) in range (0.000000, 30.000000)
MSS of errors squared = NaN
Corrected MSS of errors squared = NaN
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting J2(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     7.26155585124214e-13    0      Inf       0
    x1               -0.0012710330186426    0     -Inf       0
    x2                0.00651026297661967   0      Inf       0
    x3               -0.0137140446845646    0     -Inf       0
    x4                0.0152428894481205    0      Inf       0
    x5               -0.00944464259071733   0     -Inf       0
    x6                0.00309733476259722   0      Inf       0
    x7               -0.000420476701891743  0     -Inf       0
    x8                     5.85426356939372  0      Inf       0
    x9                    -14.6673676485813  0     -Inf       0
    x10                    20.3833417904609  0      Inf       0
    x11                   -16.9682399387734  0     -Inf       0
    x12                    8.46109667224148  0      Inf       0
    x13                   -2.33996132097703  0     -Inf       0
    x14                    0.27686448371625  0      Inf       0


Number of observations: 301, Error degrees of freedom: 286
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                 _______________    ___    ____________________    ___    ______

    Total        10.4218514727575   300    0.0347395049091915
```



```
    Model       10.4218514727575     14     0.744417962339818    Inf       0
    Residual                   0    286                     0

A = 0.000012, B = 0.100000, C = 0.000014
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100012, 0.200026, 0.300032, 0.400036, 0.500040, 0.600043,
0.700046]
List of powers for D(x): [0.100012, 0.200026, 0.300032, 0.400036, 0.500040, 0.600043,
0.700046]
Fitting J2(x) in range (0.000000, 30.000000)
MSS of errors squared = 9.200833e-02
Corrected MSS of errors squared = 1.593631e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting J2(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -1.60415231923659e-12    0     -Inf       0
    x1             -9.20441456749919e-05    0     -Inf       0
    x2              0.000525697587515834    0      Inf       0
    x3             -0.00102929867301992     0     -Inf       0
    x4              0.000849682053761697    0      Inf       0
    x5             -0.000254030090882537    0     -Inf       0
    x6                   67.298365974012    0      Inf       0
    x7                  -491.44186071745    0     -Inf       0
    x8                  1492.83617303204    0      Inf       0
    x9                 -2367.42937439713    0     -Inf       0
    x10                 2071.20077039495    0      Inf       0
    x11                -951.038962242184    0     -Inf       0
    x12                 179.574887900068    0      Inf       0


Number of observations: 301, Error degrees of freedom: 288
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  ____________    ___    ____________________    ___    ______
```



```
    Total       10.4218514727559    300    0.0347395049091862
    Model       10.4218514727559     12     0.868487622729656    Inf       0
    Residual                   0    288                     0

A = 0.000000, B = 0.100000, C = 0.000000
order Q(x) = 5.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
Fitting J2(x) in range (0.000000, 30.000000)
MSS of errors squared = 3.580406e-01
Corrected MSS of errors squared = 6.201446e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting J2(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -1.60415231923659e-12    0     -Inf       0
    x1             -9.20441456749919e-05    0     -Inf       0
    x2              0.000525697587515834    0      Inf       0
    x3             -0.00102929867301992     0     -Inf       0
    x4              0.000849682053761697    0      Inf       0
    x5             -0.000254030090882537    0     -Inf       0
    x6                   67.298365974012    0      Inf       0
    x7                  -491.44186071745    0     -Inf       0
    x8                   1492.83617303204    0      Inf       0
    x9                  -2367.42937439713    0     -Inf       0
    x10                  2071.20077039495    0      Inf       0
    x11                 -951.038962242184    0     -Inf       0
    x12                  179.574887900068    0      Inf       0


Number of observations: 301, Error degrees of freedom: 288
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ____________    ____    ____________________    ___    ______
```

Total	10.4218514727559	300	0.0347395049091862		
Model	10.4218514727559	12	0.868487622729656	Inf	0
Residual	0	288	0		

```
A = 0.000000, B = 0.100000, C = 0.000000
order Q(x) = 5.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
Fitting J2(x) in range (0.000000, 30.000000)
MSS of errors squared = 3.580406e-01
Corrected MSS of errors squared = 6.201446e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting J2(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9

Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	-2.01973318952329e-15	0	-Inf	0
x1	-1.07718302351161e-07	0	-Inf	0
x2	1.43110178533457e-07	0	Inf	0
x3	-3.89633054116047e-08	0	-Inf	0
x4	4.81708114815555e-09	0	Inf	0
x5	1.01466518285469	0	Inf	0
x6	-0.0152212926909397	0	-Inf	0
x7	0.000592403281997427	0	Inf	0
x8	-3.77721829472083e-05	0	-Inf	0
x9	1.46668021851484e-06	0	Inf	0

```
Number of observations: 301, Error degrees of freedom: 291
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```

	SumSq	DF	MeanSq	F	pValue
Total	10.4218514727574	300	0.0347395049091912		
Model	10.4218514727574	9	1.15798349697304	Inf	0
Residual	0	291	0		


```
A = 0.000278, B = 2.810089
order Q(x) = 4.000000, order D(x) = 5.000000
List of powers for Q(x): [0.000278, 0.023354, 0.145903, 0.369493]
List of powers for D(x): [0.000278, 0.023354, 0.145903, 0.369493, 0.671018]
Fitting J2(x) in range (0.000000, 30.000000)
MSS of errors squared = 4.046963e-02
Corrected MSS of errors squared = 5.723270e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the Bessel $J_2(x)$ function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i)	0	0.1		Inf	-Inf	3.52228500E-02	6, 7
A+B*log10(i)^4	0	2.81		Inf	-Inf	4.04696300E-02	4, 5
A+B*sqrt(i+C)	0.025	0.103	0.131	Inf	-Inf	4.92887700E-02	6, 7
A+B*i+C*sqrt(i-1)	0	0.1	0	Inf	-Inf	9.20083300E-02	7, 7
A+B/i	0.006	0.475		Inf	-Inf	1.08366300E-01	4, 7
A+B*sqrt(i)+C*(i-1)	0	0.1	0	Inf	-Inf	3.58040600E-01	5, 7
A+B*sqrt(i)+C*sqrt(i-1)	0	0.1	0	Inf	-Inf	3.58040600E-01	5, 7
A+B*i	0	0.1		1.80E+16	-1.04E+04	4.12667100E-01	6, 7
A+B*i+C*(i-1)	0	0.1	0	Inf	-Inf	4.15302300E-01	6, 7
A+B*i-C*(i-1)	0.317	0.144	0.246	Inf	-Inf	NaN	6, 7

RESULTS FOR THE BESSEL J3(X)

Using Power A+B* i

```
Fitting J3(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     9.02259760435892e-14    0       Inf        0
    x1              0.000742741912171055    0       Inf        0
    x2              -0.00410938322357813    0      -Inf        0
    x3               0.00936041689539658    0       Inf        0
    x4               -0.0112411054452086    0      -Inf        0
    x5               0.00751193021918971    0       Inf        0
    x6              -0.00265058782196768    0      -Inf        0
    x7              0.000386111056671235    0       Inf        0
    x8                  5.80440169605711    0       Inf        0
    x9                 -14.2449555976457    0      -Inf        0
    x10                 19.3120309767446    0       Inf        0
    x11                -15.6739426877951    0      -Inf        0
    x12                 7.62084271944232    0       Inf        0
    x13                -2.05574211064639    0      -Inf        0
    x14                0.237358400249723    0       Inf        0


Number of observations: 301, Error degrees of freedom: 286
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F      pValue
                ________________    ___    ____________________    ___    ______
```



```
    Total       9.18626714756054    300    0.0306208904918685
    Model       9.18626714756054     14     0.656161939111467    Inf       0
    Residual                   0    286                     0

A = 0.001412, B = 0.100000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.101412, 0.201412, 0.301412, 0.401412, 0.501412, 0.601412,
0.701412]
List of powers for D(x): [0.101412, 0.201412, 0.301412, 0.401412, 0.501412, 0.601412,
0.701412]
Fitting J3(x) in range (0.000000, 30.000000)
MSS of errors squared = 2.176376e-01
Corrected MSS of errors squared = 3.077860e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting J3(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     9.39059030150913e-12    0      Inf       0
    x1              1.20738230642461e-09    0      Inf       0
    x2             -7.54796004921315e-09    0     -Inf       0
    x3              6.55122228302082e-09    0      Inf       0
    x4                0.0268182373742176    0      Inf       0
    x5                 -12.1301190864753    0     -Inf       0
    x6                  335.888956901618    0      Inf       0
    x7                 -2358.33453603815    0     -Inf       0
    x8                  6269.44233097965    0      Inf       0
    x9                 -6954.54833931053    0     -Inf       0
    x10                 2720.65488830629    0      Inf       0


Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______

    Total       9.18626714747713    300    0.0306208904915904
```



```
    Model       9.18626714747713     10     0.918626714747713    Inf       0
    Residual                   0    290                     0

A = 0.066579, B = 0.338257
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.404836, 0.235708, 0.179331]
List of powers for D(x): [0.404836, 0.235708, 0.179331, 0.151143, 0.134230, 0.122955,
0.114901]
Fitting J3(x) in range (0.000000, 30.000000)
MSS of errors squared = Inf
Corrected MSS of errors squared = Inf
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting J3(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -2.09892073556909e-12    0     -Inf       0
    x1              2.91398333251731e-08    0      Inf       0
    x2             -6.03545597509154e-08    0     -Inf       0
    x3              3.14655141993133e-08    0      Inf       0
    x4                  66.2620651800218    0      Inf       0
    x5                  -480.70537327249    0     -Inf       0
    x6                  1452.80654294098    0      Inf       0
    x7                 -2294.00920370597    0     -Inf       0
    x8                  1999.29432140965    0      Inf       0
    x9                 -914.832688715781    0     -Inf       0
    x10                 172.184336152511    0      Inf       0


Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______

    Total       9.18626714754912     300    0.0306208904918304
    Model       9.18626714754912      10     0.918626714754912    Inf       0
    Residual                   0     290                     0
```



```
A = 0.000000, B = 0.100000
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.141421, 0.173205]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
Fitting J3(x) in range (0.000000, 30.000000)
MSS of errors squared = 5.338281e-02
Corrected MSS of errors squared = 7.549470e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting J3(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -1.65555140078255e-11    0     -Inf       0
    x1             -0.000394424006953832    0     -Inf       0
    x2                0.0018378678375255    0      Inf       0
    x3               -0.00314403159932709    0     -Inf       0
    x4                0.00235237637228296    0      Inf       0
    x5             -0.000651783487046811    0     -Inf       0
    x6                  1432.75267911058    0      Inf       0
    x7                 -9532.26196802774    0     -Inf       0
    x8                  26313.3968481274    0      Inf       0
    x9                 -38501.383238511     0     -Inf       0
    x10                 31490.9979531363    0      Inf       0
    x11                -13657.6394387907    0     -Inf       0
    x12                 2455.13716467259    0      Inf       0


Number of observations: 301, Error degrees of freedom: 288
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       9.18626714705185     300    0.0306208904901728
    Model       9.18626714705185      12      0.76552226225432     Inf       0
    Residual                   0     288                     0
```



```
A = 0.000000, B = 0.100000, C = 2.000000
order Q(x) = 5.000000, order D(x) = 7.000000
List of powers for Q(x): [0.173205, 0.200000, 0.223607, 0.244949, 0.264575]
List of powers for D(x): [0.173205, 0.200000, 0.223607, 0.244949, 0.264575, 0.282843,
0.300000]
Fitting J3(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.711075e-01
Corrected MSS of errors squared = 2.963669e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting J3(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE     tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    1.13262118154776e-13    0      Inf       0
    x1             0.000688035625759358    0      Inf       0
    x2             -0.00380164430503733    0     -Inf       0
    x3              0.00864233859384267    0      Inf       0
    x4              -0.0103518526771913    0     -Inf       0
    x5              0.00689589292332343    0      Inf       0
    x6             -0.00242435476914633    0     -Inf       0
    x7             0.000351716898992408    0      Inf       0
    x8                 5.47095221709184    0      Inf       0
    x9                -13.0957738458331    0     -Inf       0
    x10                17.543751809922    0      Inf       0
    x11               -14.1152548730606    0     -Inf       0
    x12                  6.811932110963    0      Inf       0
    x13               -1.82494674357546    0     -Inf       0
    x14                0.20933225479701    0      Inf       0


Number of observations: 301, Error degrees of freedom: 286
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                 _______________    ___    ____________________    ___    ______

    Total        9.18626714756065    300    0.0306208904918688
```


Model	9.18626714756065	14	0.656161939111475	Inf	0
Residual	0	286	0		

```
A = 0.000000, B = 0.100000, C = 0.002536
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.202536, 0.305072, 0.407608, 0.510144, 0.612680,
0.715216]
List of powers for D(x): [0.100000, 0.202536, 0.305072, 0.407608, 0.510144, 0.612680,
0.715216]
Fitting J3(x) in range (0.000000, 30.000000)
MSS of errors squared = 4.797372e-02
Corrected MSS of errors squared = 8.309293e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting J3(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	-3.15713249869236e-08	0	-Inf	0
x1	-4.70330297513831e-08	0	-Inf	0
x2	2.25411543448997e-07	0	Inf	0
x3	-3.76951064407266e-07	0	-Inf	0
x4	2.31570501221898e-07	0	Inf	0
x5	0.000714561360749354	0	Inf	0
x6	-0.00713156021681442	0	-Inf	0
x7	0.0316425545488547	0	Inf	0
x8	-0.0846807324430776	0	-Inf	0
x9	0.175289635405377	0	Inf	0
x10	0.900765662342957	0	Inf	0
x11	-0.0166001989003393	0	-Inf	0

```
Number of observations: 300, Error degrees of freedom: 288
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```

	SumSq	DF	MeanSq	F	pValue
Total	9.18529698161764	299	0.0307200567947078		


```
    Model       9.18529698161764     11     0.835026998328877    Inf       0
    Residual                   0    288                     0

A = 0.586223, B = 0.101834, C = 0.242040
order Q(x) = 4.000000, order D(x) = 7.000000
List of powers for Q(x): [0.688057, 0.547852, 0.407647, 0.267441]
List of powers for D(x): [0.688057, 0.547852, 0.407647, 0.267441, 0.127236, -0.012969,
-0.153174]
Fitting J3(x) in range (0.000000, 30.000000)
MSS of errors squared = NaN
Corrected MSS of errors squared = NaN
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting J3(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    1.00431027045248e-13     0      Inf       0
    x1             0.000695834811079659     0      Inf       0
    x2             -0.00393754735241566     0     -Inf       0
    x3              0.00906590974203387     0      Inf       0
    x4              -0.0109473603192749     0     -Inf       0
    x5               0.0073405457977715     0      Inf       0
    x6             -0.00259601364023097     0     -Inf       0
    x7             0.000378758014911619     0      Inf       0
    x8                 5.52868515717009     0      Inf       0
    x9                -13.5493121349547     0     -Inf       0
    x10                18.5117787891378     0      Inf       0
    x11               -15.0908601036211     0     -Inf       0
    x12                7.3588871006995     0      Inf       0
    x13               -1.98921426386303     0     -Inf       0
    x14                0.23002866633932     0      Inf       0


Number of observations: 301, Error degrees of freedom: 286
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ____________    ___    ____________________    ___    ______
```



```
    Total       9.18626714756082    300    0.0306208904918694
    Model       9.18626714756082     14     0.656161939111487    Inf       0
    Residual                   0    286                     0

A = 0.000000, B = 0.100000, C = 0.002658
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.202658, 0.303759, 0.404604, 0.505316, 0.605944,
0.706511]
List of powers for D(x): [0.100000, 0.202658, 0.303759, 0.404604, 0.505316, 0.605944,
0.706511]
Fitting J3(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.418638e-01
Corrected MSS of errors squared = 2.457152e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting J3(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     -3.0974597171372e-12    0     -Inf       0
    x1               4.50898626594081e-08    0      Inf       0
    x2              -8.77864369010372e-08    0     -Inf       0
    x3               4.33194696900064e-08    0      Inf       0
    x4                   30.4990797990681    0      Inf       0
    x5                  -171.538478181491    0     -Inf       0
    x6                   430.559285411992    0      Inf       0
    x7                  -582.774403816947    0     -Inf       0
    x8                   443.776093757623    0      Inf       0
    x9                  -179.767154204931    0     -Inf       0
    x10                  30.2455772051891    0      Inf       0


Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______
```


Total	9.18626714754948	300	0.0306208904918316		
Model	9.18626714754948	10	0.918626714754949	Inf	0
Residual	0	290	0		

```
A = 0.000000, B = 0.100000, C = 0.010666
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.152087, 0.194537]
List of powers for D(x): [0.100000, 0.152087, 0.194537, 0.231998, 0.266271, 0.298280,
0.328572]
Fitting J3(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.580726e-01
Corrected MSS of errors squared = 2.737898e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting J3(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	-2.09892073556909e-12	0	-Inf	0
x1	2.91398333251731e-08	0	Inf	0
x2	-6.03545597509154e-08	0	-Inf	0
x3	3.14655141993133e-08	0	Inf	0
x4	66.2620651800218	0	Inf	0
x5	-480.70537327249	0	-Inf	0
x6	1452.80654294098	0	Inf	0
x7	-2294.00920370597	0	-Inf	0
x8	1999.29432140965	0	Inf	0
x9	-914.832688715781	0	-Inf	0
x10	172.184336152511	0	Inf	0

```
Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```

	SumSq	DF	MeanSq	F	pValue
Total	9.18626714754912	300	0.0306208904918304		
Model	9.18626714754912	10	0.918626714754912	Inf	0


```
    Residual                  0    290                    0

A = 0.000000, B = 0.100000, C = 0.000000
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.141421, 0.173205]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
Fitting J3(x) in range (0.000000, 30.000000)
MSS of errors squared = 5.338281e-02
Corrected MSS of errors squared = 9.246174e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting J3(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                      ____________________    __    _____    ______

    (Intercept)    -8.55832891200296e-17    0     -Inf        0
    x1                                  0    0      NaN       NaN
    x2               6.3696023857332e-15    0      Inf        0
    x3              -6.20919395239003e-15    0     -Inf        0
    x4                  0.999999975232782    0      Inf        0
    x5               3.13195554867027e-08    0      Inf        0
    x6              -7.58816855126443e-09    0     -Inf        0
    x7               1.03582696892288e-09    0      Inf        0


Number of observations: 301, Error degrees of freedom: 294
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______

    Total       9.18626714756074     300    0.0306208904918691
    Model       9.18626714756074       6      1.53104452459346    Inf       0
    Residual                   0     294                     0

A = 0.000000, B = 0.100000
order Q(x) = 3.000000, order D(x) = 4.000000
List of powers for Q(x): [0.000000, 0.000821, 0.005182]
```



```
List of powers for D(x): [0.000000, 0.000821, 0.005182, 0.013139]
Fitting J3(x) in range (0.000000, 30.000000)
MSS of errors squared = 2.329207e-02
Corrected MSS of errors squared = 3.293996e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the Bessel $J_3(x)$ function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*log10(i)^4	0	0.1		Inf	-Inf	2.32920700E-02	3, 4
A+B*i+C*(i-1)	0	0.1	0.002	Inf	-Inf	4.79737200E-02	7, 7
A+B*sqrt(i)	0	0.1		Inf	-Inf	5.33828100E-02	3, 7
A+B*sqrt(i)+C*sqrt(i-1)	0	0.1	0	Inf	-Inf	5.33828100E-02	3, 7
A+B*i+C*sqrt(i-1)	0	0.1	0.002	Inf	-Inf	1.41863800E-01	7, 7
A+B*sqrt(i)+C*(i-1)	0	0.1	0.01	Inf	-Inf	1.58072600E-01	3, 7
A+B*sqrt(i+C)	0	0.1	2	Inf	-Inf	1.71107500E-01	5, 7
A+B*i	0.001	0.1		Inf	-Inf	2.17637600E-01	7, 7
A+B/i	0.066	0.338		Inf	-Inf	Inf	3, 7
A+B*i-C*(i-1)	0.586	0.101	0.242	Inf	-Inf	NaN	4, 7

RESULTS FOR THE BESSEL J4(X)

Using Power A+B* i

```
Fitting J4(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -3.18417283075703e-10    0     -Inf       0
    x1              6.97515402348515e-08    0      Inf       0
    x2             -1.10164268024451e-07    0     -Inf       0
    x3              4.3554514879354e-08     0      Inf       0
    x4                 5.70773953488567     0      Inf       0
    x5                -13.9419483754961     0     -Inf       0
    x6                 18.8922997826091     0      Inf       0
    x7                -15.3382162828327     0     -Inf       0
    x8                 7.46099486817145     0      Inf       0
    x9                -2.01340066056683     0     -Inf       0
    x10                0.232527728741069    0      Inf       0


Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq           F      pValue
                ______________    ___    ________________    ___    ______

    Total       8.389774640655    300    0.02796591546885
    Model       8.389774640655     10     0.8389774640655    Inf       0
    Residual                 0    290                   0
```



```
A = 0.000000, B = 0.100000
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.200000, 0.300000]
List of powers for D(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
Fitting J4(x) in range (0.000000, 30.000000)
MSS of errors squared = 3.025319e-01
Corrected MSS of errors squared = 4.278447e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting J4(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     -4.7556315412939e-12    0     -Inf        0
    x1               6.82421265393356e-07   0      Inf        0
    x2              -8.79057198029283e-06   0     -Inf        0
    x3               2.02206374173058e-05   0      Inf        0
    x4              -1.21119425724022e-05   0     -Inf        0
    x5                  -8.86615391140583   0     -Inf        0
    x6                   1029.24659177579   0      Inf        0
    x7                  -11494.3620039144   0     -Inf        0
    x8                   35256.1732430132   0      Inf        0
    x9                  -34108.8493083634   0     -Inf        0
    x10                                 0   0      NaN      NaN
    x11                  9327.65763111168   0      Inf        0


Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq           DF           MeanSq            F     pValue
                ____________________    ___    ____________________    ___    ______

    Total       8.38977464078134    300    0.0279659154692711
    Model       8.38977464078134     10     0.838977464078134    Inf       0
    Residual                   0    290                      0
```



```
A = 0.090090, B = 0.100000
order Q(x) = 4.000000, order D(x) = 7.000000
List of powers for Q(x): [0.190090, 0.140090, 0.123423, 0.115090]
List of powers for D(x): [0.190090, 0.140090, 0.123423, 0.115090, 0.110090, 0.106757,
0.104376]
Fitting J4(x) in range (0.000000, 30.000000)
MSS of errors squared = Inf
Corrected MSS of errors squared = Inf
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting J4(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                         Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -6.67154760913807e-11    0     -Inf       0
    x1              5.02737068451741e-08    0      Inf       0
    x2             -1.02802060933903e-07    0     -Inf       0
    x3              5.30949094067685e-08    0      Inf       0
    x4                   112.638331024621    0      Inf       0
    x5                  -838.908164573384    0     -Inf       0
    x6                   2542.3242961491    0      Inf       0
    x7                  -3998.83190757983    0     -Inf       0
    x8                   3462.61990891223    0      Inf       0
    x9                  -1572.38415434953    0     -Inf       0
    x10                  293.541689820365    0      Inf       0


Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F     pValue
                   _______________    ___    ____________________    ___    ______

    Total       8.38977464061351     300    0.0279659154687117
    Model       8.38977464061351      10     0.838977464061351    Inf       0
    Residual                   0     290                     0

A = 0.029206, B = 0.125342
order Q(x) = 3.000000, order D(x) = 7.000000
```



```
List of powers for Q(x): [0.154548, 0.206466, 0.246304]
List of powers for D(x): [0.154548, 0.206466, 0.246304, 0.279889, 0.309479, 0.336229,
0.360829]
Fitting J4(x) in range (0.000000, 30.000000)
MSS of errors squared = 3.975816e-02
Corrected MSS of errors squared = 5.622653e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting J4(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -1.00447363920038e-11    0     -Inf       0
    x1              -3.7263520155494e-06    0     -Inf       0
    x2              1.38137606614031e-05    0      Inf       0
    x3             -1.63239997788792e-05    0     -Inf       0
    x4              6.23913867544435e-06    0      Inf       0
    x5                  707.567873766394    0      Inf       0
    x6                 -5734.99632282659    0     -Inf       0
    x7                  18193.4840314558    0      Inf       0
    x8                 -29545.4993585566    0     -Inf       0
    x9                  26228.3486909903    0      Inf       0
    x10                -12158.2499288339    0     -Inf       0
    x11                 2310.34501243233    0      Inf       0


Number of observations: 301, Error degrees of freedom: 289
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______

    Total       8.38977464092802    300    0.0279659154697601
    Model       8.38977464092802     11     0.762706785538911    Inf       0
    Residual                   0    289                     0

A = 0.092548, B = 0.100000, C = 0.165698
order Q(x) = 4.000000, order D(x) = 7.000000
List of powers for Q(x): [0.200516, 0.239711, 0.270472, 0.296649]
```



```
List of powers for D(x): [0.200516, 0.239711, 0.270472, 0.296649, 0.319830, 0.340857,
0.360237]
Fitting J4(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.896505e-01
Corrected MSS of errors squared = 3.284843e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting J4(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -3.18417283075703e-10    0     -Inf       0
    x1              6.97515402348515e-08    0      Inf       0
    x2             -1.10164268024451e-07    0     -Inf       0
    x3              4.3554514879354e-08     0      Inf       0
    x4                 5.70773953488567     0      Inf       0
    x5                -13.9419483754961     0     -Inf       0
    x6                 18.8922997826091     0      Inf       0
    x7                -15.3382162828327     0     -Inf       0
    x8                 7.46099486817145     0      Inf       0
    x9                -2.01340066056683     0     -Inf       0
    x10               0.232527728741069     0      Inf       0


Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq           F     pValue
                ______________    ___    ________________    ___    ______

    Total       8.389774640655    300    0.02796591546885
    Model       8.389774640655     10     0.8389774640655    Inf       0
    Residual                 0    290                   0

A = 0.000000, B = 0.100000, C = 0.000000
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.200000, 0.300000]
List of powers for D(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
```



```
Fitting J4(x) in range (0.000000, 30.000000)
MSS of errors squared = 3.025319e-01
Corrected MSS of errors squared = 5.240006e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting J4(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE     tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     3.52642642038e-14    0      Inf        0
    x1             -0.020364014538238    0     -Inf        0
    x2              0.123065382778226    0      Inf        0
    x3             -0.309757858727234    0     -Inf        0
    x4              0.415654316457038    0      Inf        0
    x5             -0.313609601473785    0     -Inf        0
    x6              0.126144865307587    0      Inf        0
    x7             -0.021133089683167    0     -Inf        0
    x8              -6.39226944276158    0     -Inf        0
    x9               34.1175991028068    0      Inf        0
    x10             -68.7739398315677    0     -Inf        0
    x11              56.9413101981822    0      Inf        0
    x12                             0    0      NaN      NaN
    x13             -30.1449840439306    0     -Inf        0
    x14              15.2522839696052    0      Inf        0


Number of observations: 301, Error degrees of freedom: 287
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______

    Total       8.38977464065438     300    0.0279659154688479
    Model       8.38977464065438      13     0.645367280050337     Inf       0
    Residual                   0     287                     0

A = 0.000000, B = 0.149116, C = 0.167340
order Q(x) = 7.000000, order D(x) = 7.000000
```



```
List of powers for Q(x): [0.149116, 0.130892, 0.112668, 0.094444, 0.076220, 0.057996,
0.039773]
List of powers for D(x): [0.149116, 0.130892, 0.112668, 0.094444, 0.076220, 0.057996,
0.039773]
Fitting J4(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.431851e-01
Corrected MSS of errors squared = 2.480039e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting J4(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -3.18417283075703e-10    0     -Inf       0
    x1              6.97515402348515e-08    0      Inf       0
    x2             -1.10164268024451e-07    0     -Inf       0
    x3              4.3554514879354e-08     0      Inf       0
    x4                 5.70773953488567     0      Inf       0
    x5                -13.9419483754961     0     -Inf       0
    x6                 18.8922997826091     0      Inf       0
    x7                -15.3382162828327     0     -Inf       0
    x8                 7.46099486817145     0      Inf       0
    x9                -2.01340066056683     0     -Inf       0
    x10               0.232527728741069     0      Inf       0


Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq           F     pValue
                ________________    ___    ________________    ___    ______

    Total       8.389774640655     300    0.02796591546885
    Model       8.389774640655      10     0.8389774640655    Inf       0
    Residual                 0     290                   0

A = 0.000000, B = 0.100000, C = 0.000000
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.200000, 0.300000]
```



```
List of powers for D(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
Fitting J4(x) in range (0.000000, 30.000000)
MSS of errors squared = 3.025319e-01
Corrected MSS of errors squared = 5.240006e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting J4(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    8.94917773438693e-12     0      Inf       0
    x1               0.0615053990326698     0      Inf       0
    x2               -0.652151965738136     0     -Inf       0
    x3                 2.47341852727948     0      Inf       0
    x4                -4.6004830943785      0     -Inf       0
    x5                   4.55998106149      0      Inf       0
    x6               -2.32045826960268      0     -Inf       0
    x7                0.47818835082762      0      Inf       0
    x8                 731.21812534758      0      Inf       0
    x9               -6178.52816631118      0     -Inf       0
    x10               20075.6579779845      0      Inf       0
    x11              -33117.4407818038      0     -Inf       0
    x12               29728.0874706793      0      Inf       0
    x13              -13895.4393604336      0     -Inf       0
    x14               2657.44473064056      0      Inf       0


Number of observations: 301, Error degrees of freedom: 286
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F      pValue
                 _______________    ___    ____________________    ___    ______

    Total        8.38977464062877    300    0.0279659154687626
    Model        8.38977464062877     14     0.599269617187769    Inf       0
    Residual                    0    286                     0

A = 0.118904, B = 0.104654, C = 0.000000
```



```
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.223558, 0.266907, 0.300170, 0.328212, 0.352917, 0.375253,
0.395792]
List of powers for D(x): [0.223558, 0.266907, 0.300170, 0.328212, 0.352917, 0.375253,
0.395792]
Fitting J4(x) in range (0.000000, 30.000000)
MSS of errors squared = 3.034207e-02
Corrected MSS of errors squared = 5.255401e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting J4(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -9.18201727640616e-13    0     -Inf       0
    x1              -1.5600265522576e-06    0     -Inf       0
    x2              8.52514747483833e-06    0      Inf       0
    x3             -1.61403837134019e-05    0     -Inf       0
    x4              1.29527428120118e-05    0      Inf       0
    x5             -3.77734190309836e-06    0     -Inf       0
    x6                  64.6212322589909    0      Inf       0
    x7                 -464.046800247112    0     -Inf       0
    x8                  1391.63746035815    0      Inf       0
    x9                 -2183.22826781393    0     -Inf       0
    x10                 1891.9859459908    0      Inf       0
    x11                -861.330218512147    0     -Inf       0
    x12                 161.360647884541    0      Inf       0


Number of observations: 301, Error degrees of freedom: 288
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       8.38977464063113    300    0.0279659154687704
    Model       8.38977464063113     12     0.699147886719261    Inf       0
    Residual                   0    288                     0
```



```
A = 0.000000, B = 0.100000, C = 0.000000
order Q(x) = 5.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
Fitting J4(x) in range (0.000000, 30.000000)
MSS of errors squared = 7.049163e-02
Corrected MSS of errors squared = 1.220951e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting J4(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                      ____________________    __    _____    ______

    (Intercept)     1.50058561819535e-16    0      Inf        0
    x1                                 0    0      NaN      NaN
    x2             -6.25662124080979e-08    0     -Inf        0
    x3              1.48175725854924e-07    0      Inf        0
    x4             -1.4654629527914e-07     0     -Inf        0
    x5              8.44660776581081e-08    0      Inf        0
    x6             -2.74025197510622e-08    0     -Inf        0
    x7              3.87322383472096e-09    0      Inf        0
    x8                 0.999994427400859    0      Inf        0
    x9              7.30244480779931e-06    0      Inf        0
    x10            -2.18930972615988e-06    0     -Inf        0
    x11             5.23583959823262e-07    0      Inf        0
    x12            -6.41199032827459e-08    0     -Inf        0


Number of observations: 301, Error degrees of freedom: 289
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       8.38977464065495     300    0.0279659154688498
    Model       8.38977464065495      11     0.762706785514087    Inf       0
    Residual                   0     289                     0
```



```
A = 0.000000, B = 0.100000
order Q(x) = 7.000000, order D(x) = 5.000000
List of powers for Q(x): [0.000000, 0.000821, 0.005182, 0.013139, 0.023869, 0.036665,
0.051007]
List of powers for D(x): [0.000000, 0.000821, 0.005182, 0.013139, 0.023869]
Fitting J4(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.218818e-02
Corrected MSS of errors squared = 1.723669e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the Bessel $J_4(x)$ function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*log10(i)^4	0	0.1		Inf	-Inf	1.21881800E-02	7, 5
A+B*sqrt(i)+C*(i-1)	0.118	0.104	0	Inf	-Inf	3.03420700E-02	7, 7
A+B*sqrt(i)	0.029	0.125		Inf	-Inf	3.97581600E-02	3, 7
A+B*sqrt(i)+C*sqrt(i-1)	0	0.1	0	Inf	-Inf	7.04916300E-02	5, 7
A+B*i-C*(i-1)	0	0.149	0.167	Inf	-Inf	1.43185100E-01	7, 7
A+B*sqrt(i+C)	0.092	0.1	0.165	Inf	-Inf	1.89650500E-01	4, 7
A+B*i	0	0.1		Inf	-Inf	3.02531900E-01	3, 7
A+B*i+C*(i-1)	0	0.1	0	Inf	-Inf	3.02531900E-01	3, 7
A+B*i+C*sqrt(i-1)	0	0.1	0	Inf	-Inf	3.02531900E-01	3, 7
A+B/i	0.09	0.1		Inf	-Inf	Inf	4, 7

RESULTS FOR THE BESSEL J5(X)

Using Power A+B* i

```
Fitting J5(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     -1.6172442540745e-11    0     -Inf       0
    x1              -8.14637836827588e-07   0     -Inf       0
    x2               2.20300672799594e-06   0      Inf       0
    x3              -1.93429354004425e-06   0     -Inf       0
    x4               5.54993694823536e-07   0      Inf       0
    x5                    12.3890709260405   0      Inf       0
    x6                   -36.9133408862103   0     -Inf       0
    x7                    53.3822090677525   0      Inf       0
    x8                   -44.4693328004486   0     -Inf       0
    x9                    21.8261741101937   0      Inf       0
    x10                   -5.8922648557002   0     -Inf       0
    x11                  0.677449225307313   0      Inf       0


Number of observations: 301, Error degrees of freedom: 289
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF            MeanSq            F      pValue
                ___________________    ___    ____________________    ___    ______

    Total       7.51808538914194       300    0.0250602846304731
    Model       7.51808538914194        11     0.683462308103812     Inf       0
```



```
    Residual                   0    289                    0

A = 0.065000, B = 0.100000
order Q(x) = 4.000000, order D(x) = 7.000000
List of powers for Q(x): [0.165000, 0.265000, 0.365000, 0.465000]
List of powers for D(x): [0.165000, 0.265000, 0.365000, 0.465000, 0.565000, 0.665000,
0.765000]
Fitting J5(x) in range (0.000000, 30.000000)
MSS of errors squared = 5.599545e-02
Corrected MSS of errors squared = 7.918952e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting J5(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                      ____________________    __    _____    ______

    (Intercept)     -7.7659705378933e-10     0     -Inf       0
    x1               1.23195487462083e-09    0      Inf       0
    x2              -1.27195915979383e-08    0     -Inf       0
    x3               1.33551429445358e-08    0      Inf       0
    x4                0.00698350724261756    0      Inf       0
    x5                  -4.95009077723238    0     -Inf       0
    x6                   160.045583605004    0      Inf       0
    x7                  -1218.02148481256    0     -Inf       0
    x8                    3404.6940630437    0      Inf       0
    x9                   -3909.7537681462    0     -Inf       0
    x10                  1568.97870746077    0      Inf       0


Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F     pValue
                  _______________    ___    ____________________    ___    ______

    Total       7.51808538912819     300    0.0250602846304273
    Model       7.51808538912819      10     0.751808538912819    Inf       0
    Residual                   0     290                     0
```



```
A = 0.113248, B = 0.658658
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.771905, 0.442577, 0.332800]
List of powers for D(x): [0.771905, 0.442577, 0.332800, 0.277912, 0.244979, 0.223024,
0.207342]
Fitting J5(x) in range (0.000000, 30.000000)
MSS of errors squared = 7.625698e-02
Corrected MSS of errors squared = 1.078437e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting J5(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                         Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     2.49069500573412e-13    0      Inf       0
    x1               2.49018616169899e-05   0      Inf       0
    x2              -0.000198022727160473   0     -Inf       0
    x3               0.000557923288232532   0      Inf       0
    x4              -0.000733982976769954   0     -Inf       0
    x5               0.000461785227888016   0      Inf       0
    x6              -0.000112604616074125   0     -Inf       0
    x7                    63.3701718779798   0      Inf       0
    x8                   -451.438935127698   0     -Inf       0
    x9                    1345.60147176961   0      Inf       0
    x10                  -2100.24475708579   0     -Inf       0
    x11                   1811.93306893256   0      Inf       0
    x12                  -821.563933453096   0     -Inf       0
    x13                   153.342912888891   0      Inf       0


Number of observations: 301, Error degrees of freedom: 287
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______

    Total       7.51808538918058     300    0.0250602846306019
    Model       7.51808538918058      13     0.578314260706198     Inf       0
    Residual                   0     287                     0
```



```
A = 0.000000, B = 0.100000
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
Fitting J5(x) in range (0.000000, 30.000000)
MSS of errors squared = 3.825715e-02
Corrected MSS of errors squared = 5.410378e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting J5(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     1.25231762843682e-13    0      Inf       0
    x1               2.84290927366553e-05   0      Inf       0
    x2              -0.000225875036096159   0     -Inf       0
    x3               0.000635998187690148   0      Inf       0
    x4              -0.000836276641825181   0     -Inf       0
    x5               0.000525921611333161   0      Inf       0
    x6              -0.000128197151039333   0     -Inf       0
    x7                     66.750832260708   0      Inf       0
    x8                    -478.071378416761   0     -Inf       0
    x9                     1428.34854336341   0      Inf       0
    x10                   -2232.51435063783   0     -Inf       0
    x11                    1927.86943589237   0      Inf       0
    x12                   -874.738257293767   0     -Inf       0
    x13                    163.35517461805    0      Inf       0


Number of observations: 301, Error degrees of freedom: 287
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______

    Total       7.51808538911648    300    0.0250602846303883
    Model       7.51808538911648     13     0.578314260701267    Inf       0
```



```
    Residual                  0    287                    0

A = 0.001877, B = 0.100000, C = 0.000000
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.101877, 0.143298, 0.175082, 0.201877, 0.225483, 0.246826]
List of powers for D(x): [0.101877, 0.143298, 0.175082, 0.201877, 0.225483, 0.246826,
0.266452]
Fitting J5(x) in range (0.000000, 30.000000)
MSS of errors squared = Inf
Corrected MSS of errors squared = Inf
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting J5(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -3.78946261855095e-10    0     -Inf       0
    x1              3.01298732223585e-08    0      Inf       0
    x2             -4.4602297715871e-08     0     -Inf       0
    x3              1.66754686166304e-08    0      Inf       0
    x4                  5.48006432898285    0      Inf       0
    x5                 -12.8527207888313    0     -Inf       0
    x6                  16.7238022088741    0      Inf       0
    x7                 -13.0386320434623    0     -Inf       0
    x8                  6.09100754963117    0      Inf       0
    x9                 -1.57865019562427    0     -Inf       0
    x10                0.175114312147288    0      Inf       0


Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______

    Total       7.51808538914219     300    0.0250602846304739
    Model       7.51808538914219      10     0.751808538914219    Inf       0
    Residual                   0     290                     0
```



```
A = 0.000000, B = 0.110566, C = 0.000000
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.110566, 0.221132, 0.331697]
List of powers for D(x): [0.110566, 0.221132, 0.331697, 0.442263, 0.552829, 0.663395,
0.773960]
Fitting J5(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.469235e-02
Corrected MSS of errors squared = 2.544790e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting J5(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                         Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -4.92441465477686e-12    0     -Inf       0
    x1             -4.75543604243517e-07    0     -Inf       0
    x2              1.26181690556494e-06    0      Inf       0
    x3             -1.07072195136675e-06    0     -Inf       0
    x4              2.94124210981545e-07    0      Inf       0
    x5                   5.29156883057369    0      Inf       0
    x6                  -11.9817906129929    0     -Inf       0
    x7                   15.0492788203866    0      Inf       0
    x8                  -11.3237544763992    0     -Inf       0
    x9                   5.10442774415135    0      Inf       0
    x10                 -1.27633157857683    0     -Inf       0
    x11                 0.136564518889263    0      Inf       0


Number of observations: 301, Error degrees of freedom: 289
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                ________________    ___    ____________________    ___    ______

    Total       7.51808538914192    300    0.0250602846304731
    Model       7.51808538914192     11     0.683462308103811    Inf       0
    Residual                   0    289                     0

A = 0.000000, B = 0.124381, C = 0.000000
```



```
order Q(x) = 4.000000, order D(x) = 7.000000
List of powers for Q(x): [0.124381, 0.248763, 0.373144, 0.497526]
List of powers for D(x): [0.124381, 0.248763, 0.373144, 0.497526, 0.621907, 0.746288,
0.870670]
Fitting J5(x) in range (0.000000, 30.000000)
MSS of errors squared = 3.234307e-01
Corrected MSS of errors squared = 5.601983e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting J5(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     -3.1127415441016e-09    0     -Inf       0
    x1               1.43919907628225e-07   0      Inf       0
    x2              -2.13776352978843e-07   0     -Inf       0
    x3               8.04418443236719e-08   0      Inf       0
    x4                   14.7906295600666   0      Inf       0
    x5                  -45.4691034405816   0     -Inf       0
    x6                   66.5139899155777   0      Inf       0
    x7                   -55.636972328597   0     -Inf       0
    x8                   27.3215220128726   0      Inf       0
    x9                  -7.36438963867764   0     -Inf       0
    x10                 0.844273558792823   0      Inf       0


Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______

    Total       7.51808538914298     300    0.0250602846304766
    Model       7.51808538914298      10     0.751808538914298    Inf       0
    Residual                   0     290                      0

A = 0.085143, B = 0.103339, C = 0.000000
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.188482, 0.291821, 0.395161]
```



```
List of powers for D(x): [0.188482, 0.291821, 0.395161, 0.498500, 0.601839, 0.705178,
0.808517]
Fitting J5(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.406293e-02
Corrected MSS of errors squared = 2.435770e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting J5(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     8.76755116283119e-13    0      Inf       0
    x1             -6.05396471722256e-08    0     -Inf       0
    x2              2.2717198345401e-07     0      Inf       0
    x3             -2.70378953275737e-07    0     -Inf       0
    x4              1.03781975817447e-07    0      Inf       0
    x5                  63.6053896481925    0      Inf       0
    x6                  -453.79752004476    0     -Inf       0
    x7                   1354.1807521168    0      Inf       0
    x8                  -2115.66035797966    0     -Inf       0
    x9                   1826.7628432047    0      Inf       0
    x10                 -828.912199810541    0     -Inf       0
    x11                  154.821092702359    0      Inf       0


Number of observations: 301, Error degrees of freedom: 289
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F      pValue
                  _____________    ___    ____________________    ___    ______

    Total       7.5180853891558    300    0.0250602846305193
    Model       7.5180853891558     11     0.683462308105073    Inf       0
    Residual                  0    289                     0

A = 0.000000, B = 0.100000, C = 0.000000
order Q(x) = 4.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.141421, 0.173205, 0.200000]
```



```
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
Fitting J5(x) in range (0.000000, 30.000000)
MSS of errors squared = 2.802466e-02
Corrected MSS of errors squared = 4.854013e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting J5(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     2.49069500573412e-13    0      Inf       0
    x1               2.49018616169899e-05   0      Inf       0
    x2              -0.000198022727160473   0     -Inf       0
    x3               0.000557923288232532   0      Inf       0
    x4              -0.000733982976769954   0     -Inf       0
    x5               0.000461785227888016   0      Inf       0
    x6              -0.000112604616074125   0     -Inf       0
    x7                    63.3701718779798   0      Inf       0
    x8                   -451.438935127698   0     -Inf       0
    x9                    1345.60147176961   0      Inf       0
    x10                  -2100.24475708579   0     -Inf       0
    x11                   1811.93306893256   0      Inf       0
    x12                  -821.563933453096   0     -Inf       0
    x13                   153.342912888891   0      Inf       0


Number of observations: 301, Error degrees of freedom: 287
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______

    Total       7.51808538918058     300    0.0250602846306019
    Model       7.51808538918058      13     0.578314260706198     Inf       0
    Residual                   0     287                     0

A = 0.000000, B = 0.100000, C = 0.000000
order Q(x) = 6.000000, order D(x) = 7.000000
```



```
List of powers for Q(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
Fitting J5(x) in range (0.000000, 30.000000)
MSS of errors squared = 3.825715e-02
Corrected MSS of errors squared = 6.626332e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting J5(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     4.08082652468774e-16    0      Inf        0
    x1                                 0    0      NaN      NaN
    x2             -2.99111906829586e-07    0     -Inf        0
    x3              7.08795707257603e-07    0      Inf        0
    x4             -7.01689262132809e-07    0     -Inf        0
    x5              4.0491590236772e-07     0      Inf        0
    x6             -1.31520111841885e-07    0     -Inf        0
    x7              1.86096707027938e-08    0      Inf        0
    x8                 0.999999988124992    0      Inf        0
    x9              1.50207392258925e-08    0      Inf        0
    x10            -3.64457913296705e-09    0     -Inf        0
    x11             4.98846162999823e-10    0      Inf        0


Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______

    Total       7.51808538914207    300    0.0250602846304736
    Model       7.51808538914207     10     0.751808538914207    Inf       0
    Residual                   0    290                      0

A = 0.000000, B = 0.100000
order Q(x) = 7.000000, order D(x) = 4.000000
```



```
List of powers for Q(x): [0.000000, 0.000821, 0.005182, 0.013139, 0.023869, 0.036665,
0.051007]
List of powers for D(x): [0.000000, 0.000821, 0.005182, 0.013139]
Fitting J5(x) in range (0.000000, 30.000000)
MSS of errors squared = 3.411671e-02
Corrected MSS of errors squared = 4.824832e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the Bessel $J_5(x)$ function. The results are sorted in ascending order using the MSSE values.

Power	*A*	*B*	*C*	*F*	*AICc*	*MSSE*	*Q/D Orders*
A+B*i+C*sqrt(i-1)	0.085	0.103	0	Inf	-Inf	1.40629300E-02	3, 7
A+B*i+C*(i-1)	0	0.11	0	Inf	-Inf	1.46923500E-02	3, 7
A+B*sqrt(i)+C*(i-1)	0	0.1	0	Inf	-Inf	2.80246600E-02	4, 7
A+B*log10(i)^4	0	0.1		Inf	-Inf	3.41167100E-02	7, 4
A+B*sqrt(i)	0	0.1		Inf	-Inf	3.82571500E-02	6, 7
A+B*sqrt(i)+C*sqrt(i-1)	0	0.1	0	Inf	-Inf	3.82571500E-02	6, 7
A+B*i	0.065	0.1		Inf	-Inf	5.59954500E-02	4, 7
A+B/i	0.113	0.658		Inf	-Inf	7.62569800E-02	3, 7
A+B*i-C*(i-1)	0	0.124	0	Inf	-Inf	3.23430700E-01	4, 7
A+B*sqrt(i+C)	0.001	0.1	0	Inf	-Inf	Inf	6, 7

RESULTS FOR THE NATURAL LOGRITHM

Using Power A+B* i

```
Fitting ln(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)    -8.59224786729495     0     -Inf        0
    x1                                0     0      NaN      NaN
    x2                                0     0      NaN      NaN
    x3                                0     0      NaN      NaN
    x4                                0     0      NaN      NaN
    x5             -2240.76844324728    0     -Inf        0
    x6              2432.50090935225    0      Inf        0
    x7             -183.140218199546    0     -Inf        0
    x8             -104.605917124061    0     -Inf        0
    x9              364.265950363828    0      Inf        0
    x10            -594.530069535926    0     -Inf        0
    x11             568.300443796264    0      Inf        0
    x12            -447.463872890853    0     -Inf        0


Number of observations: 901, Error degrees of freedom: 892
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F      pValue
                   ______________    ___    ____________________    ___    ______
```



```
    Total       312.39398039607    900     0.3471044226623
    Model       312.39398039607      8    39.0492475495087    Inf       0
    Residual                  0    892                   0

A = 0.478954, B = 0.100000
order Q(x) = 7.000000, order D(x) = 5.000000
List of powers for Q(x): [0.578954, 0.678954, 0.778954, 0.878954, 0.978954, 1.078954,
1.178954]
List of powers for D(x): [0.578954, 0.678954, 0.778954, 0.878954, 0.978954]
Fitting ln(x) in range (1.000000, 10.000000)
MSS of errors squared = 4.527610e-13
Corrected MSS of errors squared = 6.403008e-13
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting ln(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    _________________    __    _____    ______

    (Intercept)     -17.893479692792     0     -Inf        0
    x1               17.8934797233262    0      Inf        0
    x2                              0    0      NaN      NaN
    x3                              0    0      NaN      NaN
    x4              -36.1654602095231    0     -Inf        0
    x5               1395.57758445162    0      Inf        0
    x6              -5170.67774362836    0     -Inf        0
    x7                              0    0      NaN      NaN
    x8               9123.98254153609    0      Inf        0
    x9                              0    0      NaN      NaN
    x10             -5318.41151610651    0     -Inf        0


Number of observations: 901, Error degrees of freedom: 894
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq           F     pValue
                  _________________    ___    _________________    ___    ______

    Total       312.393980394103    900    0.347104422660114
    Model       312.393980394103      6     52.0656633990171    Inf       0
```



```
    Residual                  0    894                    0

A = 0.274136, B = 0.100000
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.374136, 0.324136, 0.307469]
List of powers for D(x): [0.374136, 0.324136, 0.307469, 0.299136, 0.294136, 0.290803,
0.288422]
Fitting ln(x) in range (1.000000, 10.000000)
MSS of errors squared = 1.871343e-11
Corrected MSS of errors squared = 2.646479e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting ln(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ____________________    __    _____    ______

    (Intercept)    -20.1816391905868     0     -Inf        0
    x1                             0     0      NaN       NaN
    x2                             0     0      NaN       NaN
    x3                             0     0      NaN       NaN
    x4                             0     0      NaN       NaN
    x5                             0     0      NaN       NaN
    x6                             0     0      NaN       NaN
    x7              20.1816391916396     0      Inf        0
    x8             -142.754926687168     0     -Inf        0
    x9              660.341990981033     0      Inf        0
    x10            -1183.40067480217     0     -Inf        0
    x11             846.263542450042     0      Inf        0
    x12                            0     0      NaN       NaN
    x13            -289.742914517575     0     -Inf        0
    x14             101.880072387528     0      Inf        0


Number of observations: 901, Error degrees of freedom: 893
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                 ___________________    ___    ____________________    ___    ______
```


Total	312.393980392944	900	0.347104422658826		
Model	312.393980392944	7	44.6277114847063	Inf	0
Residual	0	893	0		

A = 0.152284, B = 0.100000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.252284, 0.293706, 0.325490, 0.352284, 0.375891, 0.397233, 0.416860]
List of powers for D(x): [0.252284, 0.293706, 0.325490, 0.352284, 0.375891, 0.397233, 0.416860]
Fitting ln(x) in range (1.000000, 10.000000)
MSS of errors squared = 4.856034e-13
Corrected MSS of errors squared = 6.867469e-13
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*sqrt(i+C)

Fitting ln(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	-8.03486778104742	0	-Inf	0
x1	-79635.8836760197	0	-Inf	0
x2	0	0	NaN	NaN
x3	0	0	NaN	NaN
x4	69990.8308755784	0	Inf	0
x5	9653.08766856994	0	Inf	0
x6	-2279.97048510983	0	-Inf	0
x7	-7664.23270000454	0	-Inf	0
x8	-4257.98218302653	0	-Inf	0
x9	-3045.28581116142	0	-Inf	0

Number of observations: 901, Error degrees of freedom: 893
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	312.393980452257	900	0.34710442272473		
Model	312.393980452257	7	44.6277114931795	Inf	0


```
    Residual                   0    893                   0

A = 0.214552, B = 0.274864, C = 1.510999
order Q(x) = 5.000000, order D(x) = 4.000000
List of powers for Q(x): [0.650105, 0.729583, 0.798339, 0.859809, 0.915913]
List of powers for D(x): [0.650105, 0.729583, 0.798339, 0.859809]
Fitting ln(x) in range (1.000000, 10.000000)
MSS of errors squared = 5.739412e-14
Corrected MSS of errors squared = 9.940954e-14
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting ln(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      -13.3437565394552     0     -Inf        0
    x1                -15.316808310446     0     -Inf        0
    x2                               0     0      NaN      NaN
    x3                               0     0      NaN      NaN
    x4                28.6605648517079     0      Inf        0
    x5               -6.67741591734955     0     -Inf        0
    x6                -7.3906512384542     0     -Inf        0
    x7               -9.45252420429962     0     -Inf        0
    x8               -3.39130254257255     0     -Inf        0
    x9               0.202707001024465     0      Inf        0
    x10             -0.0212418544025209     0     -Inf        0
    x11             0.00120555659006209     0      Inf        0


Number of observations: 901, Error degrees of freedom: 891
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq           F     pValue
                  ____________________    ___    ____________________    ___    ______

    Total       312.393980393313    900    0.347104422659236
    Model       312.393980393313      9     34.7104422659236    Inf        0
    Residual                   0    891                    0
```



```
A = 0.010022, B = 0.248431, C = 0.033960
order Q(x) = 4.000000, order D(x) = 7.000000
List of powers for Q(x): [0.258453, 0.540845, 0.823236, 1.105627]
List of powers for D(x): [0.258453, 0.540845, 0.823236, 1.105627, 1.388019, 1.670410,
1.952801]
Fitting ln(x) in range (1.000000, 10.000000)
MSS of errors squared = 1.846120e-13
Corrected MSS of errors squared = 3.197573e-13
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting ln(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ____________________    __    _____    ______

    (Intercept)    -33.2310001499109     0     -Inf        0
    x1                            0     0      NaN       NaN
    x2                            0     0      NaN       NaN
    x3             -1962.74288205097     0     -Inf        0
    x4                            0     0      NaN       NaN
    x5                            0     0      NaN       NaN
    x6                            0     0      NaN       NaN
    x7              1995.97388222427     0      Inf        0
    x8             -335.390947295864     0     -Inf        0
    x9              738.024797225112     0      Inf        0
    x10            -456.750589554951     0     -Inf        0


Number of observations: 901, Error degrees of freedom: 895
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq           F     pValue
                  _______________    ___    ________________    ___    ______

    Total       312.393980394796    900    0.347104422660885
    Model       312.393980394796      5     62.4787960789593    Inf       0
    Residual                   0    895                    0

A = 0.000000, B = 0.133726, C = 0.127432
order Q(x) = 7.000000, order D(x) = 3.000000
```



```
List of powers for Q(x): [0.133726, 0.140021, 0.146315, 0.152610, 0.158904, 0.165199,
0.171493]
List of powers for D(x): [0.133726, 0.140021, 0.146315]
Fitting ln(x) in range (1.000000, 10.000000)
MSS of errors squared = 1.911439e-12
Corrected MSS of errors squared = 3.310709e-12
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting ln(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      -5.93838297481804     0     -Inf        0
    x1               -15.7785682273234     0     -Inf        0
    x2                 6.9300941274154     0      Inf        0
    x3                               0     0      NaN      NaN
    x4                               0     0      NaN      NaN
    x5                14.3178986366884     0      Inf        0
    x6               0.480317221688685     0      Inf        0
    x7              -0.0113586937169705     0     -Inf        0
    x8                -9.30658885669531     0     -Inf        0
    x9                -12.8084624706717     0     -Inf        0
    x10               -3.96329092707304     0     -Inf        0
    x11               -8.51285579266257     0     -Inf        0
    x12                -2.4702015876063     0     -Inf        0


Number of observations: 901, Error degrees of freedom: 890
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F      pValue
                  _______________    ___    _________________    ___    ______

    Total        312.393980393368    900    0.347104422659298
    Model        312.393980393368     10     31.2393980393368    Inf        0
    Residual                    0    890                    0

A = 0.265269, B = 0.352323, C = 0.299138
order Q(x) = 7.000000, order D(x) = 5.000000
```



```
List of powers for Q(x): [0.617592, 1.269052, 1.745282, 2.192682, 2.625158, 3.048098,
3.464263]
List of powers for D(x): [0.617592, 1.269052, 1.745282, 2.192682, 2.625158]
Fitting ln(x) in range (1.000000, 10.000000)
MSS of errors squared = 5.252672e-12
Corrected MSS of errors squared = 9.097895e-12
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting ln(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)     -6.8305772097232     0     -Inf        0
    x1              -265.27724719136     0     -Inf        0
    x2                            0      0      NaN       NaN
    x3                            0      0      NaN       NaN
    x4                            0      0      NaN       NaN
    x5                            0      0      NaN       NaN
    x6               241.832539573797    0      Inf        0
    x7               30.2752848445648    0      Inf        0
    x8              -40.1985949542806    0     -Inf        0
    x9              -91.4900392951185    0     -Inf        0
    x10             -84.8060075271792    0     -Inf        0
    x11             -71.9231542894163    0     -Inf        0
    x12             -93.3169291991063    0     -Inf        0
    x13             -39.4295458654385    0     -Inf        0
    x14             -1.98607640576285    0     -Inf        0


Number of observations: 901, Error degrees of freedom: 890
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F      pValue
                  _______________    ___    ________________    ___    ______

    Total       312.39398039312     900    0.347104422659022
    Model       312.39398039312      10      31.239398039312    Inf       0
    Residual                  0     890                    0
```



```
A = 0.538391, B = 0.104487, C = 0.271731
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.642878, 0.957888, 1.262829, 1.562557, 1.858953, 2.152984,
2.445221]
List of powers for D(x): [0.642878, 0.957888, 1.262829, 1.562557, 1.858953, 2.152984,
2.445221]
Fitting ln(x) in range (1.000000, 10.000000)
MSS of errors squared = 9.158967e-14
Corrected MSS of errors squared = 1.586380e-13
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting ln(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -8.14867838692351     0     -Inf        0
    x1             -6.79622993055562     0     -Inf        0
    x2                            0     0      NaN      NaN
    x3                            0     0      NaN      NaN
    x4                            0     0      NaN      NaN
    x5              45.1012490865158     0      Inf        0
    x6             -30.1563407355358     0     -Inf        0
    x7             -6.70245626602969     0     -Inf        0
    x8             -10.3288930287248     0     -Inf        0
    x9              1.94442180000058     0      Inf        0
    x10            -9.44610757268834     0     -Inf        0
    x11              5.3135486850873     0      Inf        0
    x12             1.71741785515815     0      Inf        0


Number of observations: 901, Error degrees of freedom: 891
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F      pValue
                  ____________________    ___    ____________________    ___    ______

    Total         312.393980392892     900    0.347104422658769
    Model         312.393980392892       9     34.7104422658769    Inf        0
    Residual                     0     891                     0
```



```
A = 0.073233, B = 0.356632, C = 0.470183
order Q(x) = 6.000000, order D(x) = 6.000000
List of powers for Q(x): [0.429865, 1.047770, 1.355878, 1.600879, 1.811054, 1.998162]
List of powers for D(x): [0.429865, 1.047770, 1.355878, 1.600879, 1.811054, 1.998162]
Fitting ln(x) in range (1.000000, 10.000000)
MSS of errors squared = 2.414819e-10
Corrected MSS of errors squared = 4.182590e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting ln(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)                        0    0      NaN      NaN
    x1                                 0    0      NaN      NaN
    x2                                 0    0      NaN      NaN
    x3               1.0838277808944e-15    0      Inf        0
    x4                  1.00001468075528    0      Inf        0
    x5              -1.92541611899552e-05    0     -Inf        0
    x6               5.79837424700339e-06    0      Inf        0
    x7              -1.39807960696438e-06    0     -Inf        0
    x8               1.73111273162223e-07    0      Inf        0


Number of observations: 901, Error degrees of freedom: 895
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq           F     pValue
                  _________________    ___    ________________    ___    ______

    Total       312.393980393285    900    0.347104422659206
    Model       312.393980393285      5     62.4787960786571    Inf       0
    Residual                   0    895                    0

A = 0.000000, B = 0.100000
order Q(x) = 3.000000, order D(x) = 5.000000
List of powers for Q(x): [0.000000, 0.000821, 0.005182]
List of powers for D(x): [0.000000, 0.000821, 0.005182, 0.013139, 0.023869]
```



```
Fitting ln(x) in range (1.000000, 10.000000)
MSS of errors squared = 6.914598e-02
Corrected MSS of errors squared = 9.778718e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the natural logarithm function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i+C)	0.214	0.274	1.51	Inf	-Inf	5.73941200E-14	5, 4
A+B*sqrt(i)+C*(i-1)	0.538	0.104	0.271	Inf	-Inf	9.15896700E-14	7, 7
A+B*i+C*(i-1)	0.01	0.248	0.033	Inf	-Inf	1.84612000E-13	4, 7
A+B*i	0.478	0.1		Inf	-Inf	4.52761000E-13	7, 5
A+B*sqrt(i)	0.152	0.1		Inf	-Inf	4.85603400E-13	7, 7
A+B*i-C*(i-1)	0	0.133	0.127	Inf	-Inf	1.91143900E-12	7, 3
A+B*i+C*sqrt(i-1)	0.265	0.352	0.299	Inf	-Inf	5.25267200E-12	7, 5
A+B/i	0.274	0.1		Inf	-Inf	1.87134300E-11	3, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.073	0.356	0.47	Inf	-Inf	2.41481900E-10	6, 6
A+B*log10(i)^4	0	0.1		Inf	-Inf	6.91459800E-02	3, 5

RESULTS FOR THE COMMON LOGARITHM

Using Power A+B* i

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -3.40789134198338     0     -Inf        0
    x1                            0     0      NaN       NaN
    x2                            0     0      NaN       NaN
    x3                            0     0      NaN       NaN
    x4              4852.31126829642    0      Inf        0
    x5             -4848.90337692329    0     -Inf        0
    x6             -73.5152981573804    0     -Inf        0
    x7              261.700213711598    0      Inf        0
    x8             -506.40157477477     0     -Inf        0
    x9              1166.94387117995    0      Inf        0
    x10              536.92174900924    0      Inf        0
    x11            -14.2488353936858    0     -Inf        0


Number of observations: 901, Error degrees of freedom: 892
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF            MeanSq            F      pValue
                  ____________________    ___    ____________________    ___    ______

    Total        58.9211587779403    900    0.0654679541977114
    Model        58.9211587779403      8      7.36514484724254    Inf        0
```



```
    Residual                   0    892                    0

A = 0.492062, B = 0.123433
order Q(x) = 5.000000, order D(x) = 6.000000
List of powers for Q(x): [0.615495, 0.738929, 0.862362, 0.985796, 1.109229]
List of powers for D(x): [0.615495, 0.738929, 0.862362, 0.985796, 1.109229, 1.232663]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 5.737639e-14
Corrected MSS of errors squared = 8.114246e-14
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     -3.49921879165346    0     -Inf        0
    x1               3.45181735281987    0      Inf        0
    x2               2371.69215657384    0      Inf        0
    x3                             0     0      NaN      NaN
    x4              -4688.22592077077    0     -Inf        0
    x5                             0     0      NaN      NaN
    x6               2316.58116566388    0      Inf        0
    x7              -0.634761879906234    0     -Inf        0
    x8               -297.304136728964    0     -Inf        0
    x9               -1304.62658733281    0     -Inf        0
    x10               703.187876364102    0      Inf        0


Number of observations: 901, Error degrees of freedom: 892
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _______________    ___    ____________________    ___    ______

    Total        58.9211587791879    900    0.0654679541990976
    Model        58.9211587791879      8      7.36514484739848    Inf       0
    Residual                   0    892                    0

A = 0.453947, B = 0.965543
```

```
order Q(x) = 6.000000, order D(x) = 4.000000
List of powers for Q(x): [1.419490, 0.936719, 0.775795, 0.695333, 0.647056, 0.614871]
List of powers for D(x): [1.419490, 0.936719, 0.775795, 0.695333]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 8.254884e-14
Corrected MSS of errors squared = 1.167417e-13
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -10.0373461209231    0     -Inf       0
    x1             -12459.3781220954    0     -Inf       0
    x2             -3748.54913519234    0     -Inf       0
    x3              16217.9646034118    0      Inf       0
    x4             -441.600977622148    0     -Inf       0
    x5             -1559.29076341226    0     -Inf       0
    x6             -379.207614579982    0     -Inf       0


Number of observations: 901, Error degrees of freedom: 894
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______

    Total       58.9211587780089    900    0.0654679541977877
    Model       58.9211587780089      6      9.82019312966815    Inf       0
    Residual                   0    894                     0

A = 0.112296, B = 0.100000
order Q(x) = 3.000000, order D(x) = 3.000000
List of powers for Q(x): [0.212296, 0.253718, 0.285501]
List of powers for D(x): [0.212296, 0.253718, 0.285501]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 5.214265e-15
Corrected MSS of errors squared = 7.374084e-15
R-Squared =   1.00000000
```



```
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                   ____________________    __    _____    ______

    (Intercept)     -9.9897621855948     0     -Inf        0
    x1                             0     0      NaN       NaN
    x2                             0     0      NaN       NaN
    x3                             0     0      NaN       NaN
    x4                             0     0      NaN       NaN
    x5              9.98976218579049     0      Inf        0
    x6             -117.267506495935     0     -Inf        0
    x7              574.622609359972     0      Inf        0
    x8             -1206.38797377501     0     -Inf        0
    x9              1273.73541861871     0      Inf        0
    x10            -673.517618324661     0     -Inf        0
    x11             141.937419816223     0      Inf        0


Number of observations: 901, Error degrees of freedom: 893
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF            MeanSq            F      pValue
                 _______________    ___    ____________________    ___    ______

    Total        58.9211587783804    900    0.0654679541982004
    Model        58.9211587783804      7      8.41730839691148    Inf       0
    Residual                    0    893                     0

A = 0.117678, B = 0.100000, C = 0.053282
order Q(x) = 5.000000, order D(x) = 6.000000
List of powers for Q(x): [0.220307, 0.260970, 0.292414, 0.319005, 0.342473]
List of powers for D(x): [0.220307, 0.260970, 0.292414, 0.319005, 0.342473, 0.363712]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 9.817733e-14
Corrected MSS of errors squared = 1.700481e-13
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
```



```
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      -4.27329515932367     0     -Inf        0
    x1               -23.5520012854875     0     -Inf        0
    x2                29.5063792342807     0      Inf        0
    x3                               0     0      NaN      NaN
    x4                               0     0      NaN      NaN
    x5               -0.182206183777942    0     -Inf        0
    x6               -1.48014910390745     0     -Inf        0
    x7               -0.0187275011238678   0     -Inf        0
    x8               -13.4532856346755     0     -Inf        0
    x9               -12.3202861895383     0     -Inf        0
    x10               4.28476695019386     0      Inf        0
    x11                              0     0      NaN      NaN
    x12               1.82677386856542     0      Inf        0
    x13              0.374439212365932     0      Inf        0


Number of observations: 901, Error degrees of freedom: 890
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______

    Total        58.9211587782005    900    0.0654679541980005
    Model        58.9211587782005     10      5.89211587782005    Inf       0
    Residual                    0    890                     0

A = 0.182397, B = 0.218536, C = 0.115237
order Q(x) = 7.000000, order D(x) = 6.000000
List of powers for Q(x): [0.400933, 0.734705, 1.068478, 1.402250, 1.736023, 2.069795,
2.403567]
List of powers for D(x): [0.400933, 0.734705, 1.068478, 1.402250, 1.736023, 2.069795]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 9.984474e-13
Corrected MSS of errors squared = 1.729362e-12
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
```



```
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ____________________    __    _____    ______

    (Intercept)    -3.58156442224547     0     -Inf        0
    x1                             0     0      NaN      NaN
    x2              18552.5756618341     0      Inf        0
    x3                             0     0      NaN      NaN
    x4                             0     0      NaN      NaN
    x5             -18548.9940973911     0     -Inf        0
    x6             -219.410321902444     0     -Inf        0
    x7              2317.04229820069     0      Inf        0
    x8              3940.10448785256     0      Inf        0
    x9              5192.75351967882     0      Inf        0
    x10             1621.25068197916     0      Inf        0
    x11            -43.5731035523581     0     -Inf        0


Number of observations: 901, Error degrees of freedom: 892
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF            MeanSq            F      pValue
                  _________________    ___    ____________________    ___    ______

    Total       58.9211587840333     900    0.0654679542044815
    Model       58.9211587840333       8      7.36514484800416    Inf       0
    Residual                   0     892                     0

A = 0.530415, B = 0.100000, C = 0.000000
order Q(x) = 5.000000, order D(x) = 6.000000
List of powers for Q(x): [0.630415, 0.730415, 0.830415, 0.930415, 1.030415]
List of powers for D(x): [0.630415, 0.730415, 0.830415, 0.930415, 1.030415, 1.130415]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 4.294145e-15
Corrected MSS of errors squared = 7.437678e-15
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
```



```
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                   ____________________    __    _____    ______

    (Intercept)     -4.66030107777155    0     -Inf       0
    x1              -2.49477921858088    0     -Inf       0
    x2                -18.39478243502    0     -Inf       0
    x3               13.6466703258212    0      Inf       0
    x4               11.1271902822184    0      Inf       0
    x5              0.776002123705074    0      Inf       0
    x6              -6.54394141037586    0     -Inf       0
    x7              -17.5991614163369    0     -Inf       0
    x8              -18.4169522124534    0     -Inf       0
    x9              -4.07436788545862    0     -Inf       0
    x10            -0.132404765838152    0     -Inf       0


Number of observations: 901, Error degrees of freedom: 890
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                 _______________    ___    ____________________    ___    ______

    Total        58.9211587781619    900    0.0654679541979576
    Model        58.9211587781619     10      5.89211587781619    Inf       0
    Residual                    0    890                     0

A = 0.052728, B = 0.278588, C = 0.301774
order Q(x) = 5.000000, order D(x) = 5.000000
List of powers for Q(x): [0.331316, 0.911678, 1.315264, 1.689767, 2.049216]
List of powers for D(x): [0.331316, 0.911678, 1.315264, 1.689767, 2.049216]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 1.887376e-14
Corrected MSS of errors squared = 3.269031e-14
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting log(x) in range (1.000000, 10.000000)
```

```
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                          Estimate          SE    tStat    pValue
                      ____________________    __    _____    ______

    (Intercept)       -3.91945781420955     0     -Inf       0
    x1                -29.5630676572493     0     -Inf       0
    x2                -4.86017180317009     0     -Inf       0
    x3                 40.9709895797382     0      Inf       0
    x4                                0     0      NaN      NaN
    x5                -2.62829230422642     0     -Inf       0
    x6                -16.6393673062091     0     -Inf       0
    x7                 -39.722777207176     0     -Inf       0
    x8                -13.7209293979567     0     -Inf       0
    x9                 3.93083144894266     0      Inf       0
    x10               0.616438000252339     0      Inf       0
    x11              -0.00545527046448447     0     -Inf       0


Number of observations: 901, Error degrees of freedom: 890
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq          DF          MeanSq            F     pValue
                   _______________    ___    ________________    ___    ______

    Total          58.9211587781793    900    0.065467954197977
    Model          58.9211587781793     10     5.89211587781793    Inf       0
    Residual                      0    890                     0

A = 0.074771, B = 0.380348, C = 0.266016
order Q(x) = 5.000000, order D(x) = 6.000000
List of powers for Q(x): [0.455119, 0.878680, 1.265585, 1.633515, 1.989319]
List of powers for D(x): [0.455119, 0.878680, 1.265585, 1.633515, 1.989319, 2.336509]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 3.517210e-14
Corrected MSS of errors squared = 6.091986e-14
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
```



```
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ________________    ___    _____    ______

    (Intercept)    -4.46288544077442     0     -Inf        0
    x1             -19.3673121249338     0     -Inf        0
    x2                             0     0      NaN       NaN
    x3               23.830197567283     0      Inf        0
    x4             -12.4614564363139     0     -Inf        0
    x5             -29.2368497919062     0     -Inf        0
    x6              14.6309626360739     0      Inf        0
    x7                -17.7369638016     0     -Inf        0
    x8              10.6018422771839     0      Inf        0
    x9              -3.5242425936154     0     -Inf        0
    x10            0.502693216820963     0      Inf        0


Number of observations: 901, Error degrees of freedom: 891
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq           DF            MeanSq            F      pValue
                  _________________    ___    ____________________    ___    ______

    Total       58.9211587782023     900    0.0654679541980025
    Model       58.9211587782023       9      6.54679541980025    Inf       0
    Residual                   0     891                     0

A = 0.151262, B = 0.231038, C = 0.322361
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.382300, 0.800360, 1.007319]
List of powers for D(x): [0.382300, 0.800360, 1.007319, 1.171683, 1.312601, 1.438008,
1.552151]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 1.156091e-13
Corrected MSS of errors squared = 2.002408e-13
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
```



```
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)     -2.77115127558843     0     -Inf        0
    x1                            0     0      NaN       NaN
    x2               21741.9829195199     0      Inf        0
    x3              -20832.8000344269     0     -Inf        0
    x4              -906.411733697181     0     -Inf        0
    x5              -3604.08062955648     0     -Inf        0
    x6               7508.46853897914     0      Inf        0
    x7               3465.10266160938     0      Inf        0
    x8               131.538673981648     0      Inf        0
    x9              -0.343665767059351     0     -Inf        0


Number of observations: 901, Error degrees of freedom: 892
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______

    Total       58.9211587744444     900    0.0654679541938272
    Model       58.9211587744444       8      7.36514484680555    Inf       0
    Residual                   0     892                     0

A = 0.856073, B = 3.195061
order Q(x) = 4.000000, order D(x) = 5.000000
List of powers for Q(x): [0.856073, 0.882311, 1.021648, 1.275869]
List of powers for D(x): [0.856073, 0.882311, 1.021648, 1.275869, 1.618702]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 4.138004e-14
Corrected MSS of errors squared = 5.852022e-14
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the common logarithm function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i-C*(i-1)	0.53	0.1	0	Inf	-Inf	4.29414500E-15	5, 6
A+B*sqrt(i)	0.112	0.1		Inf	-Inf	5.21426500E-15	3, 3
A+B*i+C*sqrt(i-1)	0.052	0.278	0.301	Inf	-Inf	1.88737600E-14	5, 5
A+B*sqrt(i)+C*(i-1)	0.074	0.38	0.266	Inf	-Inf	3.51721000E-14	5, 6
A+B*log10(i)^4	0.856	3.195		Inf	-Inf	4.13800400E-14	4, 5
A+B*i	0.492	0.123		Inf	-Inf	5.73763900E-14	5, 6
A+B/i	0.453	0.965		Inf	-Inf	8.25488400E-14	6, 4
A+B*sqrt(i+C)	0.117	0.1	0.053	Inf	-Inf	9.81773300E-14	5, 6
A+B*sqrt(i)+C*sqrt(i-1)	0.151	0.231	0.322	Inf	-Inf	1.15609100E-13	3, 7
A+B*i+C*(i-1)	0.182	0.218	0.115	Inf	-Inf	9.98447400E-13	7, 6

RESULTS FOR THE COMMON LOGARITHM OF THE GAMMA FUNCTION

Using Power A+B* i

```
Fitting log10Gamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      0.821466466466011     0      Inf        0
    x1               -2.03164995681268     0     -Inf        0
    x2                1.40981546198646     0      Inf        0
    x3                               0     0      NaN      NaN
    x4               -0.199658392817594    0     -Inf        0
    x5                  3.5210710207115    0      Inf        0
    x6               -5.19738896820676     0     -Inf        0
    x7                4.12460359942692     0      Inf        0
    x8               -1.85666095880281     0     -Inf        0
    x9                0.449340566452975    0      Inf        0
    x10              -0.0456516909532447    0     -Inf        0


Number of observations: 981, Error degrees of freedom: 971
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq          F     pValue
                  _____________    ___    _________________    ___    ______

    Total        2151267.44232855    980    2195.17085951893
```


Model	2151267.44232855	9	239029.715814283	Inf	0
Residual	0	971	0		

```
A = 0.000000, B = 0.100000
order Q(x) = 4.000000, order D(x) = 6.000000
List of powers for Q(x): [0.100000, 0.200000, 0.300000, 0.400000]
List of powers for D(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000]
Fitting log10Gamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 1.963046e-06
Corrected MSS of errors squared = 2.776166e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting log10Gamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	0.088001144895251	0	Inf	0
x1	-0.112635484572341	0	-Inf	0
x2	0.803149912192791	0	Inf	0
x3	-0.778516765279196	0	-Inf	0
x4	0	0	NaN	NaN
x5	0	0	NaN	NaN
x6	0.38483623225307	0	Inf	0
x7	-26.8425659825419	0	-Inf	0
x8	188.324226586311	0	Inf	0
x9	-358.0545333214	0	-Inf	0
x10	197.187832149695	0	Inf	0

```
Number of observations: 981, Error degrees of freedom: 972
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```

	SumSq	DF	MeanSq	F	pValue
Total	2151267.4423282	980	2195.17085951857		
Model	2151267.4423282	8	268908.430291025	Inf	0
Residual	0	972	0		


```
A = 0.038954, B = 0.114883
order Q(x) = 5.000000, order D(x) = 5.000000
List of powers for Q(x): [0.153837, 0.096396, 0.077249, 0.067675, 0.061931]
List of powers for D(x): [0.153837, 0.096396, 0.077249, 0.067675, 0.061931]
Fitting log10Gamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 2.378245e-05
Corrected MSS of errors squared = 3.363347e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting log10Gamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)       1.90777772365962     0      Inf        0
    x1                 7.5174142467948     0      Inf        0
    x2                -3.7536440375383     0     -Inf        0
    x3                               0     0      NaN      NaN
    x4                -14.629082491341     0     -Inf        0
    x5                               0     0      NaN      NaN
    x6                               0     0      NaN      NaN
    x7                8.95721222739222     0      Inf        0
    x8               -12.7529169379462     0     -Inf        0
    x9                6.24011415003546     0      Inf        0
    x10              -5.02645475812461     0     -Inf        0
    x11               2.82473083979006     0      Inf        0
    x12              -1.03761146995661     0     -Inf        0
    x13              0.223707830770391     0      Inf        0
    x14             -0.0214878033110754    0     -Inf        0


Number of observations: 981, Error degrees of freedom: 969
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq          F     pValue
                 _______________    ___    _______________    ___    ______

    Total        2151267.44232626    980    2195.17085951659
    Model        2151267.44232626     11    195569.767484205    Inf       0
    Residual                    0    969                   0
```



```
A = 0.196689, B = 0.676945
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.873635, 1.154035, 1.369193, 1.550580, 1.710386, 1.854860,
1.987719]
List of powers for D(x): [0.873635, 1.154035, 1.369193, 1.550580, 1.710386, 1.854860,
1.987719]
Fitting log10Gamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 2.628136e-10
Corrected MSS of errors squared = 3.716746e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting log10Gamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      8.39361318412931     0      Inf        0
    x1              -264.376392971618     0     -Inf        0
    x2               460.673209171141     0      Inf        0
    x3                             0     0      NaN      NaN
    x4              -263.423040530374     0     -Inf        0
    x5                             0     0      NaN      NaN
    x6                             0     0      NaN      NaN
    x7               58.7325859760675     0      Inf        0
    x8               22.3614782803603     0      Inf        0
    x9              -64.2007084972414     0     -Inf        0
    x10              74.2512689257744     0      Inf        0
    x11             -39.6988197138963     0     -Inf        0
    x12              8.15911583017222     0      Inf        0


Number of observations: 99, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF         MeanSq          F     pValue
                  _____________    ___    _________________    ___    ______

    Total       220712.747500381    98    2252.17089286103
    Model       220712.747500381     9    24523.6386111535    Inf       0
```



```
    Residual                   0    89                   0

A = 0.000000, B = 0.126791, C = 1.160190
order Q(x) = 7.000000, order D(x) = 5.000000
List of powers for Q(x): [0.186352, 0.225395, 0.258610, 0.288019, 0.314692, 0.339274,
0.362192]
List of powers for D(x): [0.186352, 0.225395, 0.258610, 0.288019, 0.314692]
Fitting log10Gamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 2.098146e-08
Corrected MSS of errors squared = 3.634096e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting log10Gamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)         1.09390337872941    0      Inf       0
    x1                 -6.77411562844253    0     -Inf       0
    x2                  9.59786171889768    0      Inf       0
    x3                 -11.1097849433761    0     -Inf       0
    x4                  13.1560619095436    0      Inf       0
    x5                 -5.96433844546168    0     -Inf       0
    x6                  2.9736886005153     0      Inf       0
    x7                -0.0446668092880984   0     -Inf       0
    x8                 0.00445138124293748  0      Inf       0
    x9                -0.000188061289104892 0     -Inf       0


Number of observations: 981, Error degrees of freedom: 971
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF         MeanSq          F     pValue
                ________________    ___    _____________    ___    ______

    Total       2151267.44232852    980    2195.1708595189
    Model       2151267.44232852      9    239029.71581428    Inf       0
    Residual                   0    971                   0

A = 0.040611, B = 0.248026, C = 0.020361
```



```
order Q(x) = 5.000000, order D(x) = 4.000000
List of powers for Q(x): [0.288637, 0.557024, 0.825412, 1.093799, 1.362186]
List of powers for D(x): [0.288637, 0.557024, 0.825412, 1.093799]
Fitting log10Gamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 1.854269e-09
Corrected MSS of errors squared = 3.211688e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting log10Gamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      0.765803567490689     0      Inf       0
    x1              -0.163795759704938     0     -Inf       0
    x2               0.591700675747153     0      Inf       0
    x3              -1.19372564870729      0     -Inf       0
    x4              0.00391967574463522    0      Inf       0
    x5              -0.0553582946318991    0     -Inf       0
    x6               0.665253789790442     0      Inf       0
    x7               0.483788634888977     0      Inf       0
    x8              -0.0978386707961098    0     -Inf       0


Number of observations: 981, Error degrees of freedom: 972
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq          F     pValue
                 _______________    ___    _________________    ___    ______

    Total        2151267.44232858    980    2195.17085951896
    Model        2151267.44232858      8    268908.430291073    Inf       0
    Residual                   0    972                   0

A = 0.220968, B = 0.100000, C = 0.236886
order Q(x) = 3.000000, order D(x) = 5.000000
List of powers for Q(x): [0.320968, 0.184082, 0.047196]
List of powers for D(x): [0.320968, 0.184082, 0.047196, -0.089690, -0.226576]
Fitting log10Gamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 1.103332e-06
```



```
Corrected MSS of errors squared = 1.911028e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting log10Gamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)       -1.42123148097441     0     -Inf       0
    x1                -7.49462637195772     0     -Inf       0
    x2                 1.15076811839519     0      Inf       0
    x3                                0     0      NaN     NaN
    x4                 15.5039894753428     0      Inf       0
    x5                                0     0      NaN     NaN
    x6                -7.73996744980882     0     -Inf       0
    x7                 6.61057069084886     0      Inf       0
    x8                  6.7527129346609     0      Inf       0
    x9                0.337903244700781     0      Inf       0
    x10              -0.0131996811957404     0     -Inf       0
    x11               0.000520534216168253     0      Inf       0
    x12              -9.68295558068155e-06     0     -Inf       0


Number of observations: 981, Error degrees of freedom: 970
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F      pValue
                  _____________    ___    _________________    ___    ______

    Total       2151267.4423289    980    2195.17085951928
    Model       2151267.4423289     10     215126.74423289    Inf       0
    Residual                  0    970                   0

A = 0.044516, B = 0.289845, C = 0.010546
order Q(x) = 6.000000, order D(x) = 6.000000
List of powers for Q(x): [0.334361, 0.634753, 0.928967, 1.222164, 1.514836, 1.807171]
List of powers for D(x): [0.334361, 0.634753, 0.928967, 1.222164, 1.514836, 1.807171]
Fitting log10Gamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 4.067030e-10
Corrected MSS of errors squared = 7.044302e-10
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting log10Gamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      -0.886351513869465     0     -Inf       0
    x1               -0.365371469976944     0     -Inf       0
    x2               -0.284426554303293     0     -Inf       0
    x3                 3.20208563783325     0      Inf       0
    x4                -1.39358652889752     0     -Inf       0
    x5               -0.266464742261947     0     -Inf       0
    x6              -0.00606922735253027     0     -Inf       0
    x7              2.37050194527565e-05     0      Inf       0
    x8                 2.54862830254448     0      Inf       0
    x9                0.466584574959352     0      Inf       0
    x10               0.0182325220283437     0      Inf       0


Number of observations: 981, Error degrees of freedom: 970
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq          F     pValue
                 _______________    ___    _________________    ___    ______

    Total        2151267.44232859    980    2195.17085951896
    Model        2151267.44232859     10    215126.744232859    Inf       0
    Residual                    0    970                   0

A = 0.051926, B = 0.100000, C = 0.387237
order Q(x) = 7.000000, order D(x) = 3.000000
List of powers for Q(x): [0.151926, 0.580584, 0.999604, 1.413636, 1.824480, 2.233059,
2.639922]
List of powers for D(x): [0.151926, 0.580584, 0.999604]
Fitting log10Gamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 3.611675e-10
Corrected MSS of errors squared = 6.255604e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
```



```
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting log10Gamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     -27.2783014928659    0     -Inf        0
    x1               193.366426855826    0      Inf        0
    x2              -530.896323809137    0     -Inf        0
    x3                             0    0      NaN      NaN
    x4               947.040595162078    0      Inf        0
    x5                             0    0      NaN      NaN
    x6              -1202.26298818729    0     -Inf        0
    x7               620.030374167757    0      Inf        0
    x8               1.34817742157071    0      Inf        0
    x9              -0.833918451998216    0     -Inf        0
    x10              0.266820408592356    0      Inf        0


Number of observations: 981, Error degrees of freedom: 972
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq          F     pValue
                  _______________    ___    _______________    ___    ______

    Total       2151267.44232862    980      2195.170859519
    Model       2151267.44232862      8    268908.430291077    Inf       0
    Residual                   0    972                  0

A = 0.000000, B = 0.100000, C = 0.054362
order Q(x) = 7.000000, order D(x) = 3.000000
List of powers for Q(x): [0.100000, 0.195783, 0.250084, 0.294157, 0.332330, 0.366505,
0.397733]
List of powers for D(x): [0.100000, 0.195783, 0.250084]
Fitting log10Gamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 6.822774e-08
Corrected MSS of errors squared = 1.181739e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
```



```
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting log10Gamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)                        0    0      NaN      NaN
    x1              1.99961830648737e-08    0      Inf        0
    x2                                 0    0      NaN      NaN
    x3             -3.11758980320813e-08    0     -Inf        0
    x4              1.11644532147849e-08    0      Inf        0
    x5                 0.999999917042169    0      Inf        0
    x6              9.17996614628453e-08    0      Inf        0
    x7             -1.18243145907812e-08    0     -Inf        0
    x8              7.24554356943926e-10    0      Inf        0
    x9             -1.92452421488811e-11    0     -Inf        0
    x10             8.6457941604409e-14     0      Inf        0
    x11            3.24734067495661e-15     0      Inf        0


Number of observations: 981, Error degrees of freedom: 971
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq          F     pValue
                  _________________    ___    _________________    ___    ______

    Total        2151267.44232858    980    2195.17085951896
    Model        2151267.44232858      9    239029.715814287    Inf       0
    Residual                    0    971                   0

A = 0.000000, B = 3.500000
order Q(x) = 4.000000, order D(x) = 7.000000
List of powers for Q(x): [0.000000, 0.028741, 0.181377, 0.459862]
List of powers for D(x): [0.000000, 0.028741, 0.181377, 0.459862, 0.835415, 1.283288,
1.785239]
Fitting log10Gamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 3.544408e-06
Corrected MSS of errors squared = 5.012549e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```


Comments

The following table shows the summary results for the ten models that fit the common logarithm of the gamma function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i)	0.196	0.676		Inf	-Inf	2.62813600E-10	7, 7
A+B*sqrt(i)+C*(i-1)	0.051	0.1	0.387	Inf	-Inf	3.61167500E-10	7, 3
A+B*i+C*sqrt(i-1)	0.044	0.289	0.01	Inf	-Inf	4.06703000E-10	6, 6
A+B*i+C*(i-1)	0.04	0.248	0.02	Inf	-Inf	1.85426900E-09	5, 4
A+B*sqrt(i+C)	0	0.126	1.16	Inf	-Inf	2.09814600E-08	7, 5
A+B*sqrt(i)+C*sqrt(i-1)	0	0.1	0.054	Inf	-Inf	6.82277400E-08	7, 3
A+B*i-C*(i-1)	0.22	0.1	0.236	Inf	-Inf	1.10333200E-06	3, 5
A+B*i	0	0.1		Inf	-Inf	1.96304600E-06	4, 6
A+B*log10(i)^4	0	3.5		Inf	-Inf	3.54440800E-06	4, 7
A+B/i	0.038	0.114		Inf	-Inf	2.37824500E-05	5, 5

RESULTS FOR THE COMMON EXPONENT (10^X)

Using Power A+B* i

```
Fitting 10^x in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)     0.999999999998214    0      Inf         0
    x1              -66.7969895360496    0     -Inf         0
    x2               209.626732742231    0      Inf         0
    x3              -239.038809989091    0     -Inf         0
    x4               74.6592068102763    0      Inf         0
    x5               71.1448288200368    0      Inf         0
    x6              -67.2231626680505    0     -Inf         0
    x7               16.7236643569933    0      Inf         0
    x8               65.7018164998608    0      Inf         0
    x9              -199.47045273076     0     -Inf         0
    x10              200.576995269033    0      Inf         0
    x11                             0    0      NaN       NaN
    x12             -146.783780103911    0     -Inf         0
    x13              104.761258795901    0      Inf         0
    x14             -23.7953847837527    0     -Inf         0


Number of observations: 101, Error degrees of freedom: 87
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF          MeanSq          F      pValue
                    ____________    ___    ____________    ___    ______
```



```
    Total       645.004627337368    100    6.45004627337368
    Model       645.004627337368     13    49.6157405644129    Inf       0
    Residual                   0     87                   0

A = 0.367134, B = 0.100000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.467134, 0.567134, 0.667134, 0.767134, 0.867134, 0.967134,
1.067134]
List of powers for D(x): [0.467134, 0.567134, 0.667134, 0.767134, 0.867134, 0.967134,
1.067134]
Fitting 10^x in range (0.000000, 1.000000)
MSS of errors squared = 1.161610e-10
Corrected MSS of errors squared = 1.642765e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting 10^x in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)        1.0000000000031     0      Inf       0
    x1                0.189078264632143    0      Inf       0
    x2                -6.86506694988387    0     -Inf       0
    x3                 84.0772142233575    0      Inf       0
    x4                 -321.03400814076    0     -Inf       0
    x5                 456.634716175001    0      Inf       0
    x6                 -213.620610160512    0     -Inf       0
    x7                -0.0286381081264091    0     -Inf       0
    x8                  6.1139825087904    0      Inf       0
    x9                 -83.7463362413515    0     -Inf       0
    x10                323.415911625009    0      Inf       0
    x11                -459.461618450568    0     -Inf       0
    x12                214.668566325427    0      Inf       0


Number of observations: 101, Error degrees of freedom: 88
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq          F     pValue
```


Total	645.004627337566	100	6.45004627337566		
Model	645.004627337566	12	53.7503856114639	Inf	0
Residual	0	88	0		

```
A = 0.304803, B = 1.521183
order Q(x) = 6.000000, order D(x) = 6.000000
List of powers for Q(x): [1.825986, 1.065394, 0.811864, 0.685099, 0.609039, 0.558333]
List of powers for D(x): [1.825986, 1.065394, 0.811864, 0.685099, 0.609039, 0.558333]
Fitting 10^x in range (0.000000, 1.000000)
MSS of errors squared = 2.011185e-09
Corrected MSS of errors squared = 2.844246e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting 10^x in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	1.00000000013259	0	Inf	0
x1	-0.205580545455225	0	-Inf	0
x2	21.1287845848849	0	Inf	0
x3	-176.359067024344	0	-Inf	0
x4	493.464088285461	0	Inf	0
x5	-614.946132847315	0	-Inf	0
x6	350.878505921102	0	Inf	0
x7	-73.1467948151202	0	-Inf	0
x8	-0.46595683757024	0	-Inf	0
x9	17.5318225772412	0	Inf	0
x10	-29.0854813821837	0	-Inf	0
x11	12.8382352926064	0	Inf	0

```
Number of observations: 101, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```

	SumSq	DF	MeanSq	F	pValue


```
    Total       645.004627337404    100    6.45004627337404
    Model       645.004627337404     11    58.6367843034003    Inf       0
    Residual                   0     89                   0

A = 0.543869, B = 0.420425
order Q(x) = 7.000000, order D(x) = 4.000000
List of powers for Q(x): [0.964294, 1.138440, 1.272066, 1.384719, 1.483967, 1.573695,
1.656208]
List of powers for D(x): [0.964294, 1.138440, 1.272066, 1.384719]
Fitting 10^x in range (0.000000, 1.000000)
MSS of errors squared = 7.137793e-09
Corrected MSS of errors squared = 1.009436e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting 10^x in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     0.999999999992621    0      Inf        0
    x1              -357.298456108889    0     -Inf        0
    x2               1320.35231269768    0      Inf        0
    x3              -1446.47816604095    0     -Inf        0
    x4                             0    0      NaN      NaN
    x5               839.612757497386    0      Inf        0
    x6              -357.177821381107    0     -Inf        0
    x7                292.54324724564    0      Inf        0
    x8              -893.585957125411    0     -Inf        0
    x9               648.223948327099    0      Inf        0
    x10                            0    0      NaN      NaN
    x11              674.206217786515    0      Inf        0
    x12               -1222.808927549    0     -Inf        0
    x13              502.420408649047    0      Inf        0


Number of observations: 101, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                        SumSq          DF          MeanSq          F     pValue
                    ____________________    ___    ____________________    ___    ______
```



```
    Total       645.004627337235    100    6.45004627337235
    Model       645.004627337235     11     58.636784303385    Inf       0
    Residual                   0     89                   0

A = 0.151516, B = 0.100000, C = 0.669555
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.280727, 0.314903, 0.343076, 0.367607, 0.389624, 0.409770]
List of powers for D(x): [0.280727, 0.314903, 0.343076, 0.367607, 0.389624, 0.409770,
0.428455]
Fitting 10^x in range (0.000000, 1.000000)
MSS of errors squared = 5.093332e-08
Corrected MSS of errors squared = 8.821910e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting 10^x in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)      1.00000000012569    0      Inf       0
    x1               0.277457555648942   0      Inf       0
    x2               -1.62849521097402   0     -Inf       0
    x3                3.40718182264866   0      Inf       0
    x4               -2.38092502859325   0     -Inf       0
    x5                1.14533462158629   0      Inf       0
    x6               -0.280166276855088  0     -Inf       0
    x7                  1.7680905567585  0      Inf       0
    x8               0.284527992638486   0      Inf       0
    x9               -1.64837100206719   0     -Inf       0
    x10              0.788445111379962   0      Inf       0
    x11              -0.09458175271528   0     -Inf       0


Number of observations: 101, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF          MeanSq          F     pValue
                  _______________    ___    _______________    ___    ______
```

```
    Total       645.004627337434    100    6.45004627337434
    Model       645.004627337434     11    58.6367843034031    Inf       0
    Residual                   0     89                   0

A = 0.082618, B = 0.235787, C = 0.159722
order Q(x) = 5.000000, order D(x) = 6.000000
List of powers for Q(x): [0.318406, 0.713915, 1.109424, 1.504933, 1.900442]
List of powers for D(x): [0.318406, 0.713915, 1.109424, 1.504933, 1.900442, 2.295951]
Fitting 10^x in range (0.000000, 1.000000)
MSS of errors squared = 6.654980e-09
Corrected MSS of errors squared = 1.152676e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting 10^x in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ________________    ___    _____    ______

    (Intercept)    0.999999999996695    0      Inf       0
    x1              1412.87098839969    0      Inf       0
    x2             -9767.28620658573    0     -Inf       0
    x3              28659.3287131371    0      Inf       0
    x4             -45355.5303083513    0     -Inf       0
    x5              40563.7923285068    0      Inf       0
    x6             -19349.6229461457    0     -Inf       0
    x7              3835.69255778985    0      Inf       0
    x8             -100.036703604403    0     -Inf       0
    x9              312.614152394213    0      Inf       0
    x10            -317.122377881427    0     -Inf       0
    x11             105.520416412507    0      Inf       0


Number of observations: 101, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq          DF          MeanSq          F     pValue
                   ________________    ___    ________________    ___    ______

    Total       645.004627337111    100    6.45004627337111
    Model       645.004627337111     11    58.6367843033737    Inf       0
```



```
    Residual                  0     89                  0

A = 0.062238, B = 0.227133, C = 0.181501
order Q(x) = 7.000000, order D(x) = 4.000000
List of powers for Q(x): [0.289372, 0.335004, 0.380637, 0.426269, 0.471901, 0.517534,
0.563166]
List of powers for D(x): [0.289372, 0.335004, 0.380637, 0.426269]
Fitting 10^x in range (0.000000, 1.000000)
MSS of errors squared = 3.306578e-08
Corrected MSS of errors squared = 5.727161e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting 10^x in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   __________________    __    _____    ______

    (Intercept)    0.999999999999411     0      Inf       0
    x1             -4.63160196447215     0     -Inf       0
    x2              11.1941715276166     0      Inf       0
    x3             -13.0927319893983     0     -Inf       0
    x4              6.49446105943609     0      Inf       0
    x5              2.88994712201787     0      Inf       0
    x6             -5.16052805139882     0     -Inf       0
    x7              2.06466505333297     0      Inf       0
    x8              4.65027650591438     0      Inf       0
    x9             -11.4876780746352     0     -Inf       0
    x10             15.1920678159618     0      Inf       0
    x11            -9.83547857495268     0     -Inf       0
    x12             2.40497405303729     0      Inf       0


Number of observations: 101, Error degrees of freedom: 88
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq          DF           MeanSq           F      pValue
                   _______________    ___    _________________    ___    ______

    Total          645.00462733744    100     6.4500462733744
    Model          645.00462733744     12    53.7503856114533    Inf       0
```



```
    Residual                  0     88                    0

A = 0.149684, B = 0.229834, C = 0.000000
order Q(x) = 7.000000, order D(x) = 5.000000
List of powers for Q(x): [0.379518, 0.609352, 0.839186, 1.069020, 1.298853, 1.528687,
1.758521]
List of powers for D(x): [0.379518, 0.609352, 0.839186, 1.069020, 1.298853]
Fitting 10^x in range (0.000000, 1.000000)
MSS of errors squared = 2.848532e-09
Corrected MSS of errors squared = 4.933802e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting 10^x in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      1.00000000000021     0      Inf       0
    x1                7.35713660096297    0      Inf       0
    x2               -14.0708336097598    0     -Inf       0
    x3                20.7230249571311    0      Inf       0
    x4               -16.4514890446156    0     -Inf       0
    x5                10.9961419963106    0      Inf       0
    x6               -4.68484306807967    0     -Inf       0
    x7                1.70248398005301    0      Inf       0
    x8               -7.3627125413345     0     -Inf       0
    x9                14.7927387876326    0      Inf       0
    x10              -15.2072102018451    0     -Inf       0
    x11               25.4529263133997    0      Inf       0
    x12                 -31.1580406652    0     -Inf       0
    x13               17.3350924557674    0      Inf       0
    x14              -3.50995632967308    0     -Inf       0


Number of observations: 101, Error degrees of freedom: 86
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq          DF          MeanSq          F     pValue
                    ____________________    ___    ____________________    ___    ______
```



```
    Total       645.004627337419    100    6.45004627337419
    Model       645.004627337419     14      46.07175909553    Inf       0
    Residual                   0     86                   0

A = 0.311348, B = 0.196554, C = 0.290161
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.507903, 0.879479, 1.232112, 1.574939, 1.911500, 2.243609,
2.572346]
List of powers for D(x): [0.507903, 0.879479, 1.232112, 1.574939, 1.911500, 2.243609,
2.572346]
Fitting 10^x in range (0.000000, 1.000000)
MSS of errors squared = 2.361292e-11
Corrected MSS of errors squared = 4.089878e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting 10^x in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                  ________________    ___    _____    ______

    (Intercept)     1.00000000000218    0      Inf       0
    x1             -19.9643575173662    0     -Inf       0
    x2              196.185565642624    0      Inf       0
    x3             -1007.56278297722    0     -Inf       0
    x4              2681.41688622001    0      Inf       0
    x5             -3689.02926601697    0     -Inf       0
    x6              2499.68764685113    0      Inf       0
    x7             -661.980412530319    0     -Inf       0
    x8              16.2280442638961    0      Inf       0
    x9             -57.1423223751807    0     -Inf       0
    x10             66.3919442765943    0      Inf       0
    x11            -24.4529941307128    0     -Inf       0


Number of observations: 101, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq          F     pValue
                  ___________    ___    _______________    ___    ______
```


Total	645.004627337548	100	6.45004627337548		
Model	645.004627337548	11	58.6367843034135	Inf	0
Residual	0	89	0		

```
A = 0.119023, B = 0.207702, C = 0.085167
order Q(x) = 7.000000, order D(x) = 4.000000
List of powers for Q(x): [0.326725, 0.497925, 0.599217, 0.681940, 0.753792, 0.818226,
0.877166]
List of powers for D(x): [0.326725, 0.497925, 0.599217, 0.681940]
Fitting 10^x in range (0.000000, 1.000000)
MSS of errors squared = 2.642460e-08
Corrected MSS of errors squared = 4.576875e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting 10^x in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	0.999999999992749	0	Inf	0
x1	-2402.57230616887	0	-Inf	0
x2	3085.40449610539	0	Inf	0
x3	-847.091735319695	0	-Inf	0
x4	189.830046664802	0	Inf	0
x5	-28.9829026960168	0	-Inf	0
x6	2.4356750399256	0	Inf	0
x7	2245.21456866834	0	Inf	0
x8	-2866.80913430612	0	-Inf	0
x9	757.013931360307	0	Inf	0
x10	-149.495569163009	0	-Inf	0
x11	15.0738760783889	0	Inf	0

```
Number of observations: 101, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```

	SumSq	DF	MeanSq	F	pValue
Total	645.004627337106	100	6.45004627337106		


```
    Model       645.004627337106     11    58.6367843033733    Inf       0
    Residual                   0     89                   0

A = 0.332069, B = 0.916622
order Q(x) = 6.000000, order D(x) = 5.000000
List of powers for Q(x): [0.332069, 0.339596, 0.379570, 0.452503, 0.550857, 0.668151]
List of powers for D(x): [0.332069, 0.339596, 0.379570, 0.452503, 0.550857]
Fitting 10^x in range (0.000000, 1.000000)
MSS of errors squared = 3.566797e-08
Corrected MSS of errors squared = 5.044213e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the commn exponent function. The results are sorted in ascending order using the MSSE values.

Power	*A*	*B*	*C*	*F*	*AICc*	*MSSE*	*Q/D Orders*
A+B*sqrt(i)+C*(i-1)	0.311	0.196	0.29	Inf	-Inf	2.36129200E-11	7, 7
A+B*i	0.367	0.1		Inf	-Inf	1.16161000E-10	7, 7
A+B/i	0.304	1.521		Inf	-Inf	2.01118500E-09	6, 6
A+B*i+C*sqrt(i-1)	0.149	0.229	0	Inf	-Inf	2.84853200E-09	7, 5
A+B*i+C*(i-1)	0.082	0.235	0.159	Inf	-Inf	6.65498000E-09	5, 6
A+B*sqrt(i)	0.543	0.42		Inf	-Inf	7.13779300E-09	7, 4
A+B*sqrt(i)+C*sqrt(i-1)	0.119	0.207	0.085	Inf	-Inf	2.64246000E-08	7, 4
A+B*i-C*(i-1)	0.062	0.227	0.181	Inf	-Inf	3.30657800E-08	7, 4
A+B*log10(i)^4	0.332	0.916		Inf	-Inf	3.56679700E-08	6, 5
A+B*sqrt(i+C)	0.151	0.1	0.669	Inf	-Inf	5.09333200E-08	6, 7

RESULTS FOR THE SINE INTEGRAL

Using Power A+B* i

```
Fitting Si(x) in range (1.300000, 20.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE     tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -0.425556690751173    0      -Inf       0
    x1               2.47044534545918    0       Inf       0
    x2              -6.09782731177053    0      -Inf       0
    x3               8.30314362814464    0       Inf       0
    x4              -6.7405331930106     0      -Inf       0
    x5               3.26404532971158    0       Inf       0
    x6              -0.873329665332348   0      -Inf       0
    x7              0.0996294253655694   0       Inf       0
    x8               5.60579140933404    0       Inf       0
    x9              -13.4770435329002    0      -Inf       0
    x10              18.0113541933338    0       Inf       0
    x11             -14.4506048302733    0      -Inf       0
    x12              6.95960922946009    0       Inf       0
    x13             -1.86290822219788    0      -Inf       0
    x14              0.213783905679764   0       Inf       0


Number of observations: 188, Error degrees of freedom: 173
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F      pValue
                  ____________    ____    ____________________    ____    ______
```



```
    Total       2.24017998653672    187    0.0119795721205172
    Model       2.24017998653672     14     0.160012856181194    Inf       0
    Residual                   0    173                     0

A = 0.000000, B = 0.100000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
List of powers for D(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
Fitting Si(x) in range (1.300000, 20.000000)
MSS of errors squared = 3.594849e-04
Corrected MSS of errors squared = 5.083884e-04
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting Si(x) in range (1.300000, 20.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -0.0362976298383173     0     -Inf       0
    x1             -0.0188500025879627     0     -Inf       0
    x2                4.22116045657442     0      Inf       0
    x3               -61.6033608846453     0     -Inf       0
    x4                235.555828136321     0      Inf       0
    x5               -323.432628202178     0     -Inf       0
    x6                145.315118933014     0      Inf       0
    x7               0.132155986276034     0      Inf       0
    x8               -84.3564073151651     0     -Inf       0
    x9                2503.99480099526     0      Inf       0
    x10              -17837.3575910223     0     -Inf       0
    x11               47337.0829526069     0      Inf       0
    x12              -52140.0586995271     0     -Inf       0
    x13                20221.561764266     0      Inf       0


Number of observations: 188, Error degrees of freedom: 174
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```


	SumSq	DF	MeanSq	F	pValue
Total	2.24017998524394	187	0.011979572113604		
Model	2.24017998524394	13	0.172321537326457	Inf	0
Residual	0	174	0		

```
A = 0.410005, B = 0.784603
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [1.194608, 0.802307, 0.671540, 0.606156, 0.566926, 0.540773]
List of powers for D(x): [1.194608, 0.802307, 0.671540, 0.606156, 0.566926, 0.540773,
0.522092]
Fitting Si(x) in range (1.300000, 20.000000)
MSS of errors squared = 2.592084e-03
Corrected MSS of errors squared = 3.665760e-03
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting Si(x) in range (1.300000, 20.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	-0.0565126209889402	0	-Inf	0
x1	90.7959353724243	0	Inf	0
x2	-562.844459534331	0	-Inf	0
x3	1195.91877776555	0	Inf	0
x4	-969.91098829601	0	-Inf	0
x5	0	0	NaN	NaN
x6	395.55959345562	0	Inf	0
x7	-149.462277573786	0	-Inf	0
x8	4349.82650937993	0	Inf	0
x9	-39381.0334311569	0	-Inf	0
x10	133555.20682181	0	Inf	0
x11	-227292.266567069	0	-Inf	0
x12	209101.837648503	0	Inf	0
x13	-99740.2306687372	0	-Inf	0
x14	19407.6596148649	0	Inf	0

```
Number of observations: 188, Error degrees of freedom: 174
```



```
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       2.24017998820523    187    0.0119795721294397
    Model       2.24017998820523     13     0.172321537554248    Inf       0
    Residual                   0    174                     0

A = 0.240418, B = 0.100000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.340418, 0.381839, 0.413623, 0.440418, 0.464025, 0.485367,
0.504993]
List of powers for D(x): [0.340418, 0.381839, 0.413623, 0.440418, 0.464025, 0.485367,
0.504993]
Fitting Si(x) in range (1.300000, 20.000000)
MSS of errors squared = 2.737419e-03
Corrected MSS of errors squared = 3.871294e-03
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting Si(x) in range (1.300000, 20.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    ______    ______

    (Intercept)    0.0071018383288281    0       Inf         0
    x1             -0.506294960693741    0      -Inf         0
    x2               1.39048060716888    0       Inf         0
    x3              -1.30680626861986    0      -Inf         0
    x4              0.415534561690554    0       Inf         0
    x5               479.910832161863    0       Inf         0
    x6              -2246.77194714639    0      -Inf         0
    x7               3920.07060127484    0       Inf         0
    x8              -2738.63629927552    0      -Inf         0
    x9                             0    0       NaN       NaN
    x10              899.028380712612    0       Inf         0
    x11             -312.601584200232    0      -Inf         0


Number of observations: 188, Error degrees of freedom: 177
```



```
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       2.24017998652738    187    0.0119795721204673
    Model       2.24017998652738     10     0.224017998652738    Inf       0
    Residual                   0    177                      0

A = 0.000000, B = 0.100000, C = 2.000000
order Q(x) = 4.000000, order D(x) = 7.000000
List of powers for Q(x): [0.173205, 0.200000, 0.223607, 0.244949]
List of powers for D(x): [0.173205, 0.200000, 0.223607, 0.244949, 0.264575, 0.282843,
0.300000]
Fitting Si(x) in range (1.300000, 20.000000)
MSS of errors squared = 2.510348e-02
Corrected MSS of errors squared = 4.348051e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting Si(x) in range (1.300000, 20.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)    -0.427930999621684    0      -Inf       0
    x1               3.66170767389108    0       Inf       0
    x2              -10.1512202254442    0      -Inf       0
    x3               14.3756829940615    0       Inf       0
    x4              -11.8830014274407    0      -Inf       0
    x5               5.80740522922075    0       Inf       0
    x6              -1.56103804814469    0      -Inf       0
    x7              0.178432001047373    0       Inf       0
    x8               8.14716502100531    0       Inf       0
    x9              -21.9708640850488    0      -Inf       0
    x10              30.5058782336262    0       Inf       0
    x11             -24.8991551211206    0      -Inf       0
    x12              12.0934654792223    0       Inf       0
    x13             -3.25001178932425    0      -Inf       0
    x14             0.373482902243512    0       Inf       0
```



```
Number of observations: 188, Error degrees of freedom: 173
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ___________________    ___    ______

    Total       2.24017998653683    187    0.0119795721205178
    Model       2.24017998653683     14     0.160012856181202    Inf       0
    Residual                   0    173                     0

A = 0.029957, B = 0.102298, C = 0.000000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.132255, 0.234552, 0.336850, 0.439147, 0.541445, 0.643743,
0.746040]
List of powers for D(x): [0.132255, 0.234552, 0.336850, 0.439147, 0.541445, 0.643743,
0.746040]
Fitting Si(x) in range (1.300000, 20.000000)
MSS of errors squared = 3.431560e-04
Corrected MSS of errors squared = 5.943636e-04
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting Si(x) in range (1.300000, 20.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)    -0.428548323898187     0     -Inf       0
    x1              0.271845085890291     0      Inf       0
    x2             -2.36166226693914      0     -Inf       0
    x3              8.71050226303403      0      Inf       0
    x4             -17.6208958020763      0     -Inf       0
    x5              20.9618263112148      0      Inf       0
    x6             -14.3770076583104      0     -Inf       0
    x7              4.84399518407974      0      Inf       0
    x8             0.562128595059939      0      Inf       0
    x9             -4.86058462656482      0     -Inf       0
    x10             17.9401692450159      0      Inf       0
    x11            -36.5296925138925      0     -Inf       0
    x12             44.0162448330191      0      Inf       0
```


x13	-30.7901715943759	0	-Inf	0
x14	10.6618480833924	0	Inf	0

Number of observations: 188, Error degrees of freedom: 173
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	2.24017998653618	187	0.0119795721205143		
Model	2.24017998653618	14	0.160012856181155	Inf	0
Residual	0	173	0		

A = 0.614502, B = 0.148505, C = 0.250000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.763007, 0.661512, 0.560017, 0.458522, 0.357027, 0.255532, 0.154037]
List of powers for D(x): [0.763007, 0.661512, 0.560017, 0.458522, 0.357027, 0.255532, 0.154037]
Fitting Si(x) in range (1.300000, 20.000000)
MSS of errors squared = 3.357664e-04
Corrected MSS of errors squared = 5.815645e-04
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*i+C*sqrt(i–1)

Fitting Si(x) in range (1.300000, 20.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	-0.139448418746337	0	-Inf	0
x1	0.286066405585294	0	Inf	0
x2	-0.559987935085542	0	-Inf	0
x3	0.852230819364153	0	Inf	0
x4	-0.645966699276785	0	-Inf	0
x5	0.244460520577484	0	Inf	0
x6	-0.0369813693253091	0	-Inf	0
x7	1.97157588980792	0	Inf	0
x8	-4.03047480316782	0	-Inf	0
x9	7.10090159761033	0	Inf	0

x10	-6.78572384439967	0	-Inf	0
x11	3.69103134261753	0	Inf	0
x12	-1.08040792141346	0	-Inf	0
x13	0.132703591852692	0	Inf	0

Number of observations: 188, Error degrees of freedom: 174
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	2.24017998653667	187	0.011979572120517		
Model	2.24017998653667	13	0.172321537425898	Inf	0
Residual	0	174	0		

A = 0.000000, B = 0.100000, C = 0.213990
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.413990, 0.602627, 0.770641, 0.927980, 1.078496]
List of powers for D(x): [0.100000, 0.413990, 0.602627, 0.770641, 0.927980, 1.078496, 1.224166]
Fitting Si(x) in range (1.300000, 20.000000)
MSS of errors squared = 6.753705e-03
Corrected MSS of errors squared = 1.169776e-02
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*sqrt(i)+C*(i–1)

Fitting Si(x) in range (1.300000, 20.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	-0.422619684349003	0	-Inf	0
x1	50.5652027705901	0	Inf	0
x2	-331.979038581483	0	-Inf	0
x3	897.039485849604	0	Inf	0
x4	-1270.19699554263	0	-Inf	0
x5	997.535939286971	0	Inf	0
x6	-413.263635576388	0	-Inf	0
x7	70.7245043752207	0	Inf	0
x8	91.433827350713	0	Inf	0

x9	-577.234276717155	0	-Inf	0
x10	1525.48583411634	0	Inf	0
x11	-2129.98602927722	0	-Inf	0
x12	1657.60655289024	0	Inf	0
x13	-682.757051931212	0	-Inf	0
x14	116.448134418487	0	Inf	0

Number of observations: 188, Error degrees of freedom: 173
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	2.24017998661716	187	0.0119795721209474		
Model	2.24017998661716	14	0.16001285618694	Inf	0
Residual	0	173	0		

A = 0.156337, B = 0.336640, C = 0.000000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.492976, 0.632417, 0.739414, 0.829616, 0.909086, 0.980932, 1.047001]
List of powers for D(x): [0.492976, 0.632417, 0.739414, 0.829616, 0.909086, 0.980932, 1.047001]
Fitting Si(x) in range (1.300000, 20.000000)
MSS of errors squared = 2.427920e-04
Corrected MSS of errors squared = 4.205281e-04
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*sqrt(i)+C*sqrt(i–1)

Fitting Si(x) in range (1.300000, 20.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	-0.0887151026355799	0	-Inf	0
x1	1.33799990465402	0	Inf	0
x2	-11.547119421802	0	-Inf	0
x3	31.18904092885	0	Inf	0
x4	-37.7878149196761	0	-Inf	0
x5	21.7582941490811	0	Inf	0

x6	-4.86136110744107	0	-Inf	0
x7	16.7450639652319	0	Inf	0
x8	-193.385179689841	0	-Inf	0
x9	678.305628519121	0	Inf	0
x10	-1127.4322360645	0	-Inf	0
x11	997.116566141211	0	Inf	0
x12	-454.88149923794	0	-Inf	0
x13	84.5313138854043	0	Inf	0

Number of observations: 188, Error degrees of freedom: 174
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	2.2401799865264	187	0.011979572120462		
Model	2.2401799865264	13	0.172321537425108	Inf	0
Residual	0	174	0		

A = 0.154192, B = 0.107765, C = 0.146183
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.261957, 0.452778, 0.547581, 0.622919, 0.687529, 0.745037]
List of powers for D(x): [0.261957, 0.452778, 0.547581, 0.622919, 0.687529, 0.745037, 0.797386]
Fitting Si(x) in range (1.300000, 20.000000)
MSS of errors squared = 2.718253e-03
Corrected MSS of errors squared = 4.708152e-03
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*log10(i)^4

Fitting Si(x) in range (1.300000, 20.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	-0.118678567816826	0	-Inf	0
x1	617.590575952046	0	Inf	0
x2	-799.303750976383	0	-Inf	0
x3	227.546484828179	0	Inf	0
x4	-53.1705731495321	0	-Inf	0

x5	8.01895959287142	0	Inf	0
x6	-0.560910059962875	0	-Inf	0
x7	5142.04985927783	0	Inf	0
x8	-6711.88030268056	0	-Inf	0
x9	2012.45891649623	0	Inf	0
x10	-534.004558232708	0	-Inf	0
x11	104.531262652895	0	Inf	0
x12	-12.9006909947583	0	-Inf	0
x13	0.743288692601158	0	Inf	0

```
Number of observations: 188, Error degrees of freedom: 174
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```

	SumSq	DF	MeanSq	F	pValue
Total	2.24017998645108	187	0.0119795721200592		
Model	2.24017998645108	13	0.172321537419314	Inf	0
Residual	0	174	0		

```
A = 0.543952, B = 1.381114
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.543952, 0.555294, 0.615524, 0.725415, 0.873610, 1.050343]
List of powers for D(x): [0.543952, 0.555294, 0.615524, 0.725415, 0.873610, 1.050343,
1.248415]
Fitting Si(x) in range (1.300000, 20.000000)
MSS of errors squared = 5.395159e-03
Corrected MSS of errors squared = 7.629906e-03
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the sine Integral. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i)+C*(i-1)	0.156	0.336	0	Inf	-Inf	2.42792000E-04	7, 7
A+B*i-C*(i-1)	0.614	0.148	0.25	Inf	-Inf	3.35766400E-04	7, 7
A+B*i+C*(i-1)	0.029	0.102	0	Inf	-Inf	3.43156000E-04	7, 7
A+B*i	0	0.1		Inf	-Inf	3.59484900E-04	7, 7
A+B/i	0.41	0.784		Inf	-Inf	2.59208400E-03	6, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.154	0.107	0.146	Inf	-Inf	2.71825300E-03	6, 7
A+B*sqrt(i)	0.24	0.1		Inf	-Inf	2.73741900E-03	7, 7
A+B*log10(i)^4	0.543	1.381		Inf	-Inf	5.39515900E-03	6, 7
A+B*i+C*sqrt(i-1)	0	0.1	0.213	Inf	-Inf	6.75370500E-03	6, 7
A+B*sqrt(i+C)	0	0.1	2	Inf	-Inf	2.51034800E-02	4, 7

RESULTS FOR THE SINE FUNCTION

Using Power A+B* i

```
Fitting sin(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate          SE     tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -2.2068921216943e-11     0     -Inf       0
    x1              -0.0786726285995106     0     -Inf       0
    x2                0.918514374263889     0      Inf       0
    x3               -4.22728353259993      0     -Inf       0
    x4                9.43597675240397      0      Inf       0
    x5               -8.94182178158167      0     -Inf       0
    x6                2.93317433307807      0      Inf       0
    x7                70.854773913851       0      Inf       0
    x8               -259.835135548709      0     -Inf       0
    x9                406.72316034698       0      Inf       0
    x10              -334.554865969934      0     -Inf       0
    x11               142.853535045707      0      Inf       0
    x12              -25.0888699196799      0     -Inf       0


Number of observations: 501, Error degrees of freedom: 488
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       47.6265936125749     500    0.0952531872251497
    Model       47.6265936125749      12      3.96888280104791     Inf       0
```



```
    Residual                   0    488                    0

A = 0.418401, B = 0.111637
order Q(x) = 6.000000, order D(x) = 6.000000
List of powers for Q(x): [0.530037, 0.641674, 0.753311, 0.864947, 0.976584, 1.088220]
List of powers for D(x): [0.530037, 0.641674, 0.753311, 0.864947, 0.976584, 1.088220]
Fitting sin(x) in range (0.000000, 1.570796)
MSS of errors squared = 1.810734e-09
Corrected MSS of errors squared = 2.560765e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting sin(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                         Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -4.07284607614182e-12    0     -Inf        0
    x1                0.0480955893501182    0      Inf        0
    x2                 -1.97611116459285    0     -Inf        0
    x3                  15.2301840347718    0      Inf        0
    x4                 -41.6917226452856    0     -Inf        0
    x5                  46.6029398659128    0      Inf        0
    x6                 -18.2129015575006    0     -Inf        0
    x7                 -1.48933899471062    0     -Inf        0
    x8                   185.77464477222    0      Inf        0
    x9                 -2185.14957870701    0     -Inf        0
    x10                 6934.27867233445    0      Inf        0
    x11                -6869.07571581975    0     -Inf        0
    x12                                0    0      NaN      NaN
    x13                 1936.66074108584    0      Inf        0


Number of observations: 501, Error degrees of freedom: 488
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F     pValue
                  _______________    ___    ____________________    ___    ______

    Total       47.6265936130762     500    0.0952531872261523
    Model       47.6265936130762      12      3.96888280108968    Inf       0
```



```
    Residual                  0    488                    0

A = 0.198937, B = 0.350751
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.549688, 0.374312, 0.315854, 0.286625, 0.269087, 0.257395]
List of powers for D(x): [0.549688, 0.374312, 0.315854, 0.286625, 0.269087, 0.257395,
0.249044]
Fitting sin(x) in range (0.000000, 1.570796)
MSS of errors squared = 1.543643e-09
Corrected MSS of errors squared = 2.183041e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting sin(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                      ____________________    __    _____    ______

    (Intercept)    -3.14907506432789e-12     0     -Inf       0
    x1               -0.0471789918681063     0     -Inf       0
    x2                  0.47232572523849     0      Inf       0
    x3                 -1.57994056124427     0     -Inf       0
    x4                  2.39182240933276     0      Inf       0
    x5                 -1.69650576011482     0     -Inf       0
    x6                 0.459491044367077     0      Inf       0
    x7                  44.2484667018629     0      Inf       0
    x8                 -249.101739077195     0     -Inf       0
    x9                  575.804149397217     0      Inf       0
    x10                -660.344659250549     0     -Inf       0
    x11                 374.443886847244     0      Inf       0
    x12                -84.0501210965179     0     -Inf       0


Number of observations: 501, Error degrees of freedom: 488
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq          DF           MeanSq           F     pValue
                   _______________    ___    ____________________    ___    ______

    Total       47.6265936126044     500    0.0952531872252087
    Model       47.6265936126044      12      3.96888280105036    Inf       0
```



```
    Residual                   0    488                    0

A = 0.000000, B = 0.100000
order Q(x) = 6.000000, order D(x) = 6.000000
List of powers for Q(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949]
Fitting sin(x) in range (0.000000, 1.570796)
MSS of errors squared = 4.219952e-09
Corrected MSS of errors squared = 5.967913e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting sin(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -2.67854229987735e-09    0     -Inf        0
    x1                 -259.038151201195    0     -Inf        0
    x2                  1737.03175181744    0      Inf        0
    x3                 -3323.95295008896    0     -Inf        0
    x4                                 0    0      NaN      NaN
    x5                  6604.04380417211    0      Inf        0
    x6                 -7042.17895975531    0     -Inf        0
    x7                  2284.13800689623    0      Inf        0
    x8                  37005.8595848865    0      Inf        0
    x9                 -215413.958439786    0     -Inf        0
    x10                 444396.776377346    0      Inf        0
    x11                -355802.47954155     0     -Inf        0
    x12                                0    0      NaN      NaN
    x13                 144839.335873703    0      Inf        0
    x14                -55024.5855519719    0     -Inf        0


Number of observations: 501, Error degrees of freedom: 488
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______

    Total         47.6265936448717    500    0.0952531872897434
```


Model	47.6265936448717	12	3.96888280373931	Inf	0
Residual	0	488	0		

A = 0.589351, B = 0.100000, C = 0.974335
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.729862, 0.761813, 0.788708, 0.812383, 0.833775, 0.853440, 0.871739]
List of powers for D(x): [0.729862, 0.761813, 0.788708, 0.812383, 0.833775, 0.853440, 0.871739]
Fitting sin(x) in range (0.000000, 1.570796)
MSS of errors squared = 1.331523e-09
Corrected MSS of errors squared = 2.306265e-09
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*i+C*(i–1)

Fitting sin(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	2.5406877295714e-12	0	Inf	0
x1	-0.00456339287145751	0	-Inf	0
x2	0.066402697011615	0	Inf	0
x3	-0.410874962575317	0	-Inf	0
x4	1.37888046285497	0	Inf	0
x5	-2.36242772583948	0	-Inf	0
x6	1.96910229125777	0	Inf	0
x7	-0.622641940764061	0	-Inf	0
x8	3.13738365163391	0	Inf	0
x9	-4.2101243189536	0	-Inf	0
x10	2.74124266032682	0	Inf	0
x11	-0.684993856416449	0	-Inf	0

Number of observations: 501, Error degrees of freedom: 489
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	47.6265936125817	500	0.0952531872251634		


```
    Model       47.6265936125817     11      4.32969032841652    Inf       0
    Residual                   0    489                     0

A = 0.020432, B = 0.128153, C = 0.059953
order Q(x) = 7.000000, order D(x) = 4.000000
List of powers for Q(x): [0.148585, 0.336691, 0.524797, 0.712903, 0.901009, 1.089115,
1.277221]
List of powers for D(x): [0.148585, 0.336691, 0.524797, 0.712903]
Fitting sin(x) in range (0.000000, 1.570796)
MSS of errors squared = 1.155031e-08
Corrected MSS of errors squared = 2.000573e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting sin(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                    ____________________    ___    _____    ______

    (Intercept)    -6.09498399547333e-14     0     -Inf       0
    x1                  4.66444317192252     0      Inf       0
    x2                 -17.5169753003226     0     -Inf       0
    x3                  25.4157184629337     0      Inf       0
    x4                 -17.8881356933832     0     -Inf       0
    x5                  6.17578102427117     0      Inf       0
    x6                -0.843586867472919     0     -Inf       0
    x7                 -97.9291676831983     0     -Inf       0
    x8                  549.693287612716     0      Inf       0
    x9                 -1248.56797079592     0     -Inf       0
    x10                 1431.65656467397     0      Inf       0
    x11                -818.306115292348     0     -Inf       0
    x12                 170.168397152863     0      Inf       0
    x13                 14.2763946492685     0      Inf       0


Number of observations: 501, Error degrees of freedom: 487
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F     pValue
                  ___________________    ___    ____________________    ___    ______
```



```
    Total       47.6265936125695    500    0.0952531872251391
    Model       47.6265936125695     13      3.66358412404381    Inf       0
    Residual                   0    487                     0

A = 0.552144, B = 0.183414, C = 0.250000
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.735558, 0.668971, 0.602385, 0.535799, 0.469212, 0.402626]
List of powers for D(x): [0.735558, 0.668971, 0.602385, 0.535799, 0.469212, 0.402626,
0.336040]
Fitting sin(x) in range (0.000000, 1.570796)
MSS of errors squared = 1.060105e-09
Corrected MSS of errors squared = 1.836156e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting sin(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -4.61593454590514e-11    0     -Inf       0
    x1              -0.00107618976802544    0     -Inf       0
    x2                 0.112562716002555    0      Inf       0
    x3                -0.888145644273418    0     -Inf       0
    x4                  7.41892096518527    0      Inf       0
    x5                 -9.36552534972946    0     -Inf       0
    x6                  3.16934850914049    0      Inf       0
    x7                 -4.48442915541758    0     -Inf       0
    x8                   41.837965879641    0      Inf       0
    x9                 -124.801719170171    0     -Inf       0
    x10                 193.691512782958    0      Inf       0
    x11                -169.243695229242    0     -Inf       0
    x12                 78.5258358138599    0      Inf       0
    x13                -15.0555961615268    0     -Inf       0


Number of observations: 501, Error degrees of freedom: 487
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F     pValue
                  ___________    ___    ____________________    ___    ______
```



```
    Total       47.6265936125845    500    0.0952531872251691
    Model       47.6265936125845     13      3.66358412404496    Inf       0
    Residual                   0    487                     0

A = 0.229565, B = 0.115339, C = 0.261605
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.344904, 0.721848, 0.945547, 1.144034, 1.329470, 1.506565]
List of powers for D(x): [0.344904, 0.721848, 0.945547, 1.144034, 1.329470, 1.506565,
1.677736]
Fitting sin(x) in range (0.000000, 1.570796)
MSS of errors squared = 1.906576e-10
Corrected MSS of errors squared = 3.302286e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting sin(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -1.828495198642e-13    0     -Inf        0
    x1              -0.149023965963171    0     -Inf        0
    x2                1.49000326710088    0      Inf        0
    x3               -4.97879577656448    0     -Inf        0
    x4                7.53004738177976    0      Inf        0
    x5               -5.33625035384771    0     -Inf        0
    x6                1.44406395425349    0      Inf        0
    x7                67.3795677506176    0      Inf        0
    x8               -317.698477410629    0     -Inf        0
    x9                467.210593603629    0      Inf        0
    x10                              0    0      NaN      NaN
    x11              -616.684176599343    0     -Inf        0
    x12               555.665310392174    0      Inf        0
    x13              -154.872870628056    0     -Inf        0


Number of observations: 501, Error degrees of freedom: 488
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
```


Total	47.6265936125856	500	0.0952531872251713		
Model	47.6265936125856	12	3.9688828010488	Inf	0
Residual	0	488	0		

A = 0.028847, B = 0.100000, C = 0.000000
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.128847, 0.170268, 0.202052, 0.228847, 0.252453, 0.273796]
List of powers for D(x): [0.128847, 0.170268, 0.202052, 0.228847, 0.252453, 0.273796, 0.293422]
Fitting sin(x) in range (0.000000, 1.570796)
MSS of errors squared = 3.971808e-09
Corrected MSS of errors squared = 6.879374e-09
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*sqrt(i)+C*sqrt(i–1)

Fitting sin(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	4.29086873281323e-11	0	Inf	0
x1	-2.90745588789456	0	-Inf	0
x2	30.8393070576553	0	Inf	0
x3	-107.798207270076	0	-Inf	0
x4	169.258530075151	0	Inf	0
x5	-123.893325869626	0	-Inf	0
x6	34.5026203532957	0	Inf	0
x7	730.115032591176	0	Inf	0
x8	-5003.25121012425	0	-Inf	0
x9	12798.7746751484	0	Inf	0
x10	-15677.9978335909	0	-Inf	0
x11	9325.59465856097	0	Inf	0
x12	-2172.23706769434	0	-Inf	0

Number of observations: 501, Error degrees of freedom: 488
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue

Total	47.6265936135195	500	0.095253187227039		
Model	47.6265936135195	12	3.96888280112663	Inf	0
Residual	0	488	0		

```
A = 0.194365, B = 0.117978, C = 0.000000
order Q(x) = 6.000000, order D(x) = 6.000000
List of powers for Q(x): [0.312343, 0.361211, 0.398709, 0.430321, 0.458172, 0.483351]
List of powers for D(x): [0.312343, 0.361211, 0.398709, 0.430321, 0.458172, 0.483351]
Fitting sin(x) in range (0.000000, 1.570796)
MSS of errors squared = 9.186803e-10
Corrected MSS of errors squared = 1.591201e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting sin(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	1.19973162008208e-15	0	Inf	0
x1	0	0	NaN	NaN
x2	-4.10789173517263e-08	0	-Inf	0
x3	1.07064958573034e-07	0	Inf	0
x4	-1.2611103132514e-07	0	-Inf	0
x5	9.19889585737795e-08	0	Inf	0
x6	-3.94688204233746e-08	0	-Inf	0
x7	7.60021450265251e-09	0	Inf	0
x8	1.00000052061521	0	Inf	0
x9	-6.54396335732349e-07	0	-Inf	0
x10	1.53543629390029e-07	0	Inf	0
x11	-1.97569975877017e-08	0	-Inf	0

```
Number of observations: 501, Error degrees of freedom: 490
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```

	SumSq	DF	MeanSq	F	pValue


```
    Total       47.6265936125817    500    0.0952531872251634
    Model       47.6265936125817     10      4.76265936125817    Inf       0
    Residual                   0    490                     0

A = 0.000000, B = 0.413912
order Q(x) = 7.000000, order D(x) = 4.000000
List of powers for Q(x): [0.000000, 0.003399, 0.021450, 0.054383, 0.098797, 0.151762,
0.211123]
List of powers for D(x): [0.000000, 0.003399, 0.021450, 0.054383]
Fitting sin(x) in range (0.000000, 1.570796)
MSS of errors squared = 2.888362e-06
Corrected MSS of errors squared = 4.084760e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the sine function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i+C*sqrt(i-1)	0.229	0.115	0.261	Inf	-Inf	1.90657600E-10	6, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.194	0.117	0	Inf	-Inf	9.18680300E-10	6, 6
A+B*i-C*(i-1)	0.552	0.183	0.25	Inf	-Inf	1.06010500E-09	6, 7
A+B*sqrt(i+C)	0.589	0.1	0.974	Inf	-Inf	1.33152300E-09	7, 7
A+B/i	0.198	0.35		Inf	-Inf	1.54364300E-09	6, 7
A+B*i	0.418	0.111		Inf	-Inf	1.81073400E-09	6, 6
A+B*sqrt(i)+C*(i-1)	0.028	0.1	0	Inf	-Inf	3.97180800E-09	6, 7
A+B*sqrt(i)	0	0.1		Inf	-Inf	4.21995200E-09	6, 6
A+B*i+C*(i-1)	0.02	0.128	0.059	Inf	-Inf	1.15503100E-08	7, 4
A+B*log10(i)^4	0	0.413		Inf	-Inf	2.88836200E-06	7, 4

RESULTS FOR THE HYPERBOLIC SINE FUNCTION

Using Power A+B* i

```
Fitting sinh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -8.04819262586465e-09    0     -Inf       0
    x1                0.0284259824278277    0      Inf       0
    x2                  0.90847364344029    0      Inf       0
    x3                  1.39968346705291    0      Inf       0
    x4                 -7.27624960196728    0     -Inf       0
    x5                  8.66784724901867    0      Inf       0
    x6                 -4.37894885464301    0     -Inf       0
    x7                 0.845244764048435    0      Inf       0
    x8                  9.12126046929743    0      Inf       0
    x9                  -21.651089380892    0     -Inf       0
    x10                 24.0352397972057    0      Inf       0
    x11                -15.4626475156985    0     -Inf       0
    x12                 5.95696399905382    0      Inf       0
    x13                -1.28556497112159    0     -Inf       0
    x14                0.120353923082118    0      Inf       0


Number of observations: 501, Error degrees of freedom: 486
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq          F      pValue
                ___________________    ___    ____________________    ___    ______
```



```
    Total       169774.232792751    500    339.548465585502
    Model       169774.232792751     14    12126.7309137679    Inf       0
    Residual                   0    486                   0

A = 0.500000, B = 0.262131
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.762131, 1.024261, 1.286392, 1.548522, 1.810653, 2.072783,
2.334914]
List of powers for D(x): [0.762131, 1.024261, 1.286392, 1.548522, 1.810653, 2.072783,
2.334914]
Fitting sinh(x) in range (0.000000, 5.000000)
MSS of errors squared = 2.597451e-08
Corrected MSS of errors squared = 3.673351e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting sinh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -1.92907107370706e-09    0     -Inf        0
    x1                -0.238283703757263    0     -Inf        0
    x2                   18.221282662506    0      Inf        0
    x3                  -166.879689832552    0     -Inf        0
    x4                   493.68934100947    0      Inf        0
    x5                  -576.383585376355    0     -Inf        0
    x6                   231.60209218042    0      Inf        0
    x7                  -1.85162047695607    0     -Inf        0
    x8                   295.530152441026    0      Inf        0
    x9                  -3693.95614771385    0     -Inf        0
    x10                  12010.0324589673    0      Inf        0
    x11                 -12041.4850579571    0     -Inf        0
    x12                                 0    0      NaN      NaN
    x13                  3432.72072104958    0      Inf        0


Number of observations: 501, Error degrees of freedom: 488
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```


	SumSq	DF	MeanSq	F	pValue
Total	169774.23279267	500	339.54846558534		
Model	169774.23279267	12	14147.8527327225	Inf	0
Residual	0	488	0		

```
A = 0.420536, B = 0.502244
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.922780, 0.671658, 0.587950, 0.546097, 0.520985, 0.504243]
List of powers for D(x): [0.922780, 0.671658, 0.587950, 0.546097, 0.520985, 0.504243,
0.492285]
Fitting sinh(x) in range (0.000000, 5.000000)
MSS of errors squared = 2.138926e-06
Corrected MSS of errors squared = 3.024898e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting sinh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	9.90037219459727e-10	0	Inf	0
x1	26.2408823790982	0	Inf	0
x2	-184.88336641205	0	-Inf	0
x3	349.749257577897	0	Inf	0
x4	0	0	NaN	NaN
x5	-638.613990360631	0	-Inf	0
x6	645.171191493701	0	Inf	0
x7	-197.653449585757	0	-Inf	0
x8	1488.59185353406	0	Inf	0
x9	-9015.49496798335	0	-Inf	0
x10	18821.1915929097	0	Inf	0
x11	-15043.5367155193	0	-Inf	0
x12	0	0	NaN	NaN
x13	5991.56920109444	0	Inf	0
x14	-2241.32992006868	0	-Inf	0

```
Number of observations: 501, Error degrees of freedom: 488
```



```
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq          F     pValue
                ___________________    ___    ___________________    ___    ______

    Total       169774.232792852    500    339.548465585704
    Model       169774.232792852     12    14147.8527327377    Inf       0
    Residual                   0    488                   0

A = 0.362740, B = 0.125652
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.488392, 0.540439, 0.580376, 0.614045, 0.643707, 0.670524,
0.695185]
List of powers for D(x): [0.488392, 0.540439, 0.580376, 0.614045, 0.643707, 0.670524,
0.695185]
Fitting sinh(x) in range (0.000000, 5.000000)
MSS of errors squared = 2.372601e-06
Corrected MSS of errors squared = 3.355365e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting sinh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    1.43301029734834e-10    0      Inf        0
    x1               -0.514209690782541    0     -Inf        0
    x2                 3.72438497947748    0      Inf        0
    x3                -7.52350460348438    0     -Inf        0
    x4                                0    0      NaN      NaN
    x5                 16.0715801517014    0      Inf        0
    x6                -17.5848605327166    0     -Inf        0
    x7                 5.82696098236354    0      Inf        0
    x8                 218.443772020217    0      Inf        0
    x9                -999.017075047189    0     -Inf        0
    x10                1714.85602320179    0      Inf        0
    x11               -1181.62595481345    0     -Inf        0
    x12                               0    0      NaN      NaN
    x13                378.29228547282    0      Inf        0
    x14               -129.949349749537    0     -Inf        0
```



```
Number of observations: 501, Error degrees of freedom: 488
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq          F     pValue
                ________________    ___    ________________    ___    ______

    Total       169774.232792816    500    339.548465585632
    Model       169774.232792816     12    14147.8527327347    Inf       0
    Residual                   0    488                   0

A = 0.000000, B = 0.126605, C = 1.391221
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.195777, 0.233147, 0.265305, 0.293965, 0.320070, 0.344200,
0.366746]
List of powers for D(x): [0.195777, 0.233147, 0.265305, 0.293965, 0.320070, 0.344200,
0.366746]
Fitting sinh(x) in range (0.000000, 5.000000)
MSS of errors squared = 3.661159e-06
Corrected MSS of errors squared = 6.341313e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting sinh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -4.35357358119317e-10    0     -Inf       0
    x1               0.00166065911476189    0      Inf       0
    x2              -0.0315496789069003     0     -Inf       0
    x3                 0.32504830068097     0      Inf       0
    x4                -0.633767753187719    0     -Inf       0
    x5                 0.528787130217626    0      Inf       0
    x6                -0.206056548138178    0     -Inf       0
    x7                 0.0309749826754234   0      Inf       0
    x8                     2.073439410235   0      Inf       0
    x9                -2.03498639698789     0     -Inf       0
    x10                1.46100883797489     0      Inf       0
    x11               -0.665908502978266    0     -Inf       0
```



```
    x12                0.172962141280727    0      Inf       0
    x13               -0.0193618571810924   0     -Inf       0


Number of observations: 501, Error degrees of freedom: 487
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq          F     pValue
                ________________    ___    ________________    ___    ______

    Total       169774.232792744    500    339.548465585487
    Model       169774.232792744     13    13059.5563686726    Inf       0
    Residual                   0    487                   0

A = 0.011295, B = 0.164589, C = 0.137780
order Q(x) = 7.000000, order D(x) = 6.000000
List of powers for Q(x): [0.175884, 0.478253, 0.780622, 1.082991, 1.385360, 1.687729,
1.990098]
List of powers for D(x): [0.175884, 0.478253, 0.780622, 1.082991, 1.385360, 1.687729]
Fitting sinh(x) in range (0.000000, 5.000000)
MSS of errors squared = 1.812801e-06
Corrected MSS of errors squared = 3.139863e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting sinh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ________________    ___    _____    ______

    (Intercept)      -0.0712970143411907    0     -Inf       0
    x1              -0.000803662258734649   0     -Inf       0
    x2                0.00542547104391039   0      Inf       0
    x3               -0.0180176074942059    0     -Inf       0
    x4                0.0612303147052082    0      Inf       0
    x5                0.0260555030768531    0      Inf       0
    x6               -0.00280146477883772   0     -Inf       0
    x7                0.000208571110969221  0      Inf       0
    x8               -0.00341550337170804   0     -Inf       0
    x9                0.0318197778277133    0      Inf       0
    x10              -0.148967044866651     0     -Inf       0
```


x11	0.729326999581122	0	Inf	0
x12	0.456367163526859	0	Inf	0
x13	-0.0734192259578145	0	-Inf	0
x14	0.00828773875423669	0	Inf	0

Number of observations: 500, Error degrees of freedom: 485
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	169558.095142868	499	339.795781849435		
Model	169558.095142868	14	12111.2925102049	Inf	0
Residual	0	485	0		

A = 0.227919, B = 0.121652, C = 0.223568
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.349571, 0.247654, 0.145738, 0.043821, -0.058096, -0.160013, -0.261929]
List of powers for D(x): [0.349571, 0.247654, 0.145738, 0.043821, -0.058096, -0.160013, -0.261929]
Fitting sinh(x) in range (0.000000, 5.000000)
MSS of errors squared = NaN
Corrected MSS of errors squared = NaN
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*i+C*sqrt(i–1)

Fitting sinh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	-9.34839999611049e-12	0	-Inf	0
x1	-0.0086557594887174	0	-Inf	0
x2	0.183290890210494	0	Inf	0
x3	-1.0837724368987	0	-Inf	0
x4	2.86892609958523	0	Inf	0
x5	-3.71680281779312	0	-Inf	0
x6	2.31634717252426	0	Inf	0
x7	-0.558358844988467	0	-Inf	0

x8	14.9089562089785	0	Inf	0
x9	-41.9856693129729	0	-Inf	0
x10	53.5900110980648	0	Inf	0
x11	-36.8038144933824	0	-Inf	0
x12	13.2725457069987	0	Inf	0
x13	-1.98285824620336	0	-Inf	0

Number of observations: 501, Error degrees of freedom: 487
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	169774.232792734	500	339.548465585467		
Model	169774.232792734	13	13059.5563686718	Inf	0
Residual	0	487	0		

A = 0.125750, B = 0.118460, C = 0.000000
order Q(x) = 7.000000, order D(x) = 6.000000
List of powers for Q(x): [0.244210, 0.362670, 0.481130, 0.599590, 0.718050, 0.836511, 0.954971]
List of powers for D(x): [0.244210, 0.362670, 0.481130, 0.599590, 0.718050, 0.836511]
Fitting sinh(x) in range (0.000000, 5.000000)
MSS of errors squared = 1.245598e-05
Corrected MSS of errors squared = 2.157439e-05
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*sqrt(i)+C*(i–1)

Fitting sinh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	-4.44501828590081e-09	0	-Inf	0
x1	0.984184338042706	0	Inf	0
x2	-14.2277513878998	0	-Inf	0
x3	61.8570216062893	0	Inf	0
x4	-109.338636956053	0	-Inf	0
x5	85.4351817572556	0	Inf	0
x6	-24.6646438737787	0	-Inf	0


```
    x7                 67.4575336674504    0      Inf       0
    x8                -364.828453683656    0     -Inf       0
    x9                 820.133859976664    0      Inf       0
    x10               -950.806907744706    0     -Inf       0
    x11                588.195454907141    0      Inf       0
    x12               -178.347300746279    0     -Inf       0
    x13                19.1572199457628    0      Inf       0


Number of observations: 501, Error degrees of freedom: 487
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq          F     pValue
                ________________    ___    ________________    ___    ______

    Total       169774.232793031    500    339.548465586062
    Model       169774.232793031     13    13059.5563686947    Inf       0
    Residual                   0    487                   0

A = 0.275328, B = 0.400000, C = 0.000000
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.675328, 0.841013, 0.968148, 1.075328, 1.169755, 1.255123]
List of powers for D(x): [0.675328, 0.841013, 0.968148, 1.075328, 1.169755, 1.255123,
1.333628]
Fitting sinh(x) in range (0.000000, 5.000000)
MSS of errors squared = 9.914760e-05
Corrected MSS of errors squared = 1.717287e-04
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting sinh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -1.46761488295846e-10    0     -Inf       0
    x1                 0.119995047481029    0      Inf       0
    x2                 -3.80435839872946    0     -Inf       0
    x3                  25.9162975172607    0      Inf       0
    x4                 -70.5925208536264    0     -Inf       0
    x5                  92.0300638087154    0      Inf       0
```


x6	-57.7631198644904	0	-Inf	0
x7	14.0785712757142	0	Inf	0
x8	22.58522328193	0	Inf	0
x9	-157.625548472424	0	-Inf	0
x10	451.254296507553	0	Inf	0
x11	-670.48067136442	0	-Inf	0
x12	551.941325483276	0	Inf	0
x13	-239.61015958184	0	-Inf	0
x14	42.9483587646335	0	Inf	0

Number of observations: 501, Error degrees of freedom: 486
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	169774.232792983	500	339.548465585967		
Model	169774.232792983	14	12126.7309137845	Inf	0
Residual	0	486	0		

A = 0.000000, B = 0.435429, C = 0.070683
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.435429, 0.686473, 0.854146, 0.993285, 1.115015, 1.224632, 1.325175]
List of powers for D(x): [0.435429, 0.686473, 0.854146, 0.993285, 1.115015, 1.224632, 1.325175]
Fitting sinh(x) in range (0.000000, 5.000000)
MSS of errors squared = 1.624160e-06
Corrected MSS of errors squared = 2.813127e-06
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*log10(i)^4

Fitting sinh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	-7.30838063413437e-13	0	-Inf	0
x1	0.0086821829159018	0	Inf	0
x2	-0.0128203910903591	0	-Inf	0

x3	0.00713941145524896	0	Inf	0
x4	-0.00345192794628011	0	-Inf	0
x5	0.000766748251322857	0	Inf	0
x6	5.0456771773669	0	Inf	0
x7	-4.44501277745912	0	-Inf	0
x8	0.437489196468936	0	Inf	0
x9	-0.0413157623563869	0	-Inf	0
x10	0.00299934519047505	0	Inf	0
x11	-0.00010609139438882	0	-Inf	0

```
Number of observations: 501, Error degrees of freedom: 489
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```

	SumSq	DF	MeanSq	F	pValue
Total	169774.232792744	500	339.548465585487		
Model	169774.232792744	11	15434.0211629767	Inf	0
Residual	0	489	0		

```
A = 0.065405, B = 3.500000
order Q(x) = 5.000000, order D(x) = 6.000000
List of powers for Q(x): [0.065405, 0.094147, 0.246783, 0.525267, 0.900820]
List of powers for D(x): [0.065405, 0.094147, 0.246783, 0.525267, 0.900820, 1.348693]
Fitting sinh(x) in range (0.000000, 5.000000)
MSS of errors squared = 2.687631e-06
Corrected MSS of errors squared = 3.800884e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the hyperbolic sine function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i	0.5	0.262		Inf	-Inf	2.59745100E-08	7, 7
A+B*sqrt(i)+C*sqrt(i-1)	0	0.435	0.07	Inf	-Inf	1.62416000E-06	7, 7
A+B*i+C*(i-1)	0.011	0.164	0.137	Inf	-Inf	1.81280100E-06	7, 6
A+B/i	0.42	0.502		Inf	-Inf	2.13892600E-06	6, 7
A+B*sqrt(i)	0.362	0.125		Inf	-Inf	2.37260100E-06	7, 7
A+B*log10(i)^4	0.065	3.5		Inf	-Inf	2.68763100E-06	5, 6
A+B*sqrt(i+C)	0	0.126	1.391	Inf	-Inf	3.66115900E-06	7, 7
A+B*i+C*sqrt(i-1)	0.125	0.118	0	Inf	-Inf	1.24559800E-05	7, 6
A+B*sqrt(i)+C*(i-1)	0.275	0.4	0	Inf	-Inf	9.91476000E-05	6, 7
A+B*i-C*(i-1)	0.227	0.121	0.223	Inf	-Inf	NaN	7, 7

RESULTS FOR THE TANGENT FUNCTION

Using Power A+B* i

```
Fitting tan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE     tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    8.20139078885225e-14     0      Inf        0
    x1             -0.00160323800075477     0     -Inf        0
    x2                0.865474573582817     0      Inf        0
    x3                0.515099050278576     0      Inf        0
    x4                -2.93139588936604     0     -Inf        0
    x5                 6.22384588304203     0      Inf        0
    x6                -3.83855672112883     0     -Inf        0
    x7                0.479290390486426     0      Inf        0
    x8                0.517216524295003     0      Inf        0
    x9                 1.48481617942837     0      Inf        0
    x10               -10.0903358630299     0     -Inf        0
    x11                16.3713197655651     0      Inf        0
    x12               -14.6307704827087     0     -Inf        0
    x13                8.66140517893234     0      Inf        0
    x14               -2.15617572832214     0     -Inf        0


Number of observations: 101, Error degrees of freedom: 86
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq            F      pValue
                ___________    ____    ____________________    ____    ______
```



```
    Total       18.4802838293816    100    0.184802838293816
    Model       18.4802838293816     14     1.32002027352726    Inf       0
    Residual                   0     86                    0

A = 0.236878, B = 0.372100
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.608979, 0.981079, 1.353179, 1.725280, 2.097380, 2.469481,
2.841581]
List of powers for D(x): [0.608979, 0.981079, 1.353179, 1.725280, 2.097380, 2.469481,
2.841581]
Fitting tan(x) in range (0.000000, 1.000000)
MSS of errors squared = 7.204837e-12
Corrected MSS of errors squared = 1.018918e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting tan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      7.1796544970202e-13    0      Inf       0
    x1               -3.4771467654824e-05    0     -Inf       0
    x2                7.0429316581369e-05    0      Inf       0
    x3               -2.11087025795872e-05    0     -Inf       0
    x4                0.00127729573237577    0      Inf       0
    x5                -0.634407618152805    0     -Inf       0
    x6                  19.0746584436793    0      Inf       0
    x7                 -146.005635377334    0     -Inf       0
    x8                  421.130489872223    0      Inf       0
    x9                 -503.54501487484    0     -Inf       0
    x10                210.978622916788    0      Inf       0


Number of observations: 101, Error degrees of freedom: 90
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                  ___________    ___    ___________________    ___    ______
```



```
    Total       18.4802838293815    100    0.184802838293815
    Model       18.4802838293815     10     1.84802838293815    Inf       0
    Residual                   0     90                    0

A = 0.011415, B = 0.655309
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.666724, 0.339069, 0.229851]
List of powers for D(x): [0.666724, 0.339069, 0.229851, 0.175242, 0.142477, 0.120633,
0.105031]
Fitting tan(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.529117e-08
Corrected MSS of errors squared = 2.162498e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting tan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                          Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    7.12672578893368e-13     0      Inf       0
    x1              -0.0216947746086053     0     -Inf       0
    x2                 10.3095249160481     0      Inf       0
    x3                -58.0062963082145     0     -Inf       0
    x4                 161.649512752597     0      Inf       0
    x5                -227.196291364576     0     -Inf       0
    x6                 154.018676256358     0      Inf       0
    x7                -40.3778768655806     0     -Inf       0
    x8                 9.31516641690371     0      Inf       0
    x9                -18.5341085366465     0     -Inf       0
    x10                2.34000269889559     0      Inf       0
    x11                17.2804147964123     0      Inf       0
    x12                -9.6426162196157     0     -Inf       0


Number of observations: 101, Error degrees of freedom: 88
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                ____________    ____    ____________________    ___    ______
```



```
    Total       18.4802838293813    100    0.184802838293813
    Model       18.4802838293813     12     1.54002365244844    Inf       0
    Residual                   0     88                     0

A = 0.518948, B = 0.427171
order Q(x) = 7.000000, order D(x) = 5.000000
List of powers for Q(x): [0.946119, 1.123059, 1.258830, 1.373291, 1.474132, 1.565300,
1.649137]
List of powers for D(x): [0.946119, 1.123059, 1.258830, 1.373291, 1.474132]
Fitting tan(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.187766e-10
Corrected MSS of errors squared = 1.679754e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting tan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    7.04895397972904e-13     0      Inf       0
    x1                0.487189989402153     0      Inf       0
    x2                 2.23533238490827     0      Inf       0
    x3                -6.42165694885142     0     -Inf       0
    x4                  10.213177618321     0      Inf       0
    x5                -5.59117850809204     0     -Inf       0
    x6                -9.85903044891283     0     -Inf       0
    x7                 51.9641054362468     0      Inf       0
    x8                 -148.02286046256     0     -Inf       0
    x9                 269.968913156246     0      Inf       0
    x10               -302.936693314727     0     -Inf       0
    x11                 187.64383891795     0      Inf       0
    x12               -48.3508377888153     0     -Inf       0


Number of observations: 101, Error degrees of freedom: 88
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                ____________    ____    ____________________    ___    ______
```



```
    Total       18.4802838293769    100    0.184802838293769
    Model       18.4802838293769     12     1.54002365244807    Inf       0
    Residual                   0     88                    0

A = 0.020062, B = 0.684187, C = 0.871879
order Q(x) = 5.000000, order D(x) = 7.000000
List of powers for Q(x): [0.956144, 1.179528, 1.366344, 1.530223, 1.677982]
List of powers for D(x): [0.956144, 1.179528, 1.366344, 1.530223, 1.677982, 1.813609,
1.939677]
Fitting tan(x) in range (0.000000, 1.000000)
MSS of errors squared = 4.122283e-11
Corrected MSS of errors squared = 7.140003e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting tan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -6.19346407847808e-12    0     -Inf       0
    x1               0.00217454679828553    0      Inf       0
    x2              -0.0493837014153254     0     -Inf       0
    x3               0.977529301718513      0      Inf       0
    x4              -1.99780532215012       0     -Inf       0
    x5               1.53996679466745       0      Inf       0
    x6              -0.420644882542422      0     -Inf       0
    x7               3.41061968534738       0      Inf       0
    x8              -5.05470419571976       0     -Inf       0
    x9               4.17206901773343       0      Inf       0
    x10             -2.03228149164485       0     -Inf       0
    x11              0.471012993877281      0      Inf       0


Number of observations: 101, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF          MeanSq            F     pValue
                  _____________    ___    ____________________    ___    ______

    Total       18.4802838293816    100    0.184802838293816
```

```
    Model       18.4802838293816     11     1.68002580267105    Inf       0
    Residual                   0     89                    0

A = 0.161330, B = 0.188771, C = 0.104758
order Q(x) = 6.000000, order D(x) = 5.000000
List of powers for Q(x): [0.350102, 0.643631, 0.937160, 1.230689, 1.524218, 1.817748]
List of powers for D(x): [0.350102, 0.643631, 0.937160, 1.230689, 1.524218]
Fitting tan(x) in range (0.000000, 1.000000)
MSS of errors squared = 4.248200e-09
Corrected MSS of errors squared = 7.358099e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting tan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8

Estimated Coefficients:
                        Estimate          SE     tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     -0.0509698314045562     0     -Inf       0
    x1              -0.00276763390736128    0     -Inf       0
    x2                0.0505164590206326    0      Inf       0
    x3               0.00356843892344844    0      Inf       0
    x4              -0.00031931455798812    0     -Inf       0
    x5               -0.0279976007588578    0     -Inf       0
    x6                 0.924779647778547    0      Inf       0
    x7                 0.125481056994809    0      Inf       0
    x8               -0.0222811584740714    0     -Inf       0


Number of observations: 100, Error degrees of freedom: 91
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF         MeanSq           F     pValue
                 _____________    ___    _____________________    ___    ______

    Total        18.0954624389228    99    0.182782448878008
    Model        18.0954624389228     8     2.26193280486535    Inf       0
    Residual                    0    91                    0

A = 0.003729, B = 0.113453, C = 0.219083
order Q(x) = 4.000000, order D(x) = 4.000000
```



```
List of powers for Q(x): [0.117182, 0.011553, -0.094077, -0.199706]
List of powers for D(x): [0.117182, 0.011553, -0.094077, -0.199706]
Fitting tan(x) in range (0.000000, 1.000000)
MSS of errors squared = NaN
Corrected MSS of errors squared = NaN
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting tan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     2.12104934194455e-15    0      Inf       0
    x1               9.89778910684726e-05   0      Inf       0
    x2               -0.00260891245228195   0     -Inf       0
    x3              -0.000935150502160888   0     -Inf       0
    x4                 0.0842050578845037   0      Inf       0
    x5                -0.0683443027343324   0     -Inf       0
    x6                   1.88471853406097   0      Inf       0
    x7                  -4.13178239772796   0     -Inf       0
    x8                   8.34809574105368   0      Inf       0
    x9                  -8.99993968291739   0     -Inf       0
    x10                  5.04605864422932   0      Inf       0
    x11                 -1.15512284883046   0     -Inf       0


Number of observations: 101, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______

    Total         18.4802838293817    100    0.184802838293817
    Model         18.4802838293817     11     1.68002580267106    Inf       0
    Residual                     0     89                    0

A = 0.001416, B = 0.100000, C = 0.198160
order Q(x) = 5.000000, order D(x) = 6.000000
List of powers for Q(x): [0.101416, 0.399576, 0.581657, 0.744640, 0.897737]
List of powers for D(x): [0.101416, 0.399576, 0.581657, 0.744640, 0.897737, 1.044516]
```



```
Fitting tan(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.671391e-10
Corrected MSS of errors squared = 2.894934e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting tan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -2.94118842082977e-13    0     -Inf       0
    x1                 0.013778939707562    0      Inf       0
    x2                -0.154075394479141    0     -Inf       0
    x3                 0.556135359452176    0      Inf       0
    x4                -0.363453205539184    0     -Inf       0
    x5                  17.5091946037909    0      Inf       0
    x6                 -67.1349513822938    0     -Inf       0
    x7                  126.34975767652     0      Inf       0
    x8                 -138.195049661617    0     -Inf       0
    x9                  89.2147149377571    0      Inf       0
    x10                -31.3504333945094    0     -Inf       0
    x11                 4.57313074906942    0      Inf       0


Number of observations: 101, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                  _____________    ___    ________________    ___    ______

    Total       18.4802838293821    100    0.184802838293821
    Model       18.4802838293821     11     1.6800258026711    Inf       0
    Residual                   0     89                   0

A = 0.296305, B = 0.100000, C = 0.132119
order Q(x) = 4.000000, order D(x) = 7.000000
List of powers for Q(x): [0.396305, 0.569846, 0.733749, 0.892664]
List of powers for D(x): [0.396305, 0.569846, 0.733749, 0.892664, 1.048390, 1.201851,
1.353597]
Fitting tan(x) in range (0.000000, 1.000000)
```



```
MSS of errors squared = 6.379956e-10
Corrected MSS of errors squared = 1.105041e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting tan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     1.48473453936118e-13    0      Inf        0
    x1             -0.000941288718083327    0     -Inf        0
    x2                0.0321491981234499    0      Inf        0
    x3                -0.142517463771996    0     -Inf        0
    x4                 0.210088512778042    0      Inf        0
    x5                -0.0978793509057569    0     -Inf        0
    x6                   4.56917286198682    0      Inf        0
    x7                  -22.9606314864349    0     -Inf        0
    x8                   39.3971401239005    0      Inf        0
    x9                                  0    0      NaN      NaN
    x10                 -62.367409277686    0     -Inf        0
    x11                  60.2214702713097    0      Inf        0
    x12                 -17.8603201244095    0     -Inf        0


Number of observations: 101, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                  _________________    ___    ________________    ___    ______

    Total       18.4802838293805     100    0.184802838293805
    Model       18.4802838293805      11     1.68002580267096    Inf       0
    Residual                   0      89                    0

A = 0.007299, B = 0.100000, C = 0.111769
order Q(x) = 5.000000, order D(x) = 7.000000
List of powers for Q(x): [0.107299, 0.260490, 0.338570, 0.400889, 0.454444]
List of powers for D(x): [0.107299, 0.260490, 0.338570, 0.400889, 0.454444, 0.502172,
0.545652]
Fitting tan(x) in range (0.000000, 1.000000)
```



```
MSS of errors squared = 1.981809e-10
Corrected MSS of errors squared = 3.432594e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting tan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -1.71092456684199e-15    0     -Inf       0
    x1             -9.63985781532556e-05    0     -Inf       0
    x2              0.000158945665998394    0      Inf       0
    x3             -0.000168501161791305    0     -Inf       0
    x4               0.00051858278670031    0      Inf       0
    x5             -0.000254802326656989    0     -Inf       0
    x6             -1.79876482061812e-05    0     -Inf       0
    x7               4.4558671537584e-07    0      Inf       0
    x8                  2.25164473815963    0      Inf       0
    x9                 -1.34503999181382    0     -Inf       0
    x10                0.105695856853265    0      Inf       0
    x11               -0.0147107879773169    0     -Inf       0
    x12                0.00266082896962138    0      Inf       0
    x13               -0.000340719720494663    0     -Inf       0


Number of observations: 101, Error degrees of freedom: 87
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                  _____________    ___    _________________    ___    ______

    Total        18.4802838293815    100    0.184802838293815
    Model        18.4802838293815     13     1.42156029456781    Inf       0
    Residual                    0     87                    0

A = 0.024248, B = 3.438275
order Q(x) = 7.000000, order D(x) = 6.000000
List of powers for Q(x): [0.024248, 0.052482, 0.202426, 0.475999, 0.844929, 1.284904,
1.778003]
List of powers for D(x): [0.024248, 0.052482, 0.202426, 0.475999, 0.844929, 1.284904]
```



```
Fitting tan(x) in range (0.000000, 1.000000)
MSS of errors squared = 5.033570e-11
Corrected MSS of errors squared = 7.118543e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the tangent function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i	0.236	0.372		Inf	-Inf	7.20483700E-12	7, 7
A+B*sqrt(i+C)	0.02	0.684	0.871	Inf	-Inf	4.12228300E-11	5, 7
A+B*log10(i)^4	0.024	3.438		Inf	-Inf	5.03357000E-11	7, 6
A+B*sqrt(i)	0.518	0.427		Inf	-Inf	1.18776600E-10	7, 5
A+B*i+C*sqrt(i-1)	0.001	0.1	0.198	Inf	-Inf	1.67139100E-10	5, 6
A+B*sqrt(i)+C*sqrt(i-1)	0.007	0.1	0.111	Inf	-Inf	1.98180900E-10	5, 7
A+B*sqrt(i)+C*(i-1)	0.296	0.1	0.132	Inf	-Inf	6.37995600E-10	4, 7
A+B*i+C*(i-1)	0.161	0.188	0.104	Inf	-Inf	4.24820000E-09	6, 5
A+B/i	0.011	0.655		Inf	-Inf	1.52911700E-08	3, 7
A+B*i-C*(i-1)	0.003	0.113	0.219	Inf	-Inf	NaN	4, 4

RESULTS FOR THE HYPERBOLIC TANGENT FUNCTION

Using Power A+B* i

```
Fitting tanh(x) in range (0.000000, 3.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -1.55393703836974e-10    0     -Inf       0
    x1                0.0187119772864276    0      Inf       0
    x2                -0.224516735181885    0     -Inf       0
    x3                  3.20913885974697    0      Inf       0
    x4                 -7.05250653519415    0     -Inf       0
    x5                  6.90644914143744    0      Inf       0
    x6                 -3.40772616665961    0     -Inf       0
    x7                 0.739820886101112    0      Inf       0
    x8                  10.7176123994735    0      Inf       0
    x9                  -29.731584232802    0     -Inf       0
    x10                 40.0229808115127    0      Inf       0
    x11                -31.2676996721796    0     -Inf       0
    x12                 13.6352939049698    0      Inf       0
    x13                -2.62525458219081    0     -Inf       0


Number of observations: 301, Error degrees of freedom: 287
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ____________________    ___    ______
```



```
    Total       23.0506117213929    300    0.0768353724046429
    Model       23.0506117213929     13      1.77312397856868    Inf       0
    Residual                   0    287                     0

A = 0.366685, B = 0.231426
order Q(x) = 7.000000, order D(x) = 6.000000
List of powers for Q(x): [0.598111, 0.829536, 1.060962, 1.292387, 1.523813, 1.755239,
1.986664]
List of powers for D(x): [0.598111, 0.829536, 1.060962, 1.292387, 1.523813, 1.755239]
Fitting tanh(x) in range (0.000000, 3.000000)
MSS of errors squared = 1.217599e-09
Corrected MSS of errors squared = 1.721944e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting tanh(x) in range (0.000000, 3.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    1.14033575722486e-13     0      Inf       0
    x1              -0.0222415048434651     0     -Inf       0
    x2                 1.22746370267971     0      Inf       0
    x3                -14.1383456755636     0     -Inf       0
    x4                 60.7510720079822     0      Inf       0
    x5                 -117.83331274199     0     -Inf       0
    x6                 104.864753327026     0      Inf       0
    x7                 -34.850640446927     0     -Inf       0
    x8                0.0316281083250381    0      Inf       0
    x9                 -2.06796624357823    0     -Inf       0
    x10                16.7906906682754     0      Inf       0
    x11               -36.8612442552049     0     -Inf       0
    x12                23.1085347662824     0      Inf       0


Number of observations: 301, Error degrees of freedom: 288
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF          MeanSq            F     pValue
                  _____________    ___    ________________    ___    ______
```



```
    Total       23.0506117213935    300    0.076835372404645
    Model       23.0506117213935     12     1.92088431011612    Inf       0
    Residual                   0    288                    0

A = 0.000000, B = 1.020916
order Q(x) = 7.000000, order D(x) = 5.000000
List of powers for Q(x): [1.020916, 0.510458, 0.340305, 0.255229, 0.204183, 0.170153,
0.145845]
List of powers for D(x): [1.020916, 0.510458, 0.340305, 0.255229, 0.204183]
Fitting tanh(x) in range (0.000000, 3.000000)
MSS of errors squared = 1.764994e-05
Corrected MSS of errors squared = 2.496079e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting tanh(x) in range (0.000000, 3.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -1.63810091736571e-12    0     -Inf       0
    x1                 -8.02264501203594    0     -Inf       0
    x2                  101.087979486882    0      Inf       0
    x3                 -437.47347754677     0     -Inf       0
    x4                  909.052075422495    0      Inf       0
    x5                 -993.108282840593    0     -Inf       0
    x6                  551.650801893118    0      Inf       0
    x7                 -123.186461091962    0     -Inf       0
    x8                  31.7671690840701    0      Inf       0
    x9                 -132.603565863345    0     -Inf       0
    x10                 217.098173122209    0      Inf       0
    x11                -158.457715126175    0     -Inf       0
    x12                 43.1959515050533    0      Inf       0


Number of observations: 301, Error degrees of freedom: 288
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F     pValue
                    _________    ___    ____________________    ___    ______
```


Total	23.0506117213848	300	0.0768353724046161		
Model	23.0506117213848	12	1.9208843101154	Inf	0
Residual	0	288	0		

A = 0.006404, B = 0.100000
order Q(x) = 7.000000, order D(x) = 5.000000
List of powers for Q(x): [0.106404, 0.147826, 0.179609, 0.206404, 0.230011, 0.251353, 0.270979]
List of powers for D(x): [0.106404, 0.147826, 0.179609, 0.206404, 0.230011]
Fitting tanh(x) in range (0.000000, 3.000000)
MSS of errors squared = 1.784839e-06
Corrected MSS of errors squared = 2.524144e-06
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*sqrt(i+C)

Fitting tanh(x) in range (0.000000, 3.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	-3.17569843946465e-10	0	-Inf	0
x1	15.6491149567213	0	Inf	0
x2	-108.213269876474	0	-Inf	0
x3	244.586338875789	0	Inf	0
x4	-227.249330145755	0	-Inf	0
x5	75.2368911709601	0	Inf	0
x6	4234.85393990131	0	Inf	0
x7	-24499.4440524054	0	-Inf	0
x8	40550.2997095616	0	Inf	0
x9	0	0	NaN	NaN
x10	-61870.6794560913	0	-Inf	0
x11	58622.4518173023	0	Inf	0
x12	-17036.494753773	0	-Inf	0

Number of observations: 301, Error degrees of freedom: 289
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue


```
    Total       23.050611719609    300    0.0768353723986967
    Model       23.050611719609     11      2.09551015632809    Inf       0
    Residual                  0    289                     0

A = 0.378258, B = 0.100000, C = 0.000000
order Q(x) = 5.000000, order D(x) = 7.000000
List of powers for Q(x): [0.478258, 0.519679, 0.551463, 0.578258, 0.601865]
List of powers for D(x): [0.478258, 0.519679, 0.551463, 0.578258, 0.601865, 0.623207,
0.642833]
Fitting tanh(x) in range (0.000000, 3.000000)
MSS of errors squared = 4.569463e-08
Corrected MSS of errors squared = 7.914543e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting tanh(x) in range (0.000000, 3.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    2.51319897410502e-11     0      Inf       0
    x1             -0.00252904290876793     0     -Inf       0
    x2                0.120454583932219     0      Inf       0
    x3                 1.56751378050533     0      Inf       0
    x4                -3.38312654608382     0     -Inf       0
    x5                 3.13485206541446     0      Inf       0
    x6                -1.44609430717249     0     -Inf       0
    x7                0.311149364426396     0      Inf       0
    x8                 1.35832949794871     0      Inf       0
    x9                0.255332368643138     0      Inf       0
    x10               -2.80537665423042     0     -Inf       0
    x11                3.52802130965272     0      Inf       0
    x12               -2.65763887993998     0     -Inf       0
    x13                1.19877037860623     0      Inf       0
    x14              -0.274263413382735     0     -Inf       0


Number of observations: 301, Error degrees of freedom: 286
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F     pValue
```


Total	23.0506117213926	300	0.0768353724046421		
Model	23.0506117213926	14	1.64647226581376	Inf	0
Residual	0	286	0		

A = 0.321582, B = 0.128515, C = 0.209872
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.450097, 0.788485, 1.126872, 1.465260, 1.803648, 2.142036, 2.480423]
List of powers for D(x): [0.450097, 0.788485, 1.126872, 1.465260, 1.803648, 2.142036, 2.480423]
Fitting tanh(x) in range (0.000000, 3.000000)
MSS of errors squared = 2.387061e-10
Corrected MSS of errors squared = 4.134511e-10
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*i–C*(i–1)

Fitting tanh(x) in range (0.000000, 3.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	-3.71558741749303e-12	0	-Inf	0
x1	0.00583905824111335	0	Inf	0
x2	-0.080937855442929	0	-Inf	0
x3	0.455286131404723	0	Inf	0
x4	-0.750828892936743	0	-Inf	0
x5	0.676858256435486	0	Inf	0
x6	-0.375529281463885	0	-Inf	0
x7	0.106863903391618	0	Inf	0
x8	4.8838627316033	0	Inf	0
x9	-10.6686593517311	0	-Inf	0
x10	13.3030745541947	0	Inf	0
x11	-10.028654009158	0	-Inf	0
x12	4.26346716481907	0	Inf	0
x13	-0.802397296235352	0	-Inf	0

Number of observations: 301, Error degrees of freedom: 287
R-squared: 1, Adjusted R-Squared: 1


```
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       23.0506117213923     300    0.0768353724046411
    Model       23.0506117213923      13      1.77312397856864    Inf       0
    Residual                   0     287                     0

A = 0.037006, B = 0.226127, C = 0.000000
order Q(x) = 7.000000, order D(x) = 6.000000
List of powers for Q(x): [0.263132, 0.489259, 0.715386, 0.941512, 1.167639, 1.393766,
1.619892]
List of powers for D(x): [0.263132, 0.489259, 0.715386, 0.941512, 1.167639, 1.393766]
Fitting tanh(x) in range (0.000000, 3.000000)
MSS of errors squared = 3.114889e-09
Corrected MSS of errors squared = 5.395146e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting tanh(x) in range (0.000000, 3.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -2.00169947988069e-12    0     -Inf       0
    x1               0.00300338939757576    0      Inf       0
    x2              -0.0475045681052305     0     -Inf       0
    x3               0.318818027168672      0      Inf       0
    x4              -0.462291628313052      0     -Inf       0
    x5               0.255025732477475      0      Inf       0
    x6              -0.0430645844652006     0     -Inf       0
    x7               4.08790518783641       0      Inf       0
    x8              -8.37672030400205       0     -Inf       0
    x9               10.866942884504        0      Inf       0
    x10             -9.26966654174938       0     -Inf       0
    x11              4.98467383887496       0      Inf       0
    x12             -1.52201136166139       0     -Inf       0
    x13              0.19738134887843       0      Inf       0


Number of observations: 301, Error degrees of freedom: 287
```



```
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       23.0506117213922    300    0.0768353724046407
    Model       23.0506117213922     13      1.77312397856863    Inf       0
    Residual                   0    287                     0

A = 0.000000, B = 0.242413, C = 0.000000
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.242413, 0.484826, 0.727238, 0.969651, 1.212064, 1.454477]
List of powers for D(x): [0.242413, 0.484826, 0.727238, 0.969651, 1.212064, 1.454477,
1.696890]
Fitting tanh(x) in range (0.000000, 3.000000)
MSS of errors squared = 1.792873e-09
Corrected MSS of errors squared = 3.105346e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting tanh(x) in range (0.000000, 3.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    3.52625594315292e-10    0      Inf       0
    x1             -0.00283150213087954    0     -Inf       0
    x2                0.401680740556144    0      Inf       0
    x3                 2.97596752411273    0      Inf       0
    x4                -5.14158353121837    0     -Inf       0
    x5                 4.46680084042503    0      Inf       0
    x6                -1.97537139049346    0     -Inf       0
    x7                0.478616137718833    0      Inf       0
    x8                -2.27516425991538    0     -Inf       0
    x9                 6.50978996175315    0      Inf       0
    x10                 -8.608955250451    0     -Inf       0
    x11                 6.8086348574532    0      Inf       0
    x12               -4.40698514120293    0     -Inf       0
    x13                1.85535209822199    0      Inf       0
    x14               -0.462619821262156    0     -Inf       0
```



```
Number of observations: 301, Error degrees of freedom: 286
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______

    Total       23.0506117213934    300    0.0768353724046446
    Model       23.0506117213934     14      1.64647226581381    Inf       0
    Residual                   0    286                     0

A = 0.340713, B = 0.100000, C = 0.400000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.440713, 0.882134, 1.313918, 1.740713, 2.164320, 2.585662,
3.005288]
List of powers for D(x): [0.440713, 0.882134, 1.313918, 1.740713, 2.164320, 2.585662,
3.005288]
Fitting tanh(x) in range (0.000000, 3.000000)
MSS of errors squared = 5.073413e-10
Corrected MSS of errors squared = 8.787410e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting tanh(x) in range (0.000000, 3.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -2.22521640797171e-12    0     -Inf       0
    x1               0.00111235923101414    0      Inf       0
    x2                -0.192733543820148    0     -Inf       0
    x3                  2.33859003149195    0      Inf       0
    x4                 -5.24160456627489    0     -Inf       0
    x5                    4.170675306193    0      Inf       0
    x6                -0.814311473600872    0     -Inf       0
    x7                -0.231314668208109    0     -Inf       0
    x8                  3.65424623530469    0      Inf       0
    x9                 -27.2544320478478    0     -Inf       0
    x10                 97.7725681979958    0      Inf       0
    x11                -174.910933088833    0     -Inf       0
    x12                 169.318600068975    0      Inf       0
```


x13	-85.0642524937779	0	-Inf	0
x14	17.4442692028141	0	Inf	0

Number of observations: 301, Error degrees of freedom: 286
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	23.0506117213876	300	0.0768353724046252		
Model	23.0506117213876	14	1.6464722658134	Inf	0
Residual	0	286	0		

A = 0.120496, B = 0.176854, C = 0.358539
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.297350, 0.729144, 0.933866, 1.095211, 1.233031, 1.355415, 1.466644]
List of powers for D(x): [0.297350, 0.729144, 0.933866, 1.095211, 1.233031, 1.355415, 1.466644]
Fitting tanh(x) in range (0.000000, 3.000000)
MSS of errors squared = 9.548312e-10
Corrected MSS of errors squared = 1.653816e-09
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*log10(i)^4

Fitting tanh(x) in range (0.000000, 3.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	1.57042618211036e-10	0	Inf	0
x1	190.43965418143	0	Inf	0
x2	-253.48692598407	0	-Inf	0
x3	93.1770992249972	0	Inf	0
x4	-44.1493300412124	0	-Inf	0
x5	19.0873680345295	0	Inf	0
x6	-5.36923893582942	0	-Inf	0
x7	0.766586606086981	0	Inf	0
x8	1110.72344592179	0	Inf	0
x9	-1414.00850346384	0	-Inf	0


```
    x10                369.178544466581     0       Inf        0
    x11               -72.8532490481596     0      -Inf        0
    x12                6.56992922873773     0       Inf        0
    x13                1.21516562987998     0       Inf        0
    x14               -0.436173934161315    0      -Inf        0


Number of observations: 301, Error degrees of freedom: 286
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq           DF            MeanSq            F      pValue
                 ________________    ___    ____________________    ___    ______

    Total        23.0506117214359    300    0.0768353724047863
    Model        23.0506117214359     14      1.64647226581685    Inf        0
    Residual                    0    286                     0

A = 1.203935, B = 2.409930
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [1.203935, 1.223725, 1.328823, 1.520573, 1.779161, 2.087545,
2.433164]
List of powers for D(x): [1.203935, 1.223725, 1.328823, 1.520573, 1.779161, 2.087545,
2.433164]
Fitting tanh(x) in range (0.000000, 3.000000)
MSS of errors squared = 1.548673e-10
Corrected MSS of errors squared = 2.190154e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the hyperbolic tangent function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*log10(i)^4	1.203	2.409		Inf	-Inf	1.54867300E-10	7, 7
A+B*i+C*(i-1)	0.321	0.128	0.209	Inf	-Inf	2.38706100E-10	7, 7
A+B*sqrt(i)+C*(i-1)	0.34	0.1	0.4	Inf	-Inf	5.07341300E-10	7, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.12	0.176	0.358	Inf	-Inf	9.54831200E-10	7, 7
A+B*i	0.366	0.231		Inf	-Inf	1.21759900E-09	7, 6
A+B*i+C*sqrt(i-1)	0	0.242	0	Inf	-Inf	1.79287300E-09	6, 7
A+B*i-C*(i-1)	0.037	0.226	0	Inf	-Inf	3.11488900E-09	7, 6
A+B*sqrt(i+C)	0.378	0.1	0	Inf	-Inf	4.56946300E-08	5, 7
A+B*sqrt(i)	0.006	0.1		Inf	-Inf	1.78483900E-06	7, 5
A+B/i	0	1.02		Inf	-Inf	1.76499400E-05	7, 5

RESULTS FOR THE TWO-SIDED T-INVERSE DISTRIBUTION

Using Power A+B* i

```
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)      0.836497504355684      0      Inf        0
    x1               -9.15699341578339      0     -Inf        0
    x2                25.6490761939587      0      Inf        0
    x3               -33.0894707896416      0     -Inf        0
    x4                21.2338053201022      0      Inf        0
    x5               -5.3774556481826       0     -Inf        0
    x6              -0.161226271598928      0     -Inf        0
    x7               0.010956970870844      0      Inf        0
    x8                7.58201937901558      0      Inf        0
    x9               -18.0131164510103      0     -Inf        0
    x10               20.8717895021151      0      Inf        0
    x11              -12.7044528791837      0     -Inf        0
    x12               3.26806595244053      0      Inf        0


Number of observations: 981, Error degrees of freedom: 968
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq           DF            MeanSq            F     pValue
                ________________    ___    ____________________    ___    ______

    Total       47.9047757848054    980    0.0488824242702096
```



```
    Model       47.9047757848054     12      3.99206464873378    Inf       0
    Residual                   0    968                     0

A = 0.057896, B = 0.144208
order Q(x) = 7.000000, order D(x) = 5.000000
List of powers for Q(x): [0.202105, 0.346313, 0.490521, 0.634729, 0.778937, 0.923145,
1.067353]
List of powers for D(x): [0.202105, 0.346313, 0.490521, 0.634729, 0.778937]
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
MSS of errors squared = 6.961561e-11
Corrected MSS of errors squared = 9.845133e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    _____________________    __    _____    ______

    (Intercept)    -0.302474656620125     0     -Inf        0
    x1             -0.960122584562905     0     -Inf        0
    x2               5.33026275732446     0      Inf        0
    x3              -18.5917972764921     0     -Inf        0
    x4                20.545484991427     0      Inf        0
    x5                              0     0      NaN      NaN
    x6              -6.69575643749424     0     -Inf        0
    x7              0.489864767584984     0      Inf        0
    x8               -2.6949638477266     0     -Inf        0
    x9                7.0103083967293     0      Inf        0
    x10              5.48902082367459     0      Inf        0
    x11             -37.7509372772111     0     -Inf        0
    x12              28.5096862183103     0      Inf        0


Number of observations: 981, Error degrees of freedom: 969
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF            MeanSq            F      pValue
                  _______________    ___    ____________________    ___    ______

    Total       47.9047757848324     980    0.0488824242702372
```



```
    Model       47.9047757848324     11      4.35497961680295    Inf       0
    Residual                   0    969                     0

A = 0.099802, B = 1.522363
order Q(x) = 6.000000, order D(x) = 6.000000
List of powers for Q(x): [1.622165, 0.860984, 0.607256, 0.480393, 0.404275, 0.353529]
List of powers for D(x): [1.622165, 0.860984, 0.607256, 0.480393, 0.404275, 0.353529]
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
MSS of errors squared = 9.408634e-11
Corrected MSS of errors squared = 1.330582e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    0.198103439484202     0      Inf        0
    x1             -1.48599806602108     0     -Inf        0
    x2              2.34519912004544     0      Inf        0
    x3             -1.05664626464927     0     -Inf        0
    x4               27.541076343713     0      Inf        0
    x5             -119.320340469076     0     -Inf        0
    x6              204.712649275844     0      Inf        0
    x7             -142.286128270504     0     -Inf        0
    x8                             0     0      NaN      NaN
    x9              46.3865144258303     0      Inf        0
    x10            -16.0338252030629     0     -Inf        0


Number of observations: 981, Error degrees of freedom: 971
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                   _____________    ___    ____________________    ___    ______

    Total       47.9047757848133    980    0.0488824242702176
    Model       47.9047757848133      9      5.32275286497925    Inf       0
    Residual                   0    971                     0
```



```
A = 0.000000, B = 0.100000
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.141421, 0.173205]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
MSS of errors squared = 5.366981e-06
Corrected MSS of errors squared = 7.590058e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     0.207387501105041     0      Inf        0
    x1              -2.63928052734731     0     -Inf        0
    x2               4.42448078940577     0      Inf        0
    x3              -1.99062577855022     0     -Inf        0
    x4               77.4311123838739     0      Inf        0
    x5              -343.978338181687     0     -Inf        0
    x6               583.651950273145     0      Inf        0
    x7              -399.355994449579     0     -Inf        0
    x8                              0     0      NaN      NaN
    x9               126.397472806535     0      Inf        0
    x10             -43.1463654218147     0     -Inf        0


Number of observations: 99, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                        SumSq          DF           MeanSq            F      pValue
                    ________________    __    ____________________    ___    ______

    Total       7.69602630400818     98    0.0785308806531447
    Model       7.69602630400818      9     0.855114033778687    Inf        0
    Residual                   0     89                     0

A = 0.010881, B = 0.100000, C = 0.558449
order Q(x) = 3.000000, order D(x) = 7.000000
```



```
List of powers for Q(x): [0.135719, 0.170833, 0.199520]
List of powers for D(x): [0.135719, 0.170833, 0.199520, 0.224386, 0.246645, 0.266976,
0.285807]
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
MSS of errors squared = 2.061111e-08
Corrected MSS of errors squared = 3.569950e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      -0.721687486166002     0     -Inf        0
    x1                0.576948574231336     0      Inf        0
    x2                                0     0      NaN      NaN
    x3                -1.23565436370282     0     -Inf        0
    x4                 2.57152363539223     0      Inf        0
    x5                -2.1395216525292      0     -Inf        0
    x6                                0     0      NaN      NaN
    x7                0.203502587605134     0      Inf        0
    x8                  1.9008101878085     0      Inf        0
    x9                -0.983116464416583    0     -Inf        0
    x10               0.259752246755933     0      Inf        0
    x11               -1.11278084794401     0     -Inf        0
    x12                1.09782947520495     0      Inf        0
    x13             -0.000157584228557745   0     -Inf        0
    x14               -0.103827018526969    0     -Inf        0


Number of observations: 981, Error degrees of freedom: 968
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________     ___    ____________________    ___    ______

    Total        47.9047757847729    980    0.0488824242701765
    Model        47.9047757847729     12      3.99206464873108    Inf       0
    Residual                    0    968                     0
```



```
A = 0.000000, B = 0.200395, C = 0.163420
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.200395, 0.564209, 0.928024, 1.291839, 1.655654, 2.019468,
2.383283]
List of powers for D(x): [0.200395, 0.564209, 0.928024, 1.291839, 1.655654, 2.019468,
2.383283]
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
MSS of errors squared = 4.185507e-10
Corrected MSS of errors squared = 7.249511e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)       2.19092988295853     0      Inf       0
    x1              -0.0555523811810384    0     -Inf       0
    x2                0.317961186519862    0      Inf       0
    x3                -1.18885711872794    0     -Inf       0
    x4                -1.38996401943944    0     -Inf       0
    x5                0.136732787270005    0      Inf       0
    x6              -0.0113565353158571    0     -Inf       0
    x7               0.0281306680696109    0      Inf       0
    x8               -0.158864536047914    0     -Inf       0
    x9                0.572129830613047    0      Inf       0
    x10               0.600281866730981    0      Inf       0
    x11             -0.0416694907275086    0     -Inf       0


Number of observations: 981, Error degrees of freedom: 969
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                 _______________    ___    ____________________    ___    ______

    Total        47.9047757848114    980    0.0488824242702157
    Model        47.9047757848114     11      4.35497961680104    Inf       0
    Residual                    0    969                     0
```



```
A = 0.197684, B = 0.122641, C = 0.244915
order Q(x) = 6.000000, order D(x) = 5.000000
List of powers for Q(x): [0.320325, 0.198052, 0.075778, -0.046496, -0.168770, -
0.291044]
List of powers for D(x): [0.320325, 0.198052, 0.075778, -0.046496, -0.168770]
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
MSS of errors squared = 9.936866e-11
Corrected MSS of errors squared = 1.721116e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)       -1.43408795429111     0     -Inf        0
    x1                 2.18677891927342     0      Inf        0
    x2                -1.69317365879945     0     -Inf        0
    x3                                0     0      NaN      NaN
    x4                -2.07429209175379     0     -Inf        0
    x5                 6.74606688269603     0      Inf        0
    x6                -4.84331807476932     0     -Inf        0
    x7              0.000335124279612671    0      Inf        0
    x8                    2.368981921178    0      Inf        0
    x9                -1.83935271441315     0     -Inf        0
    x10               0.603243823940772     0      Inf        0
    x11                0.89883702598872     0      Inf        0
    x12               -3.41211217831933     0     -Inf        0
    x13                2.46776508012289     0      Inf        0


Number of observations: 981, Error degrees of freedom: 968
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                 ___________________    ___    ____________________    ___    ______

    Total       47.9047757848422    980    0.0488824242702471
    Model       47.9047757848422     12      3.99206464873685    Inf       0
    Residual                   0    968                     0
```



```
A = 0.059398, B = 0.184463, C = 0.136013
order Q(x) = 7.000000, order D(x) = 6.000000
List of powers for Q(x): [0.243861, 0.564337, 0.805139, 1.032832, 1.253740, 1.470311,
1.683802]
List of powers for D(x): [0.243861, 0.564337, 0.805139, 1.032832, 1.253740, 1.470311]
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
MSS of errors squared = 5.750159e-11
Corrected MSS of errors squared = 9.959567e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ________________    ___    _____    ______

    (Intercept)     1.15693926683385    0       Inf        0
    x1             -144.834385128895    0      -Inf        0
    x2              452.827848038334    0       Inf        0
    x3                             0    0       NaN      NaN
    x4             -1314.3731963174     0      -Inf        0
    x5              1536.95521592652    0       Inf        0
    x6             -531.752350628014    0      -Inf        0
    x7              132.413134952648    0       Inf        0
    x8             -578.055268993054    0      -Inf        0
    x9               849.69330624786    0       Inf        0
    x10            -424.001986076546    0      -Inf        0
    x11                            0    0       NaN      NaN
    x12             -25.447630413489    0      -Inf        0
    x13             46.4000072322429    0       Inf        0


Number of observations: 981, Error degrees of freedom: 969
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F      pValue
                  _______________    ___    ____________________    ___    ______

    Total        47.904775785512    980    0.0488824242709306
    Model        47.904775785512     11      4.35497961686473    Inf        0
```



```
    Residual                  0    969                     0

A = 0.144106, B = 0.147557, C = 0.000000
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.291663, 0.352783, 0.399682, 0.439219, 0.474053, 0.505545]
List of powers for D(x): [0.291663, 0.352783, 0.399682, 0.439219, 0.474053, 0.505545,
0.534504]
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
MSS of errors squared = 2.007892e-10
Corrected MSS of errors squared = 3.477771e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -0.793428779332711     0     -Inf        0
    x1              0.704520083313841     0      Inf        0
    x2               1.06768174945043     0      Inf        0
    x3                             0     0      NaN      NaN
    x4              -15.7832109384656     0     -Inf        0
    x5               29.3297803756985     0      Inf        0
    x6              -15.2037451744517     0     -Inf        0
    x7              0.0677444588008329    0      Inf        0
    x8               2.07698876619554     0      Inf        0
    x9              -3.55828476474727     0     -Inf        0
    x10              3.27887406484451     0      Inf        0
    x11              5.62971344506776     0      Inf        0
    x12             -13.8375855973866     0     -Inf        0
    x13              7.45827786928696     0      Inf        0


Number of observations: 981, Error degrees of freedom: 968
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F     pValue
                  _______________    ___    ____________________    ___    ______

    Total         47.9047757847998    980    0.0488824242702039
```



```
    Model       47.9047757847998     12      3.99206464873332    Inf       0
    Residual                   0    968                     0

A = 0.087769, B = 0.106187, C = 0.395489
order Q(x) = 7.000000, order D(x) = 6.000000
List of powers for Q(x): [0.193956, 0.633429, 0.830996, 0.985150, 1.116188, 1.232213,
1.337460]
List of powers for D(x): [0.193956, 0.633429, 0.830996, 0.985150, 1.116188, 1.232213]
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
MSS of errors squared = 6.242375e-11
Corrected MSS of errors squared = 1.081211e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)                        0    0      NaN      NaN
    x1              -1.55152885716826e-06    0     -Inf        0
    x2                                 0    0      NaN      NaN
    x3                                 0    0      NaN      NaN
    x4               3.25847018574496e-06    0      Inf        0
    x5                                 0    0      NaN      NaN
    x6              -3.08743566080988e-06    0     -Inf        0
    x7                1.3804755327399e-06    0      Inf        0
    x8                   0.99999683143781    0      Inf        0
    x9                                 0    0      NaN      NaN
    x10              5.77507938487138e-06    0      Inf        0
    x11                                0    0      NaN      NaN
    x12             -6.96545845073554e-06    0     -Inf        0
    x13              5.99666628301916e-06    0      Inf        0
    x14             -1.63772418642872e-06    0     -Inf        0


Number of observations: 981, Error degrees of freedom: 972
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ___________________    ___    ____________________    ___    ______
```



```
    Total       47.9047757848119    980    0.0488824242702162
    Model       47.9047757848119      8      5.98809697310149    Inf       0
    Residual                   0    972                     0

A = 0.000000, B = 0.100000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.000000, 0.000821, 0.005182, 0.013139, 0.023869, 0.036665,
0.051007]
List of powers for D(x): [0.000000, 0.000821, 0.005182, 0.013139, 0.023869, 0.036665,
0.051007]
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
MSS of errors squared = 1.338433e-04
Corrected MSS of errors squared = 1.892830e-04
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the two-sided t-invserse function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i+C*sqrt(i-1)	0.059	0.184	0.136	Inf	-Inf	5.75015900E-11	7, 6
A+B*sqrt(i)+C*sqrt(i-1)	0.087	0.106	0.395	Inf	-Inf	6.24237500E-11	7, 6
A+B*i	0.057	0.144		Inf	-Inf	6.96156100E-11	7, 5
A+B/i	0.099	1.522		Inf	-Inf	9.40863400E-11	6, 6
A+B*i-C*(i-1)	0.197	0.122	0.244	Inf	-Inf	9.93686600E-11	6, 5
A+B*sqrt(i)+C*(i-1)	0.144	0.147	0	Inf	-Inf	2.00789200E-10	6, 7
A+B*i+C*(i-1)	0	0.2	0.163	Inf	-Inf	4.18550700E-10	7, 7
A+B*sqrt(i+C)	0.01	0.1	0.558	Inf	-Inf	2.06111100E-08	3, 7
A+B*sqrt(i)	0	0.1		Inf	-Inf	5.36698100E-06	3, 7
A+B*log10(i)^4	0	0.1		Inf	-Inf	1.33843300E-04	7, 7

RESULTS FOR THE ONE-SIDED T-INVERSE DISTRIBUTION

Using Power A+B* i

```
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)       1.52877196108537      0      Inf       0
    x1               -7.40882958015674      0     -Inf       0
    x2                14.7726130833848      0      Inf       0
    x3               -15.2009618027443      0     -Inf       0
    x4                8.09315588371896      0      Inf       0
    x5               -1.7854107808168       0     -Inf       0
    x6                4.78783963770226      0      Inf       0
    x7               -9.41494205854173      0     -Inf       0
    x8                9.57180066772459      0      Inf       0
    x9               -5.05724792417194      0     -Inf       0
    x10               1.11537005474117      0      Inf       0
    x11            -0.00271628165599271     0     -Inf       0


Number of observations: 981, Error degrees of freedom: 969
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ____________________    ___    ______

    Total       15.7800443490302    980    0.0161020860704389
    Model       15.7800443490302     11      1.43454948627547    Inf       0
```



```
    Residual                  0    969                    0

A = 0.000000, B = 0.113640
order Q(x) = 5.000000, order D(x) = 6.000000
List of powers for Q(x): [0.113640, 0.227280, 0.340919, 0.454559, 0.568199]
List of powers for D(x): [0.113640, 0.227280, 0.340919, 0.454559, 0.568199, 0.681839]
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
MSS of errors squared = 6.862124e-11
Corrected MSS of errors squared = 9.704509e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)       1.59106314148774    0      Inf        0
    x1               -0.638377496532929   0     -Inf        0
    x2                3.14853947682664    0      Inf        0
    x3               -18.3856184394215    0     -Inf        0
    x4                40.3628439887868    0      Inf        0
    x5               -26.3690179677418    0     -Inf        0
    x6                0.38810612397954    0      Inf        0
    x7               -2.27666049177107    0     -Inf        0
    x8                11.3874713931377    0      Inf        0
    x9               -26.1664503462198    0     -Inf        0
    x10               18.6250130274469    0      Inf        0
    x11                             0    0      NaN      NaN
    x12              -0.911521982272278   0     -Inf        0


Number of observations: 981, Error degrees of freedom: 969
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______

    Total       15.7800443490638    980    0.0161020860704732
    Model       15.7800443490638     11      1.43454948627852    Inf       0
    Residual                   0    969                     0
```



```
A = 0.000000, B = 2.000000
order Q(x) = 5.000000, order D(x) = 7.000000
List of powers for Q(x): [2.000000, 1.000000, 0.666667, 0.500000, 0.400000]
List of powers for D(x): [2.000000, 1.000000, 0.666667, 0.500000, 0.400000, 0.333333,
0.285714]
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
MSS of errors squared = 2.450928e-11
Corrected MSS of errors squared = 3.466135e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ____________________    __    _____    ______

    (Intercept)     1.62995838498931     0      Inf        0
    x1             -28.5430803071815     0     -Inf        0
    x2              63.3428180343543     0      Inf        0
    x3                             0     0      NaN      NaN
    x4             -80.0117784563243     0     -Inf        0
    x5                             0     0      NaN      NaN
    x6              83.1633501239254     0      Inf        0
    x7             -39.5809956979476     0     -Inf        0
    x8              17.4960348024552     0      Inf        0
    x9             -38.7951255876156     0     -Inf        0
    x10                            0     0      NaN      NaN
    x11             48.9413945182553     0      Inf        0
    x12                            0     0      NaN      NaN
    x13            -50.8203723748492     0     -Inf        0
    x14             24.1780253620709     0      Inf        0


Number of observations: 981, Error degrees of freedom: 970
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                 _______________    ___    ____________________    ___    ______

    Total        15.7800443490354    980    0.0161020860704443
```



```
    Model       15.7800443490354     10      1.57800443490354    Inf       0
    Residual                   0    970                     0

A = 0.000000, B = 0.100000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949,
0.264575]
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
MSS of errors squared = 7.432402e-09
Corrected MSS of errors squared = 1.051100e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      1.96935740128013     0       Inf        0
    x1               -253.48092931631     0      -Inf        0
    x2               870.566161009448     0       Inf        0
    x3              -1075.13756811955     0      -Inf        0
    x4               497.448857064756     0       Inf        0
    x5                             0     0       NaN      NaN
    x6              -41.3605731150202     0      -Inf        0
    x7               88.6844486548612     0       Inf        0
    x8              -204.208455018459     0      -Inf        0
    x9                             0     0       NaN      NaN
    x10              357.445987023511     0       Inf        0
    x11             -333.468153298488     0      -Inf        0
    x12               92.545328671766     0       Inf        0


Number of observations: 99, Error degrees of freedom: 88
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F      pValue
                    _____________    __    ____________________    ___    ______
```

Total	2.40972607707393	98	0.0245890416027952		
Model	2.40972607707393	10	0.240972607707393	Inf	0
Residual	0	88	0		

```
A = 0.000000, B = 0.100000, C = 1.728105
order Q(x) = 6.000000, order D(x) = 6.000000
List of powers for Q(x): [0.165170, 0.193083, 0.217442, 0.239335, 0.259386, 0.277995]
List of powers for D(x): [0.165170, 0.193083, 0.217442, 0.239335, 0.259386, 0.277995]
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
MSS of errors squared = 1.383725e-10
Corrected MSS of errors squared = 2.396682e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	1.23699603635656	0	Inf	0
x1	-14.0031411910915	0	-Inf	0
x2	30.3432863257649	0	Inf	0
x3	-24.0895448647869	0	-Inf	0
x4	0	0	NaN	NaN
x5	12.9772892867467	0	Inf	0
x6	-6.72754994712944	0	-Inf	0
x7	10.4263533440527	0	Inf	0
x8	-20.8351013218518	0	-Inf	0
x9	15.2957311849803	0	Inf	0
x10	-0.0209197247329545	0	-Inf	0
x11	-7.92175535973914	0	-Inf	0
x12	4.09785142920234	0	Inf	0
x13	-0.000602793084040433	0	-Inf	0

```
Number of observations: 981, Error degrees of freedom: 968
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```

	SumSq	DF	MeanSq	F	pValue


```
    Total       15.7800443490858    980    0.0161020860704957
    Model       15.7800443490858     12      1.31500369575715    Inf       0
    Residual                   0    968                     0

A = 0.235492, B = 0.186619, C = 0.000000
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.422110, 0.608729, 0.795348, 0.981966, 1.168585, 1.355204]
List of powers for D(x): [0.422110, 0.608729, 0.795348, 0.981966, 1.168585, 1.355204,
1.541822]
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
MSS of errors squared = 9.555266e-12
Corrected MSS of errors squared = 1.655021e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                   _________________    __    _____    ______

    (Intercept)     1.31457334514128    0      Inf        0
    x1             -26.7300119360709    0     -Inf        0
    x2              50.549585520633     0      Inf        0
    x3                            0    0      NaN       NaN
    x4             -75.3512589617303    0     -Inf        0
    x5              72.0598572865147    0      Inf        0
    x6             -21.8902932661077    0     -Inf        0
    x7              19.7118110948385    0      Inf        0
    x8             -38.2134122277554    0     -Inf        0
    x9              6.47387437172582    0      Inf        0
    x10             42.9464104862758    0      Inf        0
    x11            -43.1588982762208    0     -Inf        0
    x12             13.247734598187     0      Inf        0


Number of observations: 981, Error degrees of freedom: 969
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                   _________________    ___    _____________    ___    ______
```

```
    Total       15.7800443490162    980    0.0161020860704247
    Model       15.7800443490162     11       1.4345494862742    Inf       0
    Residual                   0    969                     0

A = 0.224898, B = 0.109309, C = 0.000000
order Q(x) = 6.000000, order D(x) = 6.000000
List of powers for Q(x): [0.334207, 0.443516, 0.552825, 0.662134, 0.771443, 0.880752]
List of powers for D(x): [0.334207, 0.443516, 0.552825, 0.662134, 0.771443, 0.880752]
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
MSS of errors squared = 6.391525e-12
Corrected MSS of errors squared = 1.107045e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     0.690308856514507    0       Inf       0
    x1               -1.4396705328121    0      -Inf       0
    x2               1.08568329646199    0       Inf       0
    x3              -0.298148474356974    0      -Inf       0
    x4               2.79700241705124    0       Inf       0
    x5              -4.68213670569951    0      -Inf       0
    x6               5.31575423509694    0       Inf       0
    x7              -3.60837265873991    0      -Inf       0
    x8               1.46568026325221    0       Inf       0
    x9              -0.324477430754422    0      -Inf       0
    x10              0.030462749875892    0       Inf       0


Number of observations: 981, Error degrees of freedom: 970
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                ________________    ___    ____________________    ___    ______

    Total       15.780044349033    980    0.0161020860704418
    Model       15.780044349033     10       1.5780044349033    Inf       0
    Residual                  0    970                     0
```



```
A = 0.044396, B = 0.100000, C = 0.098251
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.144396, 0.342647, 0.483344]
List of powers for D(x): [0.144396, 0.342647, 0.483344, 0.614572, 0.740899, 0.864093,
0.985062]
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
MSS of errors squared = 2.543609e-09
Corrected MSS of errors squared = 4.405659e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)       1.46702060343898     0      Inf       0
    x1               -38.5246449836669     0     -Inf       0
    x2                123.943941894659     0      Inf       0
    x3               -167.951153913096     0     -Inf       0
    x4                109.888368086149     0      Inf       0
    x5               -28.8159450649577     0     -Inf       0
    x6              -0.0430452566153475    0     -Inf       0
    x7                25.4365554912918     0      Inf       0
    x8               -79.4709810144681     0     -Inf       0
    x9                105.397595476231     0      Inf       0
    x10               -68.169764876426     0     -Inf       0
    x11                17.8121698253       0      Inf       0


Number of observations: 981, Error degrees of freedom: 969
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                 _______________    ___    ____________________    ___    ______

    Total        15.7800443490798    980    0.0161020860704896
    Model        15.7800443490798     11      1.43454948627998    Inf       0
    Residual                    0    969                     0
```



```
A = 0.333025, B = 0.100000, C = 0.108801
order Q(x) = 6.000000, order D(x) = 5.000000
List of powers for Q(x): [0.433025, 0.583247, 0.723832, 0.859428, 0.991836, 1.121979]
List of powers for D(x): [0.433025, 0.583247, 0.723832, 0.859428, 0.991836]
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
MSS of errors squared = 1.146141e-10
Corrected MSS of errors squared = 1.985174e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      2.15966326866004     0      Inf        0
    x1              -383.167698942334     0     -Inf        0
    x2               436.148602916083     0      Inf        0
    x3                             0     0      NaN      NaN
    x4               1015.81572505563     0      Inf        0
    x5                             0     0      NaN      NaN
    x6              -2957.03601445611     0     -Inf        0
    x7               1886.93525279201     0      Inf        0
    x8              -305.344654296339     0     -Inf        0
    x9               2521.26285863527     0      Inf        0
    x10             -4351.00635517155     0     -Inf        0
    x11                            0     0      NaN      NaN
    x12              4516.53134623129     0      Inf        0
    x13             -2380.57915530456     0     -Inf        0


Number of observations: 981, Error degrees of freedom: 970
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq           DF            MeanSq            F      pValue
                  ____________________    ____    ____________________    ____    ______

    Total        15.7800443486371     980    0.0161020860700378
    Model        15.7800443486371      10      1.57800443486371    Inf        0
    Residual                    0     970                      0
```



```
A = 0.500000, B = 0.100000, C = 0.000000
order Q(x) = 7.000000, order D(x) = 6.000000
List of powers for Q(x): [0.600000, 0.641421, 0.673205, 0.700000, 0.723607, 0.744949,
0.764575]
List of powers for D(x): [0.600000, 0.641421, 0.673205, 0.700000, 0.723607, 0.744949]
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
MSS of errors squared = 8.717360e-10
Corrected MSS of errors squared = 1.509891e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)       1.92310974629927    0      Inf       0
    x1                -355.57913928288    0     -Inf       0
    x2                429.309734498611    0      Inf       0
    x3                -84.5605792362826   0     -Inf       0
    x4                8.91655277170115    0      Inf       0
    x5                178.665002123943    0      Inf       0
    x6                -214.811006396839   0     -Inf       0
    x7                41.0329028022751    0      Inf       0
    x8                -3.72588323847895   0     -Inf       0
    x9               -0.170341323503187   0     -Inf       0
    x10             0.00778819981247088   0      Inf       0


Number of observations: 981, Error degrees of freedom: 970
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _______________    ___    ___________________    ___    ______

    Total       15.7800443490042     980    0.0161020860704124
    Model       15.7800443490042      10      1.57800443490042    Inf       0
    Residual                   0     970                     0

A = 0.177989, B = 1.098095
order Q(x) = 4.000000, order D(x) = 6.000000
```



```
List of powers for Q(x): [0.177989, 0.187006, 0.234894, 0.322266]
List of powers for D(x): [0.177989, 0.187006, 0.234894, 0.322266, 0.440093, 0.580609]
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
MSS of errors squared = 6.204015e-10
Corrected MSS of errors squared = 8.773802e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the one-sided t-invserse function. The results are sorted in ascending order using the MSSE values.

Power	*A*	*B*	*C*	*F*	*AICc*	*MSSE*	*Q/D Orders*
A+B*i-C*(i-1)	0.224	0.109	0	Inf	-Inf	6.39152500E-12	6, 6
A+B*i+C*(i-1)	0.235	0.186	0	Inf	-Inf	9.55526600E-12	6, 7
A+B/i	0	2		Inf	-Inf	2.45092800E-11	5, 7
A+B*i	0	0.113		Inf	-Inf	6.86212400E-11	5, 6
A+B*sqrt(i)+C*(i-1)	0.333	0.1	0.108	Inf	-Inf	1.14614100E-10	6, 5
A+B*sqrt(i+C)	0	0.1	1.728	Inf	-Inf	1.38372500E-10	6, 6
A+B*log10(i)^4	0.177	1.098		Inf	-Inf	6.20401500E-10	4, 6
A+B*sqrt(i)+C*sqrt(i-1)	0.5	0.1	0	Inf	-Inf	8.71736000E-10	7, 6
A+B*i+C*sqrt(i-1)	0.044	0.1	0.098	Inf	-Inf	2.54360900E-09	3, 7
A+B*sqrt(i)	0	0.1		Inf	-Inf	7.43240200E-09	7, 7

RESULTS FOR THE TRIGAMMA FUNCTION

Using Power A+B* i

```
Fitting tigamma(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)      0.316267477286059      0      Inf        0
    x1               -1.47094967133996      0     -Inf        0
    x2                2.93427884850287      0      Inf        0
    x3               -3.25206906099128      0     -Inf        0
    x4                2.16117720594072      0      Inf        0
    x5               -0.860911551955789     0     -Inf        0
    x6                0.190368328830336     0      Inf        0
    x7               -0.018031780743235     0     -Inf        0
    x8                5.43947819585191      0      Inf        0
    x9               -12.3012214011848      0     -Inf        0
    x10               14.8163331818008      0      Inf        0
    x11              -10.031826848517       0     -Inf        0
    x12               3.62133233438471      0      Inf        0
    x13              -0.544174368807734     0     -Inf        0


Number of observations: 991, Error degrees of freedom: 977
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F      pValue
                  ____________    ___    ____________________    ___    ______
```



```
    Total       15.7946327360629    990    0.0159541744808716
    Model       15.7946327360629     13      1.21497174892791    Inf       0
    Residual                   0    977                     0

A = 0.000000, B = 0.100000
order Q(x) = 7.000000, order D(x) = 6.000000
List of powers for Q(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
List of powers for D(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000]
Fitting tigamma(x) in range (1.000000, 100.000000)
MSS of errors squared = 2.923851e-07
Corrected MSS of errors squared = 4.134949e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting trigamma(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                         Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     8.70556238382505e-06    0      Inf        0
    x1             -2.94939118242585e-06    0     -Inf        0
    x2              2.95514261885602e-05    0      Inf        0
    x3             -3.54236153284696e-05    0     -Inf        0
    x4                0.0225943559647527    0      Inf        0
    x5                  -1.1172383486968    0     -Inf        0
    x6                  5.24405887793829    0      Inf        0
    x7                                 0    0      NaN      NaN
    x8                 -13.3562259923466    0     -Inf        0
    x9                                 0    0      NaN      NaN
    x10                 10.206811177671    0      Inf        0


Number of observations: 991, Error degrees of freedom: 982
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                  _________________    ___    ________________    ___    ______

    Total       15.7946327360623    990    0.015954174480871
```



```
    Model       15.7946327360623      8     1.97432909200778    Inf       0
    Residual                   0    982                    0

A = 0.000125, B = 0.110440
order Q(x) = 3.000000, order D(x) = 7.000000
List of powers for Q(x): [0.110566, 0.055345, 0.036939]
List of powers for D(x): [0.110566, 0.055345, 0.036939, 0.027735, 0.022213, 0.018532,
0.015903]
Fitting trigamma(x) in range (1.000000, 100.000000)
MSS of errors squared = 3.064597e-08
Corrected MSS of errors squared = 4.333995e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting trigamma(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    0.00517203693153136     0      Inf       0
    x1             -0.0767006551728714     0     -Inf       0
    x2               0.237329570906909     0      Inf       0
    x3              -0.307863058975598     0     -Inf       0
    x4               0.183932735349704     0      Inf       0
    x5             -0.0418777782158866     0     -Inf       0
    x6                46.6616203194089     0      Inf       0
    x7               -267.491087578643     0     -Inf       0
    x8                626.153957339918     0      Inf       0
    x9               -725.172268686161     0     -Inf       0
    x10               414.537241400024     0      Inf       0
    x11              -93.6894584483868     0     -Inf       0


Number of observations: 991, Error degrees of freedom: 979
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                        SumSq          DF           MeanSq           F     pValue
                    ____________________    ___    ____________________    ___    ______

    Total       15.7946327360993     990    0.0159541744809084
    Model       15.7946327360993      11      1.43587570328176    Inf       0
```



```
    Residual                   0    979                     0

A = 0.000000, B = 0.100000
order Q(x) = 5.000000, order D(x) = 6.000000
List of powers for Q(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607]
List of powers for D(x): [0.100000, 0.141421, 0.173205, 0.200000, 0.223607, 0.244949]
Fitting trigamma(x) in range (1.000000, 100.000000)
MSS of errors squared = 3.044938e-08
Corrected MSS of errors squared = 4.306193e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting trigamma(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    0.0314329772993327    0      Inf       0
    x1              -0.47432824086615    0     -Inf       0
    x2               1.52921036974095    0      Inf       0
    x3              -2.07530512223015    0     -Inf       0
    x4                1.3001708346513    0      Inf       0
    x5             -0.310872799700768    0     -Inf       0
    x6                 27.07938666242    0      Inf       0
    x7              -109.478093871334    0     -Inf       0
    x8                175.90403034915    0      Inf       0
    x9              -126.671093415423    0     -Inf       0
    x10              34.1655830222138    0      Inf       0


Number of observations: 100, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                 ___________________    __    ____________________    ___    ______

    Total        3.21395261132971    99    0.0324641677912091
    Model        3.21395261132971    10     0.321395261132971    Inf       0
    Residual                    0    89                     0

A = 0.000000, B = 0.100000, C = 0.009563
```



```
order Q(x) = 5.000000, order D(x) = 5.000000
List of powers for Q(x): [0.100477, 0.141759, 0.173481, 0.200239, 0.223821]
List of powers for D(x): [0.100477, 0.141759, 0.173481, 0.200239, 0.223821]
Fitting trigamma(x) in range (1.000000, 100.000000)
MSS of errors squared = 2.849308e-08
Corrected MSS of errors squared = 4.935147e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting trigamma(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     -0.0470695718097521     0     -Inf       0
    x1                0.229469381254758     0      Inf       0
    x2               -0.425867485787886     0     -Inf       0
    x3                0.394709123716743     0      Inf       0
    x4               -0.197565222387741     0     -Inf       0
    x5                0.051410964615587     0      Inf       0
    x6              -0.00548881261066801     0     -Inf       0
    x7                 6.21811516368628     0      Inf       0
    x8                -15.4017952701282     0     -Inf       0
    x9                 20.147636422455     0      Inf       0
    x10               -14.8785913540267     0     -Inf       0
    x11                5.88605300542037     0      Inf       0
    x12               -0.971173810319844     0     -Inf       0


Number of observations: 991, Error degrees of freedom: 978
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F      pValue
                  _______________    ___    ____________________    ___    ______

    Total       15.794632736061     990    0.0159541744808697
    Model       15.794632736061      12      1.31621939467175    Inf        0
    Residual                  0     978                     0

A = 0.005821, B = 0.100000, C = 0.000000
order Q(x) = 6.000000, order D(x) = 6.000000
```



```
List of powers for Q(x): [0.105821, 0.205821, 0.305821, 0.405821, 0.505821, 0.605821]
List of powers for D(x): [0.105821, 0.205821, 0.305821, 0.405821, 0.505821, 0.605821]
Fitting trigamma(x) in range (1.000000, 100.000000)
MSS of errors squared = 6.859719e-09
Corrected MSS of errors squared = 1.188138e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting trigamma(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)                        0    0      NaN      NaN
    x1              -0.000902626282364408    0     -Inf        0
    x2                0.00628263991486252    0      Inf        0
    x3               -0.0182153984040929     0     -Inf        0
    x4                0.0281571643744327     0      Inf        0
    x5               -0.0244738658513339     0     -Inf        0
    x6                0.0113409306142504     0      Inf        0
    x7               -0.00218886860578883    0     -Inf        0
    x8               -0.0057454577048371     0     -Inf        0
    x9                0.0348719421545257     0      Inf        0
    x10              -0.103419072939238      0     -Inf        0
    x11                1.05351044229409      0      Inf        0
    x12               0.0207821609316605     0      Inf        0


Number of observations: 991, Error degrees of freedom: 979
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______

    Total       15.7946327360624     990    0.0159541744808711
    Model       15.7946327360624      11       1.4358757032784     Inf       0
    Residual                   0     979                     0

A = 0.006705, B = 0.100000, C = 0.134738
order Q(x) = 7.000000, order D(x) = 5.000000
```



```
List of powers for Q(x): [0.106705, 0.071967, 0.037229, 0.002491, -0.032248, -
0.066986, -0.101724]
List of powers for D(x): [0.106705, 0.071967, 0.037229, 0.002491, -0.032248]
Fitting trigamma(x) in range (1.000000, 100.000000)
MSS of errors squared = 6.062081e-07
Corrected MSS of errors squared = 1.049983e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting trigamma(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)     -0.106540200225157     0     -Inf       0
    x1                0.52113810672475     0      Inf       0
    x2               -1.07508656235934     0     -Inf       0
    x3                1.21060363068301     0      Inf       0
    x4               -0.804970456254268    0     -Inf       0
    x5                0.317083957684188    0      Inf       0
    x6               -0.0687630114222293    0     -Inf       0
    x7               0.00635323755540073    0      Inf       0
    x8                  6.54839004502022    0      Inf       0
    x9                 -18.4333540491779    0     -Inf       0
    x10                28.9006073361024    0      Inf       0
    x11               -27.2269268344798    0     -Inf       0
    x12                 15.385588435747    0      Inf       0
    x13               -4.81710838880901    0     -Inf       0
    x14               0.642913671274129    0      Inf       0


Number of observations: 991, Error degrees of freedom: 976
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF          MeanSq            F     pValue
                  _______________    ___    _________________    ___    ______

    Total       15.7946327360623     990    0.015954174480871
    Model       15.7946327360623      14     1.12818805257588    Inf       0
    Residual                   0     976                    0
```



```
A = 0.000000, B = 0.100000, C = 0.000000
order Q(x) = 7.000000, order D(x) = 7.000000
List of powers for Q(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
List of powers for D(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000, 0.600000,
0.700000]
Fitting trigamma(x) in range (1.000000, 100.000000)
MSS of errors squared = 3.913079e-08
Corrected MSS of errors squared = 6.777652e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting trigamma(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -0.0190963366481445     0     -Inf        0
    x1               0.161580590202673     0      Inf        0
    x2              -0.314387510516258     0     -Inf        0
    x3                               0     0      NaN      NaN
    x4               0.637164836049296     0      Inf        0
    x5              -0.781774809330593     0     -Inf        0
    x6               0.388553800742398     0      Inf        0
    x7              -0.0720949022711135    0     -Inf        0
    x8                15.2285427694952     0      Inf        0
    x9               -61.4288329596906     0     -Inf        0
    x10               113.337256620568     0      Inf        0
    x11              -108.741157180665     0     -Inf        0
    x12               53.0374638265428     0      Inf        0
    x13              -10.4332400465311     0     -Inf        0


Number of observations: 991, Error degrees of freedom: 978
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F      pValue
                  _____________    ___    ____________________    ___    ______

    Total       15.794632736062    990    0.0159541744808707
    Model       15.794632736062     12      1.31621939467183    Inf        0
```

```
    Residual                  0    978                     0

A = 0.000000, B = 0.100000, C = 0.021862
order Q(x) = 7.000000, order D(x) = 6.000000
List of powers for Q(x): [0.100000, 0.163283, 0.216928, 0.265585, 0.311053, 0.354257,
0.395744]
List of powers for D(x): [0.100000, 0.163283, 0.216928, 0.265585, 0.311053, 0.354257]
Fitting trigamma(x) in range (1.000000, 100.000000)
MSS of errors squared = 5.590151e-09
Corrected MSS of errors squared = 9.682426e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting trigamma(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ________________    ___    _____    ______

    (Intercept)    0.877629955452427    0      Inf        0
    x1             -35.0858759681486    0     -Inf        0
    x2              176.573052345166    0      Inf        0
    x3             -370.331015905447    0     -Inf        0
    x4              389.476681430699    0      Inf        0
    x5             -204.095571052351    0     -Inf        0
    x6              42.5853516020921    0      Inf        0
    x7              20.9051324057242    0      Inf        0
    x8             -47.0405142838625    0     -Inf        0
    x9                             0    0      NaN      NaN
    x10             58.9901317938976    0      Inf        0
    x11                            0    0      NaN      NaN
    x12            -60.2071033395883    0     -Inf        0
    x13             28.3521999780955    0      Inf        0


Number of observations: 991, Error degrees of freedom: 979
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq          DF           MeanSq            F     pValue
                   ________________    ___    ____________________    ___    ______

    Total          15.7946327361047    990    0.0159541744809138
```


Model	15.7946327361047	11	1.43587570328225	Inf	0
Residual	0	979	0		

A = 0.010270, B = 0.100000, C = 0.000000
order Q(x) = 6.000000, order D(x) = 7.000000
List of powers for Q(x): [0.110270, 0.151692, 0.183475, 0.210270, 0.233877, 0.255219]
List of powers for D(x): [0.110270, 0.151692, 0.183475, 0.210270, 0.233877, 0.255219, 0.274845]
Fitting trigamma(x) in range (1.000000, 100.000000)
MSS of errors squared = 9.794456e-08
Corrected MSS of errors squared = 1.696449e-07
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*log10(i)^4

Fitting trigamma(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	0	0	NaN	NaN
x1	2.7620855861624e-10	0	Inf	0
x2	-3.10047119200234e-10	0	-Inf	0
x3	4.1151964058492e-11	0	Inf	0
x4	-2.35927653300156e-12	0	-Inf	0
x5	4.7057406156676e-14	0	Inf	0
x6	0.999999999193013	0	Inf	0
x7	1.02749095568335e-09	0	Inf	0
x8	-2.52533476213327e-10	0	-Inf	0
x9	2.89889991575354e-11	0	Inf	0

Number of observations: 991, Error degrees of freedom: 982
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	15.7946327360623	990	0.015954174480871		
Model	15.7946327360623	8	1.97432909200779	Inf	0
Residual	0	982	0		


```
A = 0.000000, B = 3.500000
order Q(x) = 5.000000, order D(x) = 4.000000
List of powers for Q(x): [0.000000, 0.028741, 0.181377, 0.459862, 0.835415]
List of powers for D(x): [0.000000, 0.028741, 0.181377, 0.459862]
Fitting trigamma(x) in range (1.000000, 100.000000)
MSS of errors squared = 2.192617e-05
Corrected MSS of errors squared = 3.100829e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the trigamma function. The results are sorted in ascending order using the MSSE values.

Power	*A*	*B*	*C*	*F*	*AICc*	*MSSE*	*Q/D Orders*
A+B*sqrt(i)+C*(i-1)	0	0.1	0.021	Inf	-Inf	5.59015100E-09	7, 6
A+B*i+C*(i-1)	0.005	0.1	0	Inf	-Inf	6.85971900E-09	6, 6
A+B*sqrt(i+C)	0	0.1	0.009	Inf	-Inf	2.84930800E-08	5, 5
A+B*sqrt(i)	0	0.1		Inf	-Inf	3.04493800E-08	5, 6
A+B/i	0	0.11		Inf	-Inf	3.06459700E-08	3, 7
A+B*i+C*sqrt(i-1)	0	0.1	0	Inf	-Inf	3.91307900E-08	7, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.01	0.1	0	Inf	-Inf	9.79445600E-08	6, 7
A+B*i	0	0.1		Inf	-Inf	2.92385100E-07	7, 6
A+B*i-C*(i-1)	0.006	0.1	0.134	Inf	-Inf	6.06208100E-07	7, 5
A+B*log10(i)^4	0	3.5		Inf	-Inf	2.19261700E-05	5, 4

CONCLUSIONS

Overall, the Pade-Shammas polynomials did well in fitting many tested equations. The Pade-Shammas polynomials struggled in fitting the following equations:

- The Bessel functions.
- The Fresnel cosine and sine integrals.

So how do the various Shammas Polynomial powers compare with each other. The next table shows the frequency of these powers in attaining the top 3 spots. The counts colored in red indicate the top of their placement.

Power	*Top 1*	*Top 2*	*Top 3*
A+B*i	3	1	1
A+B/i	4	3	5

Power	*Top 1*	*Top 2*	*Top 3*
A+B*sqrt(i)	0	3	2
A+B*sqrt(i+C)	5	2	3
A+B*i+C*(i–1)	4	3	3
A+B*i–C*(i–1)	1	8	5
A+B*i+C*sqrt(i–1)	3	3	4
A+B*sqrt(i)+C*(i–1)	6	1	4
A+B*sqrt(i)+C*sqrt(i–1)	6	5	2
A+B*log10(i)^4	0	3	3

The top Shammas polynomial powers are A+B*sqrt(i)+C*(i–1) and A+B*sqrt(i)+C*sqrt(i–1). The second top Shammas polynomial power is A+B*i–C*(i–1). The third top Shammas polynomial powers are A+B*i–C*(i–1) and A+B/i.

Overall the power expressions A+B*sqrt(i)+C*sqrt(i–1), A+B*sqrt(i)+C*(i–1), and A+B*i–C*(i–1) are the favorites.

LOOKING AT THE EFFECT OF NORMALIZATION THE DATA

In this section we look at the effect of normalizing data (in Part I). The next table shows the best normalized and non-normalized MSSE values (regardless of the Shammas polynomial power, to make things a bit simpler). The lines in blue favor data normalization while the lines in red favor non-normalizaton. The criterion for signmicant difference is a difference in the exponents by three orders.

Function	*MSSE (normalized)*	*MSSE*
acosh	4.72541900E-06	1.44319900E-07
acos	4.72541900E-06	6.35783500E-06
asin	3.84022700E-07	2.93704600E-06
atan	1.08003700E-12	1.32405500E-11
asinh	7.83524800E-07	3.26304300E-09
atanh	7.59681300E-06	1.00365500E-04
Ci	8.63345800E-03	3.22542500E-04
cos	1.18441600E-13	2.30806400E-12
cosh	2.26390500E-11	3.18572100E-09
digamma	1.92179900E-08	3.93283800E-11
erf	2.74329100E-09	3.00722900E-09
exp	1.13495600E-13	9.26021800E-12
Fresnel cos	3.15519000E-02	2.31725600E-02

Function	*MSSE (normalized)*	*MSSE*
Fresnel sin	2.47606100E-02	2.89307800E-02
J0(x)	3.96863400E-02	1.99099500E-02
J1(x)	4.22681800E-02	5.61194500E-02
J2(x)	6.91673700E-02	3.52228500E-02
J3(x)	6.91673700E-02	2.32920700E-02
J4(x)	4.23174300E-02	1.21881800E-02
J5(x)	4.72908700E-02	1.40629300E-02
ln	1.04284400E-11	5.73941200E-14
log10	1.38373400E-11	4.29414500E-15
log10gamma	2.52067800E-09	2.62813600E-10
10^x	1.21838900E-14	2.36129200E-11
Si	9.78105700E-05	2.42792000E-04
sin	1.21654700E-13	1.90657600E-10
sinh	5.14194300E-12	2.59745100E-08
tan	3.21995800E-15	7.20483700E-12
tanh	2.08310500E-11	1.54867300E-10
tinv2	8.64941700E-10	5.75015900E-11
tinv1	7.08397900E-10	6.39152500E-12
trigamma	7.08397900E-10	5.59015100E-09

The above table shows that for most functions the MSSE values are not that statistically different. There is a slight advantage for normalized data for a few fitted functions. I was expecting more advantage in normalizing data.

NEXT

The next study I recommend for you to look at is Part II. In Part II the Pade-Shammas polynomials perform curve fitting on the same set of functions using different expressions for the power functions gx1() and gx2() and using separate sets of values for A, B, and C for these functions.

DOCUMENT HISTORY

Date	*Version*	*Comments*
10/4/2020	1.00.00	Initial release.

