Pade-Shammas Polynomial Approximations Part I

By

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INTRODUCTION

This study looks at the collaboration of two mathematical frameworks—the Pade polynomials and the Shammas Polynomials. The study combines both frameworks to approximate several common functions. This study uses x and y data (calculated from the functions being approximated) that are transformed into the range of (1, 2). This transformation should limit rounding errors in calculations. Please do not be intimidated by the page count of this study. Most of the pages contain calculation results. You can quickly browse through this document. You are

probably best served by focusing on the tables that summarize the results of fitting various functions with Pade-Shammas polynomials.

The Pade Polynomials

Pade polynomials are polynomial ratios defined as:

$$y = P_{m,n}(x) = \frac{Q_m(x)}{D_n(x)} \quad (1)$$

Where $Q_m(x)$ is defined as:

$$Q_m(x) = a_0 + \sum_{i=1}^{m} a_i x^i \quad (2)$$

And $D_n(x)$ is defined as:

$$D_n(x) = 1 + \sum_{i=1}^{n} b_i x^i \quad (3)$$

The Pade polynomials have more flexibility that ordinary legacy polynomials. The trick is to find the optimum orders of the polynomials $Q_m(x)$ and $D_n(x)$. The general multiple regression model used to fit Pade polynomials is:

$$y = a_0 + \sum_{i=1}^{m} a_i x^i - \sum_{i=1}^{n} b_i y x^i \quad (4)$$

The Shammas Polynomials

In the HHC 2008 conference in Corvallis, Oregon, I introduced the Shammas polynomials as polynomials with non-integer powers. I explained that the powers of such polynomial change using some math expressions that involves the polynomial term sequence number. Moreover, the study explained that such expressions must never produce the same powers of two or more terms. In the 2008 presentation I showed that the Shammas polynomials can have multiple terms with a smaller power range compared to regular polynomials. This feature allows better curve fitting with Shammas polynomials that avoids the instability of raising large or small numbers to high powers. If you are not familiar with Shammas polynomials, then you can download my 2008 presentation using the following link:

http://www.namirshammas.com/NEW/ShammasPolynomials.pdf.

A Shammas polynomial has the following general form:

$$y = a_0 + \sum_{i=1}^{n} a_i x^{gx(i,A,B,C)} \quad (5)$$

Where A, B, and C are fixed constants. Using C is optional. The function gx(i, A, B, C) supplies the math expressions used to calculate the powers of the Shammas polynomial. The powers can be integers. In most cases, the powers are non-integers.

The Pade-Shammas Polynomials

The Pade-Shammas polynomials combine Pade and Shammas polynomials concepts. They have the following multiple regression general form:

$$y = a_0 + \sum_{i=1}^{m} a_i\, x^{gx1(i,A,B,C)} - \sum_{i=1}^{n} b_i\, y\, x^{gx2(i,A,B,C)} \qquad (6)$$

Notice that the integral powers in normal Pade polynoomials are replaced with power functions gx1() and gx2() that can be identical or different. The latter type gives the Pade-Shammas polynomials, in general, more flexibility in fitting a given set of (x, y) data. This statement is truer when the values of A, B, and C (when used) are different for the gx1() and gx2() functions. Part II of this study uses this scheme to give more flexibility to Pade-Shammas polynomials. Also note that Part IB conducts the Pade-Shammas polynomial fitting on the same functions but without normalizing the x and y data.

When using the Pade-Shammas polynomials to calculate new values employee the following equation:

$$y = [a_0 + \sum_{i=1}^{m} a_i\, x^{gx1(i,A,B,C)}] / [1 + \sum_{i=1}^{n} b_i\, x^{gx2(i,A,B,C)}] \qquad (7)$$

I am appending my last name to the name Pade in an effort to make sure it is a unique combined name. Using a more general descriptive name risks conflicts with other mathematicians that may use the same descriptive name for different types of Pade polynomials. I certainly would like to avoid unpleasant communications with these fine mathematicians!

This study looks at Pade-Shammas polynomials used to approximate common functions that include:

- Trigonometric functions and their inverses.
- Hyperbolic functions.
- Logarithmic functions.
- Exponential functions.
- Bessel functions $J_0(x)$ to $J_5(x)$.
- The sine and cosine integrals.

- The Fresnel sine and cosine.
- Inverse student-t functions.
- The common logarithm of the gamma function.
- The digamma function.
- The trigamma function.

The digamma function is defined as:

$$\psi(x) = \Gamma'(x) / \Gamma(x) = \frac{d \ln(\Gamma(x))}{dx} \quad (8)$$

The following Matlab function implements the code for the digamma function:

```
function y = digamma(x)
%DIGAMMA Summary of this function goes here
%   Detailed explanation goes here
  h = 0.001;
  fp = gammaln(x+h);
  fm = gammaln(x-h);
  y = (fp -fm)/2/h;
end
```

The above implementation of the digamma function was suggested by Albert Chan, a member of the hp museum web site, in a post he wrote on that site. The above code gives slightly more accurate results than the expression (gamma(x+h)-gamma(x-h)/(2*h)/gamma(x).

The trigamma function is defined as:

$$\psi_1(x) = \frac{d^2}{dx^2} \ln(\Gamma(x)) \quad (9)$$

The following Matlab function implements the code for the trigamma function:

```
function y = trigamma(x)
%DIGAMMA Summary of this function goes here
%   Detailed explanation goes here
  h = 0.001;
  fp = gammaln(x+h);
  fm = gammaln(x-h);
  f0 = gammaln(x);
  y = (fp -2*f0 + fm)/h/h;
end
```

The study works with Pade-Shammas Polynomials that use the following ten equations to calculate the powers of Shammas polynomial terms:

Power = A + B*i (10)

Power = A + B/i (11)

Power = A+B*sqrt(i) (12)

Power = A+B*sqrt(i+C) (13)

Power = A+B*i+C*(i–1) (14)

Power = A+B*i–C*(i–1) (15)

Power = A+B*i+C*sqrt(i–1) (16)

Power = A+B*sqrt(i)+C*(i-1) (17)

Power = A+B*sqrt(i)+C*sqrt(i–1) (18)

Power = A + B * $[\log_{10}(i)]^4$ (19)

Where A, B, and C are constants, and i a variable that represents the term number in a Pade-Shammas polynomial. The last equation introduces a new power expression not used in the Shammas Polynomial approximation study.

There are several choices that we can use to determine the best Pade-Shammas polynomial orders and the best values for parameters A, B, and C. The best one is to use an optimization function to determine the best Pade-Shammas polynomial order and the best values for parameters A, B, and C. The function would use ranges for the polynomial orders as well as the values for A, B, C.

The study uses the Matlab particle swarm optimization function to select the best Shammas polynomial orders (for $Q_n(x)$ and $D_m(x)$) along with the best values for A, B, and C.

The approximations that I obtain are typically for a defined and suitable interval. It is your responsibility to implement expanded versions of the approximation functions that take wider ranges if arguments and map them onto the interval used. For example, given that my approximation for $\log_{10}(x)$ uses the range (1, 10), to calculate $\log_{10}(235)$ use:

$\log_{10}(235)$ =

$\log_{10}(2.35 * 100)$ =

$\log_{10}(2.35) + \log_{10}(100)$ =

$\log_{10}(2.35) + 2$

The argument of the $\log_{10}(x)$ function in the last line falls in the interval (1, 10).

✍ Please note that I use the following model to perform the multiple regression:

$$y = a_0 + \sum_{i=1}^{m} a_i\, x^{gx1(i,A,B,C)} + \sum_{i=1}^{n} b_i\, y\, x^{gx2(i,A,B,C)} \quad (20)$$

I added the second set of summations instead of subtracting them (as in equation 6). Therefore, when you use the regression coefficients, use the following equation:

$$y = [a_0 + \sum_{i=1}^{m} a_i\, x^{gx1(i,A,B,C)}] / [1 - \sum_{i=1}^{n} b_i\, x^{gx2(i,A,B,C)}] \quad (21)$$

Notice the minus sign after the constant 1 in equation 21. I use equation 21 in the Matlab functions PadeShamPoly2() and PadeShamPoly3() that I present in the next section.

✍ The functions that perform the Pade-Shammas approximations convert both the x and y data into values that fall in the range of (1, 2). The rational behind the transformation of data onto the range (1, 2) is to prevent the cuver fitting process from wild swings and excessive rounding errors. The reports for the curve fitting results contain the values of minimum values for x and y (xmin and ymin, respectively) and maximum values for x and y (xmax and ymax, respectively). To use the Pade-Shammas polynomials perform the following:

1. Transform you x value(s) into the range of (1, 2) by using the expression (x – xmin)/(xmax – xmin) + 1. Use the xmin and xmax in the associated approximation results. You can still use your transformed x value if it reasonably falls outside the range (1, 2).
2. Calculate the value for yc. If the result is reasonably outside the range (1, 2) then you are fine.
3. Reverse mapping the value of y using the expression (yc – 1)*(ymax – ymin)+ymin. Use the ymin and ymax in the associated approximation results.

MATLAB CODE

The algorithm in this study uses particle swarm optimization to obtain the best values for parameters A and B in a prespecified range of Pade-Shammas polynomial orders for $Q_n(x)$ and $D_m(x)$.

This study uses the function PadeShamPoly2 () to perform various Pade-Shammas polynomial curve fitting:

```
function PadeShamPoly2(fx,gx,xRange,Lb,Ub,runNum,sFxName,diaryFilename)
%PADESHAMPOLY2 Summary of this function goes here
%   Detailed explanation goes here
clc
global bDeleteIfExists
global bUseDiary
global xdata
global ydata
global orderA
global orderB
global ggx

warning('off','all')
if isempty(sFxName)
  sFxName = getFuncName(fx);
end
xdata = xRange';
ydata = xdata;
for i=1:length(xdata)
  ydata(i)=fx(xdata(i));
end
ggx = gx;

xmin = min(xdata);
xmax = max(xdata);
ymin = min(ydata);
ymax = max(ydata);
xdata = (xdata - xmin)/(xmax - xmin) + 1;
ydata = (ydata - ymin)/(ymax - ymin) + 1;
fprintf('Fitting %s in range (%f, %f)\n', sFxName, xmin ,xmax);

options = optimoptions('particleswarm', 'Display', 'iter');
[x,psAICc] = particleswarm(@optimFunc,4,Lb,Ub,options);
A = x(1);
B = x(2);
orderA = round(x(3));
orderB = round(x(4));
if bUseDiary
  diaryFilename = strrep(diaryFilename, ".txt", strcat("_",
num2str(orderA),"_", num2str(orderB),"_run", num2str(runNum),".txt"));
  if exist(diaryFilename, 'file')==2
    if bDeleteIfExists
      delete(diaryFilename);
    else
      return;
    end
  end
end
X = [];
for i=1:orderA
  xs = xdata.^gx(i,A,B);
  X = [X;xs'];
end
```



```
for i=1:orderB
  xs = ydata.*(xdata.^gx(i,A,B));
  X = [X;xs'];
end
X = X';
lm = fitlm(X,ydata);
if bUseDiary
  diary(diaryFilename)
end
fprintf('Fitting %s in range (%f, %f)\n', sFxName, xmin ,xmax);
sp = getFuncName(gx);
fprintf('Pade-Shammas polynomial power is %s\n', sp);
format long
disp(lm);
anva = anova(lm,'summary');
disp(anva);
format short
fprintf('A = %f, B = %f\n', A, B);
fprintf('order Q(x) = %f, order D(x) = %f\n', orderA, orderB);
fprintf("Xmin = %f and Xmax = %f\n", xmin, xmax);
fprintf("Ymin = %f and Ymax = %f\n", ymin, ymax);
lstPwrA = zeros(orderA,1);
for i=1:orderA
  lstPwrA(i) = gx(i,A,B);
end
lstPwrB = zeros(orderB,1);
for i=1:orderB
  lstPwrB(i) = gx(i,A,B);
end
fprintf('List of powers for Q(x): [');
for i=1:orderA-1
 fprintf('%f, ', lstPwrA(i));
end
fprintf('%f]\n', lstPwrA(orderA));
fprintf('List of powers for D(x): [');
for i=1:orderB-1
 fprintf('%f, ', lstPwrB(i));
end
fprintf('%f]\n', lstPwrB(orderB));
fprintf('Fitting %s in range (%f, %f)\n', sFxName, xmin ,xmax);
n = length(xdata);
sumsqr = 0;
for i=1:n
  qx = lm.Coefficients{1,1};
  for j=2:orderA+1
    qx = qx + lm.Coefficients{j,1} * xdata(i)^lstPwrA(j-1);
  end
  dx = 1;
  k = 1;
  for j=orderA+2:orderA+1+orderB
    dx = dx - lm.Coefficients{j,1} * xdata(i)^lstPwrB(k);
    k = k + 1;
  end
  yc = qx/dx;
  sumsqr = sumsqr + (ydata(i) - yc)^2;
end
k = orderA + orderB + 1;
```



```
fprintf('MSS of errors squared = %e\n', sqrt(sumsqr)/n);
fprintf('Corrected MSS of errors squared = %e\n', sqrt(sumsqr)/n*sqrt(2));
fprintf("R-Squared = %12.8f\n", lm.Rsquared.Ordinary);
fprintf("R-Squared Adjusted = %12.8f\n", lm.Rsquared.Adjusted);
AIC = lm.ModelCriterion.AIC;
AICc = AIC + 2*k*(1 + (k+1)/(n-k-1));
fprintf('Particle swarm AICc =  %e\n', psAICc);
fprintf('AIC =  %e\n', AIC);
fprintf('AICc =  %e\n', AICc);

if bUseDiary
  diary off
end
end

function AICc = optimFunc(x)
  global xdata
  global ydata
  global orderA
  global orderB
  global ggx

  A = x(1);
  B = x(2);
  orderA = round(x(3));
  orderB = round(x(4));
  X = [];
  for i=1:orderA
    xs = xdata.^ggx(i,A,B);
    X = [X;xs'];
  end
  for i=1:orderB
    xs = ydata.*(xdata.^ggx(i,A,B));
    X = [X;xs'];
  end
  X = X';
  lm = fitlm(X,ydata);

  n = length(xdata);
  k = orderA + orderB + 1;
  AIC = lm.ModelCriterion.AIC;
  AICc = AIC + 2*k*(1 + (k+1)/(n-k-1));
  if isinf(AICc), AICc = -1e+99; end
end

function sFx = getFuncName(fx)
  sFx = func2str(fx);
  if sFx(1:2)=="@("
    i = strfind(sFx,")");
    sFx = sFx(i(1)+1:end);
  elseif sFx(1)=="@"
    sFx = strcat(sFx(2:end),".m");
  else
    % return sFx as is
  end
end
```


The parameters of function PadeShamPoly2 () are:

- The parameter fx is the handle (or inline function) for the function being approximated. An example is @(x)cos(x) which also shows the ***recommended format*** for the argument of parameter fx.
- The parameter gx is the handle (or inline function) for the function that calculates the powers of the Shammas polynomial. An example is @(i,A,B)A+B*sqrt(i) which also shows the ***recommended format*** for the argument of parameter gx.
- The parameter xRange is the array that specifies the minimum value, increment value, and maximum value for the range of approximation.
- The parameter Lb is is the array of lower limits for the parameters A and B, and the Shammas polynomial order. An example is [0.1 0 3].
- The parameter Ub is is the array of upper limits for the parameters A and B, and the Shammas polynomial order. An example is [1 5 7].
- The parameter runNum specifies the run number. The arguments for this parameter have nothing to do with the calculations and serve in fine tuning the name of the diary files, when used.
- The optional parameter sFxName is the name of the approximated function. An example is “cos(x)”.
- The parameter diaryFilename is the name of the diary file. An example is “cos_1.txt”.

The above listing performs the following tasks:

1. Initialize the data for the curve fitting. The function uses the global variables xdata and ydata to store the data for the polynomial fitting.
2. Store the handle of function gx in the global handle ggx.
3. Normalize the x and y data to have their values in the range (1, 2).
4. Set the optimization options and then call the Matlab function particleswarm(). The function call returns the optimized values of A and B and the optimum value for *corrected* Akaike information criterion (AICc). The arguments for this function call are:
    a. The handle to the local function optimFunc() that calculates the root mean sum of errors squared.
    b. The number of optimized variables which is 2.
    c. The lower and upper bounds arrays, Lb and Ub, respectively,
    d. The optimization parameters for function particleswarm().

5. Retrieve the optimum values and perform a Shammas polynomial fit for the best values of A and B.
6. Display the results of the regression and its associated ANOVA table.
7. Display the list of Shammas polynomial powers.
8. Display the range of the approximated function.
9. Calculate and display the value of the mean square root of the sum of squared errors. This statistic serves as a check that the Shammas polynomial performs well in checking the training data.
10. Calculate and display the value of the *corrected* mean square root of the sum of squared errors. This value equals the mean square root of the sum of squared errors multiplied by the square root of the number of Shammas polynomial parameters (i.e. A, B, and C, if used). In the case of PadeShamPoly2(), the function uses square root of 2.
11. Calculate and display the *corrected* Akaike information criterion. This statistic is calculated using:

$$AIC = n * \ln(SSE/n) + 2*k \quad (8)$$

$$AICc = AIC + 2*k*(k+1)/(n-k-1) \quad (9)$$

Where n is the number of observations, k is the total number of regression coefficients (including the intercept), and SSE is the sum of squared errors. The program obtains the value of AIC using lm.ModelCriterion.AIC. The program uses equation (3) to calculate the value for AICc.

12. Close the diary file, if one is used.

The function optimFunc() obtains the array x containing the current values of A and B, and the best Shammas polynomial order. The function calculates the transformed variables needed to perform a curve fit for a Shammas polynomial. This task calls the Matlab function fitlm(). The optimFunc() function returns the AICc as its result. I am using this statistic since the optimization is dealing with different Shammas polynomial orders and thus a varying number of polynomial coefficients.

The function getFuncName() returns a string-type function name given a handle of a function. The best way to take advantage of this function is to supply arguments like @(x)cos(x) and @(x,A,B)A+B*sqrt(x). Such arguments allow the function to discard the part that declares the variable(s) and return the part that comes after the first closed parenthesis (e.g. cos(x) and A+B*sqrt(x) for the above examples). If

you supply an argument like @fx1 which refers to the file fx1.m that defines the function fx1() then the function getFuncName() returns *fx1.m.* This string value indicates that you a rereferencing a separate Matlab file that implements the code for fx1.

It is worth going over the code that estmates the values of y to calculate the sum of error squared:

```
n = length(xdata);
sumsqr = 0;
for i=1:n
  qx = lm.Coefficients{1,1};
  for j=2:orderA+1
    qx = qx + lm.Coefficients{j,1} * xdata(i)^lstPwrA(j-1);
  end
  dx = 1;
  k = 1;
  for j=orderA+2:orderA+1+orderB
    dx = dx - lm.Coefficients{j,1} * xdata(i)^lstPwrB(k);
    k = k + 1;
  end
  yc = qx/dx;
  sumsqr = sumsqr + (ydata(i) - yc)^2;
end
```

The above code used the arrays lstPwrA and lstPwrB to store the powers used in the Shammas polynomials $Q_n(x)$ and $D_m(x)$, respectively. The first for loop calculate the value for Shammas polynomials $Q_n(x)$ using variable qx. Notice that the code initializes the value of qx using the intercept of the regression model. The loop accesses the powers of $Q_n(x)$ using lstPwrA(j-1). The second for loop calculates the value for Shammas polynomials $D_m(x)$using variable dx. Notice that the code initializes the value of dx using the constant 1. The loop accesses the powers of $Q_n(x)$ using lstPwrB(k). Here k is initialized with the value 1 and is incremented in the for loop. Also noticed that the value of variable dx is decremented with the expression lm.Coefficients{j,1} * xdata(i)^lstPwrB(k). I have explained the above code because you may need similar code to estimate other values of y given an array (or scalar) of x, as shown below:

```
n = length(your_xdata);
for i=1:n
  qx = lm.Coefficients{1,1};
  for j=2:orderA+1
    qx = qx + lm.Coefficients{j,1} * your_xdata(i)^lstPwrA(j-1);
  end
  dx = 1;
  k = 1;
  for j=orderA+2:orderA+1+orderB
    dx = dx - lm.Coefficients{j,1} * your_xdata(i)^lstPwrB(k);
```



```
    k = k + 1;
  end
  your_y(i) = qx/dx;
end
```

Under the current calculation scheme, note that, the function PadeShamPoly2 () does not explicitly iterate over different values of A and B. It delegates that task to the Matlab function particleswarm().

The study also uses the function PadeShamPoly3() to work with Shammas polynomial powers that use parameters A, B, and C:

```
function PadeShamPoly3(fx,gx,xRange,Lb,Ub,runNum,sFxName,diaryFilename)
%PADESHAMPOLY3 Summary of this function goes here
%   Detailed explanation goes here
clc
global bDeleteIfExists
global bUseDiary
global xdata
global ydata
global orderA
global orderB
global ggx

warning('off','all')
if isempty(sFxName)
  sFxName = getFuncName(fx);
end
xdata = xRange';
ydata = xdata;
for i=1:length(xdata)
  ydata(i)=fx(xdata(i));
end
ggx = gx;

xmin = min(xdata);
xmax = max(xdata);
ymin = min(ydata);
ymax = max(ydata);
xdata = (xdata - xmin)/(xmax - xmin) + 1;
ydata = (ydata - ymin)/(ymax - ymin) + 1;
fprintf('Fitting %s in range (%f, %f)\n', sFxName, xmin ,xmax);

options = optimoptions('particleswarm', 'Display', 'iter');
[x,psAICc] = particleswarm(@optimFunc,5,Lb,Ub,options);
A = x(1);
B = x(2);
C = x(3);
orderA = round(x(4));
orderB = round(x(5));
if bUseDiary
  diaryFilename = strrep(diaryFilename, ".txt", strcat("_",
num2str(orderA),"_", num2str(orderB),"_run", num2str(runNum),".txt"));
  if exist(diaryFilename, 'file')==2
    if bDeleteIfExists
```



```
      delete(diaryFilename);
    else
      return;
    end
  end
end
X = [];
for i=1:orderA
  xs = xdata.^gx(i,A,B,C);
  X = [X;xs'];
end
for i=1:orderB
  xs = ydata.*(xdata.^gx(i,A,B,C));
  X = [X;xs'];
end
X = X';
lm = fitlm(X,ydata);

if bUseDiary
  diary(diaryFilename)
end
fprintf('Fitting %s in range (%f, %f)\n', sFxName, xmin ,xmax);
sp = getFuncName(gx);
fprintf('Pade-Shammas polynomial power is %s\n', sp);
format long
disp(lm);
anva = anova(lm,'summary');
disp(anva);
format short
fprintf('A = %f, B = %f, C = %f\n', A, B, C);
fprintf('order Q(x) = %f, order D(x) = %f\n', orderA, orderB);
fprintf("Xmin = %f and Xmax = %f\n", xmin, xmax);
fprintf("Ymin = %f and Ymax = %f\n", ymin, ymax);
lstPwrA = zeros(orderA,1);
for i=1:orderA
  lstPwrA(i) = gx(i,A,B,C);
end
lstPwrB = zeros(orderB,1);
for i=1:orderB
  lstPwrB(i) = gx(i,A,B,C);
end
fprintf('List of powers for Q(x): [');
for i=1:orderA-1
 fprintf('%f, ', lstPwrA(i));
end
fprintf('%f]\n', lstPwrA(orderA));
fprintf('List of powers for D(x): [');
for i=1:orderB-1
 fprintf('%f, ', lstPwrB(i));
end
fprintf('%f]\n', lstPwrB(orderB));
fprintf('Fitting %s in range (%f, %f)\n', sFxName, xmin ,xmax);
n = length(xdata);
sumsqr = 0;
for i=1:n
  qx = lm.Coefficients{1,1};
  for j=2:orderA+1
```



```
    qx = qx + lm.Coefficients{j,1} * xdata(i)^lstPwrA(j-1);
  end
  dx = 1;
  k = 1;
  for j=orderA+2:orderA+1+orderB
    dx = dx - lm.Coefficients{j,1} * xdata(i)^lstPwrB(k));
    k = k + 1;
  end
  yc = qx/dx;
  sumsqr = sumsqr + (ydata(i) - yc)^2;
end
k = orderA + orderB + 1;
fprintf('MSS of errors squared = %e\n', sqrt(sumsqr)/n);
fprintf('Corrected MSS of errors squared = %e\n', sqrt(sumsqr)/n*sqrt(3));
fprintf("R-Squared = %12.8f\n", lm.Rsquared.Ordinary);
fprintf("R-Squared Adjusted = %12.8f\n", lm.Rsquared.Adjusted);
AIC = lm.ModelCriterion.AIC;
AICc = AIC + 2*k*(1 + (k+1)/(n-k-1));
fprintf('Particle swarm AICc =  %e\n', psAICc);
fprintf('AIC =  %e\n', AIC);
fprintf('AICc =  %e\n', AICc);

if bUseDiary
  diary off
end
end

function AICc = optimFunc(x)
  global xdata
  global ydata
  global orderA
  global orderB
  global ggx

  A = x(1);
  B = x(2);
  C = x(3);
  orderA = round(x(4));
  orderB = round(x(5));
  X = [];
  for i=1:orderA
    xs = xdata.^ggx(i,A,B,C);
    X = [X;xs'];
  end
  for i=1:orderB
    xs = ydata.*(xdata.^ggx(i,A,B,C));
    X = [X;xs'];
  end
  X = X';
  lm = fitlm(X,ydata);

  n = length(xdata);
  k = orderA + orderB + 1;
  AIC = lm.ModelCriterion.AIC;
  AICc = AIC + 2*k*(1 + (k+1)/(n-k-1));
  if isinf(AICc), AICc = -1e+99; end
end
```



```
function sFx = getFuncName(fx)
  sFx = func2str(fx);
  if sFx(1:2)=="@("
    i = strfind(sFx,")");
    sFx = sFx(i(1)+1:end);
  elseif sFx(1)=="@"
    sFx = strcat(sFx(2:end),".m");
  else
    % return sFx as is
  end
end
```

The parameters of function PadeShamPoly3() are:

- The parameter fx is the handle (or inline function) for the function being approximated. An example is @(x)cos(x) which also shows the ***recommended format*** for the argument of parameter fx.
- The parameter gx is the handle (or inline function) for the function that calculates the powers of the Shammas polynomial. An example is @(i,A,B,c)A+B*sqrt(i+C) which also shows the ***recommended format*** for the argument of parameter gx.
- The parameter xRange is the array that specifies the minimum value, increment value, and maximum value for the range of approximation.
- The parameter Lb is is the array of lower limits for the parameters A, B, C and the Shammas polynomial order. An example is [0.1 0 1 3].
- The parameter Ub is is the array of upper limits for the parameters A, B and C, and the Shammas polynomial order. An example is [1 3 3 5].
- The parameter runNum specifies the run number. The arguments for this parameter have nothing to do with the calculations and serve in fine tuning the name of the diary files, when used.
- The optional parameter sFxName is the name of the approximated function. An example is “cos(x)”.
- The parameter diaryFilename is the name of the diary file. An example is “cos_1.txt”.

The above listing performs basically the same tasks as function PadeShamPoly2 () with the exception that it optimizes the parameter C and manages values related to that parameter. The function optimFunc() is the function to be optimized to get the best values for the Shammas polynomial power parameters A, B, and C, and the best Shammas polynomial order. The function getFuncName() returns a string-type function name given a handle of a function.

Using the calculation scheme, note that function PadeShamPoly3() does not explicitly iterate over different values of A, B and C. It delegates that task to the Matlab function particleswarm().

The following Matlab script goAll performs the various Pade-Shammas polynomial fittings for the various tested functions:

```
% Version 1.0.0 8/9/2020
global bUseDiary
global bDeleteIfExists

bUseDiary = true;
bDeleteIfExists = true; % or false
selIdx = 0; % note a zero value will execute all the models
runNum = 1;
bShutdown = false;

tic;

if selIdx==0 || selIdx==1
  lstA = [0 0.5];
  lstB = [0.1 0.5];
  lstOrder = [3 7];
  Lb = [lstA(1) lstB(1) lstOrder(1) lstOrder(1)];
  Ub = [lstA(2) lstB(2) lstOrder(2) lstOrder(2)];
  gx = @(i,A,B)A+B*i;

PadeShamPoly2(@(x)acos(x),gx,[0:.01:1],Lb,Ub,runNum,"arccos(x)","arccos_1.txt
")

PadeShamPoly2(@(x)asin(x),gx,[0:.01:1],Lb,Ub,runNum,"arcsin(x)","arcsin_1.txt
")

PadeShamPoly2(@(x)atan(x),gx,[0:.01:1],Lb,Ub,runNum,"arctan(x)","arctan_1.txt
")

PadeShamPoly2(@(x)sin(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"sin(x)","sin_1.txt
")

PadeShamPoly2(@(x)cos(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"cos(x)","cos_1.txt
")
  PadeShamPoly2(@(x)tan(x),gx,[0:.01:1],Lb,Ub,runNum,"tan(x)","tan_1.txt")
  PadeShamPoly2(@(x)sinh(x),gx,[0:.01:5],Lb,Ub,runNum,"sinh(x)","sinh_1.txt")
  PadeShamPoly2(@(x)cosh(x),gx,[0:.01:5],Lb,Ub,runNum,"cosh(x)","cosh_1.txt")
  PadeShamPoly2(@(x)tanh(x),gx,[0:.01:3],Lb,Ub,runNum,"tanh(x)","tanh_1.txt")
  PadeShamPoly2(@(x)erf(x),gx,[0:.01:2.1],Lb,Ub,runNum,"erf(x)","erf_1.txt")
  PadeShamPoly2(@(x)exp(x),gx,[0:.01:2],Lb,Ub,runNum,"exp(x)","exp_1.txt")
  PadeShamPoly2(@(x)log(x),gx,[1:.01:10],Lb,Ub,runNum,"ln(x)","ln_1.txt")
  PadeShamPoly2(@(x)log10(x),gx,[1:.01:10],Lb,Ub,runNum,"log(x)","log_1.txt")
  PadeShamPoly2(@(x)10.^x,gx,[0:.01:1],Lb,Ub,runNum,"10^x","pwr10_1.txt")

PadeShamPoly2(@(x)tinv(0.95,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.95,x)","ti
nv1_1.txt")
```



```
PadeShamPoly2(@(x)tinv(0.975,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.975,x)","
tinv2_1.txt")

PadeShamPoly2(@(x)log10(gamma(x)),gx,[2:0.1:100],Lb,Ub,runNum,"log10Gamma(x)"
,"log10Gamma_1.txt")

PadeShamPoly2(@(x)digamma(x),gx,[2:0.1:100],Lb,Ub,runNum,"digamma(x)","digamm
a_1.txt")

PadeShamPoly2(@(x)trigamma(x),gx,[1:0.1:100],Lb,Ub,runNum,"tigamma(x)","triga
mma_1.txt")

PadeShamPoly2(@(x)besselj(0,x),gx,[2:0.1:30],Lb,Ub,runNum,"J0(x)","J0x_1.txt"
)

PadeShamPoly2(@(x)besselj(1,x),gx,[0:0.1:30],Lb,Ub,runNum,"J1(x)","J1x_1.txt"
)

PadeShamPoly2(@(x)besselj(2,x),gx,[0:0.1:30],Lb,Ub,runNum,"J2(x)","J2x_1.txt"
)

PadeShamPoly2(@(x)besselj(3,x),gx,[0:0.1:30],Lb,Ub,runNum,"J3(x)","J3x_1.txt"
)

PadeShamPoly2(@(x)besselj(4,x),gx,[0:0.1:30],Lb,Ub,runNum,"J4(x)","J4x_1.txt"
)

PadeShamPoly2(@(x)besselj(5,x),gx,[0:0.1:30],Lb,Ub,runNum,"J5(x)","J5x_1.txt"
)

PadeShamPoly2(@(x)integral(@(z)sin(z.^2),0,x),gx,[1:0.01:5],Lb,Ub,runNum,"Fre
snelSine(x)","FresnelSine_1.txt")

PadeShamPoly2(@(x)integral(@(z)cos(z.^2),0,x),gx,[0.5:0.01:5],Lb,Ub,runNum,"F
resnelCosine(x)","FresnelCosine_1.txt")

PadeShamPoly2(@(x)integral(@(z)sin(z)./z,0,x),gx,[1.3:0.1:20],Lb,Ub,runNum,"S
i(x)","Si_1.txt")
  PadeShamPoly2(@(x)0.57721566+log(x)-integral(@(z)(1-
cos(z))./z,0,x),gx,[0.5:0.1:20],Lb,Ub,runNum,"Ci(x)","CI_1.txt")

PadeShamPoly2(@(x)asinh(x),gx,[0:0.1:100],Lb,Ub,runNum,"asinh(x)","asinh_1.tx
t")

PadeShamPoly2(@(x)acosh(x),gx,[1:0.1:100],Lb,Ub,runNum,"acosh(x)","acosh_1.tx
t")

PadeShamPoly2(@(x)atanh(x),gx,[0:0.001:0.999],Lb,Ub,runNum,"atanh(x)","atanh_
1.txt")
end

if selIdx==0 || selIdx==2
  lstA = [0 2];
  lstB = [0.1 2];
  lstOrder = [3 7];
  Lb = [lstA(1) lstB(1) lstOrder(1) lstOrder(1)];
```



```
  Ub = [lstA(2) lstB(2) lstOrder(2) lstOrder(2)];
  gx = @(i,A,B)A+B/i;

PadeShamPoly2(@(x)acos(x),gx,[0:.01:1],Lb,Ub,runNum,"arccos(x)","arccos_2.txt
")

PadeShamPoly2(@(x)asin(x),gx,[0:.01:1],Lb,Ub,runNum,"arcsin(x)","arcsin_2.txt
")

PadeShamPoly2(@(x)atan(x),gx,[0:.01:1],Lb,Ub,runNum,"arctan(x)","arctan_2.txt
")

PadeShamPoly2(@(x)sin(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"sin(x)","sin_2.txt
")

PadeShamPoly2(@(x)cos(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"cos(x)","cos_2.txt
")
  PadeShamPoly2(@(x)tan(x),gx,[0:.01:1],Lb,Ub,runNum,"tan(x)","tan_2.txt")
  PadeShamPoly2(@(x)sinh(x),gx,[0:.01:5],Lb,Ub,runNum,"sinh(x)","sinh_2.txt")
  PadeShamPoly2(@(x)cosh(x),gx,[0:.01:5],Lb,Ub,runNum,"cosh(x)","cosh_2.txt")
  PadeShamPoly2(@(x)tanh(x),gx,[0:.01:3],Lb,Ub,runNum,"tanh(x)","tanh_2.txt")
  PadeShamPoly2(@(x)erf(x),gx,[0:.01:2.1],Lb,Ub,runNum,"erf(x)","erf_2.txt")
  PadeShamPoly2(@(x)exp(x),gx,[0:.01:2],Lb,Ub,runNum,"exp(x)","exp_2.txt")
  PadeShamPoly2(@(x)log(x),gx,[1:.01:10],Lb,Ub,runNum,"ln(x)","ln_2.txt")
  PadeShamPoly2(@(x)log10(x),gx,[1:.01:10],Lb,Ub,runNum,"log(x)","log_2.txt")
  PadeShamPoly2(@(x)10.^x,gx,[0:.01:1],Lb,Ub,runNum,"10^x","pwr10_2.txt")

PadeShamPoly2(@(x)tinv(0.95,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.95,x)","ti
nv1_2.txt")

PadeShamPoly2(@(x)tinv(0.975,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.975,x)","
tinv2_2.txt")

PadeShamPoly2(@(x)log10(gamma(x)),gx,[2:0.1:100],Lb,Ub,runNum,"log10Gamma(x)"
,"log10Gamma_2.txt")

PadeShamPoly2(@(x)digamma(x),gx,[2:0.1:100],Lb,Ub,runNum,"digamma(x)","digamm
a_2.txt")

PadeShamPoly2(@(x)trigamma(x),gx,[1:0.1:100],Lb,Ub,runNum,"trigamma(x)","trig
amma_2.txt")

PadeShamPoly2(@(x)besselj(0,x),gx,[2:0.1:30],Lb,Ub,runNum,"J0(x)","J0x_2.txt"
)

PadeShamPoly2(@(x)besselj(1,x),gx,[0:0.1:30],Lb,Ub,runNum,"J1(x)","J1x_2.txt"
)

PadeShamPoly2(@(x)besselj(2,x),gx,[0:0.1:30],Lb,Ub,runNum,"J2(x)","J2x_2.txt"
)

PadeShamPoly2(@(x)besselj(3,x),gx,[0:0.1:30],Lb,Ub,runNum,"J3(x)","J3x_2.txt"
)

PadeShamPoly2(@(x)besselj(4,x),gx,[0:0.1:30],Lb,Ub,runNum,"J4(x)","J4x_2.txt"
)
```



```
PadeShamPoly2(@(x)besselj(5,x),gx,[0:0.1:30],Lb,Ub,runNum,"J5(x)","J5x_2.txt"
)

PadeShamPoly2(@(x)integral(@(z)sin(z.^2),0,x),gx,[1:0.01:5],Lb,Ub,runNum,"Fre
snelSine(x)","FresnelSine_2.txt")

PadeShamPoly2(@(x)integral(@(z)cos(z.^2),0,x),gx,[0.5:0.01:5],Lb,Ub,runNum,"F
resnelCosine(x)","FresnelCosine_2.txt")

PadeShamPoly2(@(x)integral(@(z)sin(z)./z,0,x),gx,[1.3:0.1:20],Lb,Ub,runNum,"S
i(x)","Si_2.txt")
  PadeShamPoly2(@(x)0.57721566+log(x)-integral(@(z)(1-
cos(z))./z,0,x),gx,[0.5:0.1:20],Lb,Ub,runNum,"Ci(x)","Ci_2.txt")

PadeShamPoly2(@(x)asinh(x),gx,[0:0.1:100],Lb,Ub,runNum,"asinh(x)","asinh_2.tx
t")

PadeShamPoly2(@(x)acosh(x),gx,[1:0.1:100],Lb,Ub,runNum,"acosh(x)","acosh_2.tx
t")

PadeShamPoly2(@(x)atanh(x),gx,[0:0.001:0.999],Lb,Ub,runNum,"atanh(x)","atanh_
2.txt")
end

if selIdx==0 || selIdx==3
  lstA = [0 1.4];
  lstB = [0.1 1];
  lstOrder = [3 7];
  Lb = [lstA(1) lstB(1) lstOrder(1) lstOrder(1)];
  Ub = [lstA(2) lstB(2) lstOrder(2) lstOrder(2)];
  gx = @(i,A,B)A+B*sqrt(i);

PadeShamPoly2(@(x)acos(x),gx,[0:.01:1],Lb,Ub,runNum,"arccos(x)","arccos_3.txt
")

PadeShamPoly2(@(x)asin(x),gx,[0:.01:1],Lb,Ub,runNum,"arcsin(x)","arcsin_3.txt
")

PadeShamPoly2(@(x)atan(x),gx,[0:.01:1],Lb,Ub,runNum,"arctan(x)","arctan_3.txt
")

PadeShamPoly2(@(x)sin(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"sin(x)","sin_3.txt
")

PadeShamPoly2(@(x)cos(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"cos(x)","cos_3.txt
")
  PadeShamPoly2(@(x)tan(x),gx,[0:.01:1],Lb,Ub,runNum,"tan(x)","tan_3.txt")
  PadeShamPoly2(@(x)sinh(x),gx,[0:.01:5],Lb,Ub,runNum,"sinh(x)","sinh_3.txt")
  PadeShamPoly2(@(x)cosh(x),gx,[0:.01:5],Lb,Ub,runNum,"cosh(x)","cosh_3.txt")
  PadeShamPoly2(@(x)tanh(x),gx,[0:.01:3],Lb,Ub,runNum,"tanh(x)","tanh_3.txt")
  PadeShamPoly2(@(x)erf(x),gx,[0:.01:2.1],Lb,Ub,runNum,"erf(x)","erf_3.txt")
  PadeShamPoly2(@(x)exp(x),gx,[0:.01:2],Lb,Ub,runNum,"exp(x)","exp_3.txt")
  PadeShamPoly2(@(x)log(x),gx,[1:.01:10],Lb,Ub,runNum,"ln(x)","ln_3.txt")
  PadeShamPoly2(@(x)log10(x),gx,[1:.01:10],Lb,Ub,runNum,"log(x)","log_3.txt")
  PadeShamPoly2(@(x)10.^x,gx,[0:.01:1],Lb,Ub,runNum,"10^x","pwr10_3.txt")
```



```
PadeShamPoly2(@(x)tinv(0.95,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.95,x)","ti
nv1_3.txt")

PadeShamPoly2(@(x)tinv(0.975,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.975,x)","
tinv2_3.txt")

PadeShamPoly2(@(x)log10(gamma(x)),gx,[2:0.1:100],Lb,Ub,runNum,"log10Gamma(x)"
,"log10Gamma_3.txt")

PadeShamPoly2(@(x)digamma(x),gx,[2:0.1:100],Lb,Ub,runNum,"digamma(x)","digamm
a_3.txt")

PadeShamPoly2(@(x)trigamma(x),gx,[1:0.1:100],Lb,Ub,runNum,"trigamma(x)","trig
amma_3.txt")

PadeShamPoly2(@(x)besselj(0,x),gx,[2:0.1:30],Lb,Ub,runNum,"J0(x)","J0x_3.txt"
)

PadeShamPoly2(@(x)besselj(1,x),gx,[0:0.1:30],Lb,Ub,runNum,"J1(x)","J1x_3.txt"
)

PadeShamPoly2(@(x)besselj(2,x),gx,[0:0.1:30],Lb,Ub,runNum,"J2(x)","J2x_3.txt"
)

PadeShamPoly2(@(x)besselj(3,x),gx,[0:0.1:30],Lb,Ub,runNum,"J3(x)","J3x_3.txt"
)

PadeShamPoly2(@(x)besselj(4,x),gx,[0:0.1:30],Lb,Ub,runNum,"J4(x)","J4x_3.txt"
)

PadeShamPoly2(@(x)besselj(5,x),gx,[0:0.1:30],Lb,Ub,runNum,"J5(x)","J5x_3.txt"
)

PadeShamPoly2(@(x)integral(@(z)sin(z.^2),0,x),gx,[1:0.01:5],Lb,Ub,runNum,"Fre
snelSine(x)","FresnelSine_3.txt")

PadeShamPoly2(@(x)integral(@(z)cos(z.^2),0,x),gx,[0.5:0.01:5],Lb,Ub,runNum,"F
resnelCosine(x)","FresnelCosine_3.txt")

PadeShamPoly2(@(x)integral(@(z)sin(z)./z,0,x),gx,[1.3:0.1:20],Lb,Ub,runNum,"S
i(x)","Si_3.txt")
  PadeShamPoly2(@(x)0.57721566+log(x)-integral(@(z)(1-
cos(z))./z,0,x),gx,[0.5:0.1:20],Lb,Ub,runNum,"Ci(x)","Ci_3.txt")

PadeShamPoly2(@(x)asinh(x),gx,[0:0.1:100],Lb,Ub,runNum,"asinh(x)","asinh_3.tx
t")

PadeShamPoly2(@(x)acosh(x),gx,[1:0.1:100],Lb,Ub,runNum,"acosh(x)","acosh_3.tx
t")

PadeShamPoly2(@(x)atanh(x),gx,[0:0.001:0.999],Lb,Ub,runNum,"atanh(x)","atanh_
3.txt")
end

if selIdx==0 || selIdx==4
  lstA = [0 1];
```



```
  lstB = [0.1 1];
  lstC = [0 2];
  lstOrder = [3 7];
  Lb = [lstA(1) lstB(1) lstC(1) lstOrder(1) lstOrder(1)];
  Ub = [lstA(2) lstB(2) lstC(2) lstOrder(2) lstOrder(2)];
  gx = @(i,A,B,C)A+B*sqrt(i+C);

PadeShamPoly3(@(x)acos(x),gx,[0:.01:1],Lb,Ub,runNum,"arccos(x)","arccos_4.txt
")

PadeShamPoly3(@(x)asin(x),gx,[0:.01:1],Lb,Ub,runNum,"arcsin(x)","arcsin_4.txt
")

PadeShamPoly3(@(x)atan(x),gx,[0:.01:1],Lb,Ub,runNum,"arctan(x)","arctan_4.txt
")

PadeShamPoly3(@(x)sin(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"sin(x)","sin_4.txt
")

PadeShamPoly3(@(x)cos(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"cos(x)","cos_4.txt
")
  PadeShamPoly3(@(x)tan(x),gx,[0:.01:1],Lb,Ub,runNum,"tan(x)","tan_4.txt")
  PadeShamPoly3(@(x)sinh(x),gx,[0:.01:5],Lb,Ub,runNum,"sinh(x)","sinh_4.txt")
  PadeShamPoly3(@(x)cosh(x),gx,[0:.01:5],Lb,Ub,runNum,"cosh(x)","cosh_4.txt")
  PadeShamPoly3(@(x)tanh(x),gx,[0:.01:3],Lb,Ub,runNum,"tanh(x)","tanh_4.txt")
  PadeShamPoly3(@(x)erf(x),gx,[0:.01:2.1],Lb,Ub,runNum,"erf(x)","erf_4.txt")
  PadeShamPoly3(@(x)exp(x),gx,[0:.01:2],Lb,Ub,runNum,"exp(x)","exp_4.txt")
  PadeShamPoly3(@(x)log(x),gx,[1:.01:10],Lb,Ub,runNum,"ln(x)","ln_4.txt")
  PadeShamPoly3(@(x)log10(x),gx,[1:.01:10],Lb,Ub,runNum,"log(x)","log_4.txt")
  PadeShamPoly3(@(x)10.^x,gx,[0:.01:1],Lb,Ub,runNum,"10^x","pwr10_4.txt")

PadeShamPoly3(@(x)tinv(0.95,x),gx,[2:100],Lb,Ub,runNum,"tinv(0.95,x)","tinv1_
4.txt")

PadeShamPoly3(@(x)tinv(0.975,x),gx,[2:100],Lb,Ub,runNum,"tinv(0.975,x)","tinv
2_4.txt")

PadeShamPoly3(@(x)log10(gamma(x)),gx,[2:100],Lb,Ub,runNum,"log10Gamma(x)","lo
g10Gamma_4.txt")

PadeShamPoly3(@(x)digamma(x),gx,[2:100],Lb,Ub,runNum,"digamma(x)","digamma_4.
txt")

PadeShamPoly3(@(x)trigamma(x),gx,[1:100],Lb,Ub,runNum,"trigamma(x)","trigamma
_4.txt")

PadeShamPoly3(@(x)besselj(0,x),gx,[2:0.1:30],Lb,Ub,runNum,"J0(x)","J0x_4.txt"
)

PadeShamPoly3(@(x)besselj(1,x),gx,[0:0.1:30],Lb,Ub,runNum,"J1(x)","J1x_4.txt"
)

PadeShamPoly3(@(x)besselj(2,x),gx,[0:0.1:30],Lb,Ub,runNum,"J2(x)","J2x_4.txt"
)

PadeShamPoly3(@(x)besselj(3,x),gx,[0:0.1:30],Lb,Ub,runNum,"J3(x)","J3x_4.txt"
)
```



```
PadeShamPoly3(@(x)besselj(4,x),gx,[0:0.1:30],Lb,Ub,runNum,"J4(x)","J4x_4.txt"
)

PadeShamPoly3(@(x)besselj(5,x),gx,[0:0.1:30],Lb,Ub,runNum,"J5(x)","J5x_4.txt"
)

PadeShamPoly3(@(x)integral(@(z)sin(z.^2),0,x),gx,[1:0.01:5],Lb,Ub,runNum,"Fre
snelSine(x)","FresnelSine_4.txt")

PadeShamPoly3(@(x)integral(@(z)cos(z.^2),0,x),gx,[0.5:0.01:5],Lb,Ub,runNum,"F
resnelCosine(x)","FresnelCosine_4.txt")

PadeShamPoly3(@(x)integral(@(z)sin(z)./z,0,x),gx,[1.3:0.1:20],Lb,Ub,runNum,"S
i(x)","Si_4.txt")
  PadeShamPoly3(@(x)0.57721566+log(x)-integral(@(z)(1-
cos(z))./z,0,x),gx,[0.5:0.1:20],Lb,Ub,runNum,"Ci(x)","Ci_4.txt")

PadeShamPoly3(@(x)asinh(x),gx,[0:0.1:100],Lb,Ub,runNum,"asinh(x)","asinh_4.tx
t")

PadeShamPoly3(@(x)acosh(x),gx,[1:0.1:100],Lb,Ub,runNum,"acosh(x)","acosh_4.tx
t")

PadeShamPoly3(@(x)atanh(x),gx,[0:0.001:0.999],Lb,Ub,runNum,"atanh(x)","atanh_
4.txt")
end

if selIdx==0 || selIdx==5
  lstA = [0 0.5];
  lstB = [0.1 0.27];
  lstC = [0 0.27];
  lstOrder = [3 7];
  Lb = [lstA(1) lstB(1) lstC(1) lstOrder(1) lstOrder(1)];
  Ub = [lstA(2) lstB(2) lstC(2) lstOrder(2) lstOrder(2)];
  gx = @(i,A,B,C)A+B*i+C*(i-1);

PadeShamPoly3(@(x)acos(x),gx,[0:.01:1],Lb,Ub,runNum,"arccos(x)","arccos_5.txt
")

PadeShamPoly3(@(x)asin(x),gx,[0:.01:1],Lb,Ub,runNum,"arcsin(x)","arcsin_5.txt
")

PadeShamPoly3(@(x)atan(x),gx,[0:.01:1],Lb,Ub,runNum,"arctan(x)","arctan_5.txt
")

PadeShamPoly3(@(x)sin(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"sin(x)","sin_5.txt
")

PadeShamPoly3(@(x)cos(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"cos(x)","cos_5.txt
")
  PadeShamPoly3(@(x)tan(x),gx,[0:.01:1],Lb,Ub,runNum,"tan(x)","tan_5.txt")
  PadeShamPoly3(@(x)sinh(x),gx,[0:.01:5],Lb,Ub,runNum,"sinh(x)","sinh_5.txt")
  PadeShamPoly3(@(x)cosh(x),gx,[0:.01:5],Lb,Ub,runNum,"cosh(x)","cosh_5.txt")
  PadeShamPoly3(@(x)tanh(x),gx,[0:.01:3],Lb,Ub,runNum,"tanh(x)","tanh_5.txt")
  PadeShamPoly3(@(x)erf(x),gx,[0:.01:2.1],Lb,Ub,runNum,"erf(x)","erf_5.txt")
  PadeShamPoly3(@(x)exp(x),gx,[0:.01:2],Lb,Ub,runNum,"exp(x)","exp_5.txt")
```



```
  PadeShamPoly3(@(x)log(x),gx,[1:.01:10],Lb,Ub,runNum,"ln(x)","ln_5.txt")
  PadeShamPoly3(@(x)log10(x),gx,[1:.01:10],Lb,Ub,runNum,"log(x)","log_5.txt")
  PadeShamPoly3(@(x)10.^x,gx,[0:.01:1],Lb,Ub,runNum,"10^x","pwr10_5.txt")

PadeShamPoly3(@(x)tinv(0.95,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.95,x)","ti
nv1_5.txt")

PadeShamPoly3(@(x)tinv(0.975,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.975,x)","
tinv2_5.txt")

PadeShamPoly3(@(x)log10(gamma(x)),gx,[2:0.1:100],Lb,Ub,runNum,"log10Gamma(x)"
,"log10Gamma_5.txt")

PadeShamPoly3(@(x)digamma(x),gx,[2:0.1:100],Lb,Ub,runNum,"digamma(x)","digamm
a_5.txt")

PadeShamPoly3(@(x)trigamma(x),gx,[1:0.1:100],Lb,Ub,runNum,"trigamma(x)","trig
amma_5.txt")

PadeShamPoly3(@(x)besselj(0,x),gx,[2:0.1:30],Lb,Ub,runNum,"J0(x)","J0x_5.txt"
)

PadeShamPoly3(@(x)besselj(1,x),gx,[0:0.1:30],Lb,Ub,runNum,"J1(x)","J1x_5.txt"
)

PadeShamPoly3(@(x)besselj(2,x),gx,[0:0.1:30],Lb,Ub,runNum,"J2(x)","J2x_5.txt"
)

PadeShamPoly3(@(x)besselj(3,x),gx,[0:0.1:30],Lb,Ub,runNum,"J3(x)","J3x_5.txt"
)

PadeShamPoly3(@(x)besselj(4,x),gx,[0:0.1:30],Lb,Ub,runNum,"J4(x)","J4x_5.txt"
)

PadeShamPoly3(@(x)besselj(5,x),gx,[0:0.1:30],Lb,Ub,runNum,"J5(x)","J5x_5.txt"
)

PadeShamPoly3(@(x)integral(@(z)sin(z.^2),0,x),gx,[1:0.01:5],Lb,Ub,runNum,"Fre
snelSine(x)","FresnelSine_5.txt")

PadeShamPoly3(@(x)integral(@(z)cos(z.^2),0,x),gx,[0.5:0.01:5],Lb,Ub,runNum,"F
resnelCosine(x)","FresnelCosine_5.txt")

PadeShamPoly3(@(x)integral(@(z)sin(z)./z,0,x),gx,[1.3:0.1:20],Lb,Ub,runNum,"S
i(x)","Si_5.txt")
  PadeShamPoly3(@(x)0.57721566+log(x)-integral(@(z)(1-
cos(z))./z,0,x),gx,[0.5:0.1:20],Lb,Ub,runNum,"Ci(x)","Ci_5.txt")

PadeShamPoly3(@(x)asinh(x),gx,[0:0.1:100],Lb,Ub,runNum,"asinh(x)","asinh_5.tx
t")

PadeShamPoly3(@(x)acosh(x),gx,[1:0.1:100],Lb,Ub,runNum,"acosh(x)","acosh_5.tx
t")

PadeShamPoly3(@(x)atanh(x),gx,[0:0.001:0.999],Lb,Ub,runNum,"atanh(x)","atanh_
5.txt")
end
```



```
if selIdx==0 || selIdx==6
  lstA = [0 0.75];
  lstB = [0.1 0.25];
  lstC = [0 0.25];
  lstOrder = [3 7];
  Lb = [lstA(1) lstB(1) lstC(1) lstOrder(1) lstOrder(1)];
  Ub = [lstA(2) lstB(2) lstC(2) lstOrder(2) lstOrder(2)];
  gx = @(i,A,B,C)A+B*i-C*(i-1);

PadeShamPoly3(@(x)acos(x),gx,[0:.01:1],Lb,Ub,runNum,"arccos(x)","arccos_6.txt
")

PadeShamPoly3(@(x)asin(x),gx,[0:.01:1],Lb,Ub,runNum,"arcsin(x)","arcsin_6.txt
")

PadeShamPoly3(@(x)atan(x),gx,[0:.01:1],Lb,Ub,runNum,"arctan(x)","arctan_6.txt
")

PadeShamPoly3(@(x)sin(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"sin(x)","sin_6.txt
")

PadeShamPoly3(@(x)cos(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"cos(x)","cos_6.txt
")
  PadeShamPoly3(@(x)tan(x),gx,[0:.01:1],Lb,Ub,runNum,"tan(x)","tan_6.txt")
  PadeShamPoly3(@(x)sinh(x),gx,[0:.01:5],Lb,Ub,runNum,"sinh(x)","sinh_6.txt")
  PadeShamPoly3(@(x)cosh(x),gx,[0:.01:5],Lb,Ub,runNum,"cosh(x)","cosh_6.txt")
  PadeShamPoly3(@(x)tanh(x),gx,[0:.01:3],Lb,Ub,runNum,"tanh(x)","tanh_6.txt")
  PadeShamPoly3(@(x)erf(x),gx,[0:.01:2.1],Lb,Ub,runNum,"erf(x)","erf_6.txt")
  PadeShamPoly3(@(x)exp(x),gx,[0:.01:2],Lb,Ub,runNum,"exp(x)","exp_6.txt")
  PadeShamPoly3(@(x)log(x),gx,[1:.01:10],Lb,Ub,runNum,"ln(x)","ln_6.txt")
  PadeShamPoly3(@(x)log10(x),gx,[1:.01:10],Lb,Ub,runNum,"log(x)","log_6.txt")
  PadeShamPoly3(@(x)10.^x,gx,[0:.01:1],Lb,Ub,runNum,"10^x","pwr10_6.txt")

PadeShamPoly3(@(x)tinv(0.95,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.95,x)","ti
nv1_6.txt")

PadeShamPoly3(@(x)tinv(0.975,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.975,x)","
tinv2_6.txt")

PadeShamPoly3(@(x)log10(gamma(x)),gx,[2:0.1:100],Lb,Ub,runNum,"log10Gamma(x)"
,"log10Gamma_6.txt")

PadeShamPoly3(@(x)digamma(x),gx,[2:0.1:100],Lb,Ub,runNum,"digamma(x)","digamm
a_6.txt")

PadeShamPoly3(@(x)trigamma(x),gx,[1:0.1:100],Lb,Ub,runNum,"trigamma(x)","trig
amma_6.txt")

PadeShamPoly3(@(x)besselj(0,x),gx,[2:0.1:30],Lb,Ub,runNum,"J0(x)","J0x_6.txt"
)

PadeShamPoly3(@(x)besselj(1,x),gx,[0:0.1:30],Lb,Ub,runNum,"J1(x)","J1x_6.txt"
)

PadeShamPoly3(@(x)besselj(2,x),gx,[0:0.1:30],Lb,Ub,runNum,"J2(x)","J2x_6.txt"
)
```



```
PadeShamPoly3(@(x)besselj(3,x),gx,[0:0.1:30],Lb,Ub,runNum,"J3(x)","J3x_6.txt"
)

PadeShamPoly3(@(x)besselj(4,x),gx,[0:0.1:30],Lb,Ub,runNum,"J4(x)","J4x_6.txt"
)

PadeShamPoly3(@(x)besselj(5,x),gx,[0:0.1:30],Lb,Ub,runNum,"J5(x)","J5x_6.txt"
)

PadeShamPoly3(@(x)integral(@(z)sin(z.^2),0,x),gx,[1:0.01:5],Lb,Ub,runNum,"Fre
snelSine(x)","FresnelSine_6.txt")

PadeShamPoly3(@(x)integral(@(z)cos(z.^2),0,x),gx,[0.5:0.01:5],Lb,Ub,runNum,"F
resnelCosine(x)","FresnelCosine_6.txt")

PadeShamPoly3(@(x)integral(@(z)sin(z)./z,0,x),gx,[1.3:0.1:20],Lb,Ub,runNum,"S
i(x)","Si_6.txt")
  PadeShamPoly3(@(x)0.57721566+log(x)-integral(@(z)(1-
cos(z))./z,0,x),gx,[0.5:0.1:20],Lb,Ub,runNum,"Ci(x)","Ci_6.txt")

PadeShamPoly3(@(x)asinh(x),gx,[0:0.1:100],Lb,Ub,runNum,"asinh(x)","asinh_6.tx
t")

PadeShamPoly3(@(x)acosh(x),gx,[1:0.1:100],Lb,Ub,runNum,"acosh(x)","acosh_6.tx
t")

PadeShamPoly3(@(x)atanh(x),gx,[0:0.001:0.999],Lb,Ub,runNum,"atanh(x)","atanh_
6.txt")
end

if selIdx==0 || selIdx==7
  lstA = [0 0.34];
  lstB = [0.1 0.39];
  lstC = [0 0.36];
  lstOrder = [3 7];
  Lb = [lstA(1) lstB(1) lstC(1) lstOrder(1) lstOrder(1)];
  Ub = [lstA(2) lstB(2) lstC(2) lstOrder(2) lstOrder(2)];
  gx = @(i,A,B,C)A+B*i+C*sqrt(i-1);

PadeShamPoly3(@(x)acos(x),gx,[0:.01:1],Lb,Ub,runNum,"arccos(x)","arccos_7.txt
")

PadeShamPoly3(@(x)asin(x),gx,[0:.01:1],Lb,Ub,runNum,"arcsin(x)","arcsin_7.txt
")

PadeShamPoly3(@(x)atan(x),gx,[0:.01:1],Lb,Ub,runNum,"arctan(x)","arctan_7.txt
")

PadeShamPoly3(@(x)sin(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"sin(x)","sin_7.txt
")

PadeShamPoly3(@(x)cos(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"cos(x)","cos_7.txt
")
  PadeShamPoly3(@(x)tan(x),gx,[0:.01:1],Lb,Ub,runNum,"tan(x)","tan_7.txt")
  PadeShamPoly3(@(x)sinh(x),gx,[0:.01:5],Lb,Ub,runNum,"sinh(x)","sinh_7.txt")
  PadeShamPoly3(@(x)cosh(x),gx,[0:.01:5],Lb,Ub,runNum,"cosh(x)","cosh_7.txt")
```



```
  PadeShamPoly3(@(x)tanh(x),gx,[0:.01:3],Lb,Ub,runNum,"tanh(x)","tanh_7.txt")
  PadeShamPoly3(@(x)erf(x),gx,[0:.01:2.1],Lb,Ub,runNum,"erf(x)","erf_7.txt")
  PadeShamPoly3(@(x)exp(x),gx,[0:.01:2],Lb,Ub,runNum,"exp(x)","exp_7.txt")
  PadeShamPoly3(@(x)log(x),gx,[1:.01:10],Lb,Ub,runNum,"ln(x)","ln_7.txt")
  PadeShamPoly3(@(x)log10(x),gx,[1:.01:10],Lb,Ub,runNum,"log(x)","log_7.txt")
  PadeShamPoly3(@(x)10.^x,gx,[0:.01:1],Lb,Ub,runNum,"10^x","pwr10_7.txt")

PadeShamPoly3(@(x)tinv(0.95,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.95,x)","ti
nv1_7.txt")

PadeShamPoly3(@(x)tinv(0.975,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.975,x)","
tinv2_7.txt")

PadeShamPoly3(@(x)log10(gamma(x)),gx,[2:0.1:100],Lb,Ub,runNum,"log10Gamma(x)"
,"log10Gamma_7.txt")

PadeShamPoly3(@(x)digamma(x),gx,[2:0.1:100],Lb,Ub,runNum,"digamma(x)","digamm
a_7.txt")

PadeShamPoly3(@(x)trigamma(x),gx,[1:0.1:100],Lb,Ub,runNum,"trigamma(x)","trig
amma_7.txt")

PadeShamPoly3(@(x)besselj(0,x),gx,[2:0.1:30],Lb,Ub,runNum,"J0(x)","J0x_7.txt"
)

PadeShamPoly3(@(x)besselj(1,x),gx,[0:0.1:30],Lb,Ub,runNum,"J1(x)","J1x_7.txt"
)

PadeShamPoly3(@(x)besselj(2,x),gx,[0:0.1:30],Lb,Ub,runNum,"J2(x)","J2x_7.txt"
)

PadeShamPoly3(@(x)besselj(3,x),gx,[0:0.1:30],Lb,Ub,runNum,"J3(x)","J3x_7.txt"
)

PadeShamPoly3(@(x)besselj(4,x),gx,[0:0.1:30],Lb,Ub,runNum,"J4(x)","J4x_7.txt"
)

PadeShamPoly3(@(x)besselj(5,x),gx,[0:0.1:30],Lb,Ub,runNum,"J5(x)","J5x_7.txt"
)

PadeShamPoly3(@(x)integral(@(z)sin(z.^2),0,x),gx,[1:0.01:5],Lb,Ub,runNum,"Fre
snelSine(x)","FresnelSine_7.txt")

PadeShamPoly3(@(x)integral(@(z)cos(z.^2),0,x),gx,[0.5:0.01:5],Lb,Ub,runNum,"F
resnelCosine(x)","FresnelCosine_7.txt")

PadeShamPoly3(@(x)integral(@(z)sin(z)./z,0,x),gx,[1.3:0.1:20],Lb,Ub,runNum,"S
i(x)","Si_7.txt")
  PadeShamPoly3(@(x)0.57721566+log(x)-integral(@(z)(1-
cos(z))./z,0,x),gx,[0.5:0.1:20],Lb,Ub,runNum,"Ci(x)","Ci_7.txt")

PadeShamPoly3(@(x)asinh(x),gx,[0:0.1:100],Lb,Ub,runNum,"asinh(x)","asinh_7.tx
t")

PadeShamPoly3(@(x)acosh(x),gx,[1:0.1:100],Lb,Ub,runNum,"acosh(x)","acosh_7.tx
t")
```



```
PadeShamPoly3(@(x)atanh(x),gx,[0:0.001:0.999],Lb,Ub,runNum,"atanh(x)","atanh_
7.txt")
end

if selIdx==0 || selIdx==8
  lstA = [0 0.55];
  lstB = [0.1 0.4];
  lstC = [0 0.4];
  lstOrder = [3 7];
  Lb = [lstA(1) lstB(1) lstC(1) lstOrder(1) lstOrder(1)];
  Ub = [lstA(2) lstB(2) lstC(2) lstOrder(2) lstOrder(2)];
  gx = @(i,A,B,C)A+B*sqrt(i)+C*(i-1);

PadeShamPoly3(@(x)acos(x),gx,[0:.01:1],Lb,Ub,runNum,"arccos(x)","arccos_8.txt
")

PadeShamPoly3(@(x)asin(x),gx,[0:.01:1],Lb,Ub,runNum,"arcsin(x)","arcsin_8.txt
")

PadeShamPoly3(@(x)atan(x),gx,[0:.01:1],Lb,Ub,runNum,"arctan(x)","arctan_8.txt
")

PadeShamPoly3(@(x)sin(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"sin(x)","sin_8.txt
")

PadeShamPoly3(@(x)cos(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"cos(x)","cos_8.txt
")
  PadeShamPoly3(@(x)tan(x),gx,[0:.01:1],Lb,Ub,runNum,"tan(x)","tan_8.txt")
  PadeShamPoly3(@(x)sinh(x),gx,[0:.01:5],Lb,Ub,runNum,"sinh(x)","sinh_8.txt")
  PadeShamPoly3(@(x)cosh(x),gx,[0:.01:5],Lb,Ub,runNum,"cosh(x)","cosh_8.txt")
  PadeShamPoly3(@(x)tanh(x),gx,[0:.01:3],Lb,Ub,runNum,"tanh(x)","tanh_8.txt")
  PadeShamPoly3(@(x)erf(x),gx,[0:.01:2.1],Lb,Ub,runNum,"erf(x)","erf_8.txt")
  PadeShamPoly3(@(x)exp(x),gx,[0:.01:2],Lb,Ub,runNum,"exp(x)","exp_8.txt")
  PadeShamPoly3(@(x)log(x),gx,[1:.01:10],Lb,Ub,runNum,"ln(x)","ln_8.txt")
  PadeShamPoly3(@(x)log10(x),gx,[1:.01:10],Lb,Ub,runNum,"log(x)","log_8.txt")
  PadeShamPoly3(@(x)10.^x,gx,[0:.01:1],Lb,Ub,runNum,"10^x","pwr10_8.txt")

PadeShamPoly3(@(x)tinv(0.95,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.95,x)","ti
nv1_8.txt")

PadeShamPoly3(@(x)tinv(0.975,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.975,x)","
tinv2_8.txt")

PadeShamPoly3(@(x)log10(gamma(x)),gx,[2:0.1:100],Lb,Ub,runNum,"log10Gamma(x)"
,"log10Gamma_8.txt")

PadeShamPoly3(@(x)digamma(x),gx,[2:0.1:100],Lb,Ub,runNum,"digamma(x)","digamm
a_8.txt")

PadeShamPoly3(@(x)trigamma(x),gx,[1:0.1:100],Lb,Ub,runNum,"trigamma(x)","trig
amma_8.txt")

PadeShamPoly3(@(x)besselj(0,x),gx,[2:0.1:30],Lb,Ub,runNum,"J0(x)","J0x_8.txt"
)
```



```
PadeShamPoly3(@(x)besselj(1,x),gx,[0:0.1:30],Lb,Ub,runNum,"J1(x)","J1x_8.txt"
)

PadeShamPoly3(@(x)besselj(2,x),gx,[0:0.1:30],Lb,Ub,runNum,"J2(x)","J2x_8.txt"
)

PadeShamPoly3(@(x)besselj(3,x),gx,[0:0.1:30],Lb,Ub,runNum,"J3(x)","J3x_8.txt"
)

PadeShamPoly3(@(x)besselj(4,x),gx,[0:0.1:30],Lb,Ub,runNum,"J4(x)","J4x_8.txt"
)

PadeShamPoly3(@(x)besselj(5,x),gx,[0:0.1:30],Lb,Ub,runNum,"J5(x)","J5x_8.txt"
)

PadeShamPoly3(@(x)integral(@(z)sin(z.^2),0,x),gx,[1:0.01:5],Lb,Ub,runNum,"Fre
snelSine(x)","FresnelSine_8.txt")

PadeShamPoly3(@(x)integral(@(z)cos(z.^2),0,x),gx,[0.5:0.01:5],Lb,Ub,runNum,"F
resnelCosine(x)","FresnelCosine_8.txt")

PadeShamPoly3(@(x)integral(@(z)sin(z)./z,0,x),gx,[1.3:0.1:20],Lb,Ub,runNum,"S
i(x)","Si_8.txt")
  PadeShamPoly3(@(x)0.57721566+log(x)-integral(@(z)(1-
cos(z))./z,0,x),gx,[0.5:0.1:20],Lb,Ub,runNum,"Ci(x)","Ci_8.txt")

PadeShamPoly3(@(x)asinh(x),gx,[0:0.1:100],Lb,Ub,runNum,"asinh(x)","asinh_8.tx
t")

PadeShamPoly3(@(x)acosh(x),gx,[1:0.1:100],Lb,Ub,runNum,"acosh(x)","acosh_8.tx
t")

PadeShamPoly3(@(x)atanh(x),gx,[0:0.001:0.999],Lb,Ub,runNum,"atanh(x)","atanh_
8.txt")
end

if selIdx==0 || selIdx==9
  lstA = [0 0.5];
  lstB = [0.1 0.7];
  lstC = [0 0.7];
  lstOrder = [3 7];
  Lb = [lstA(1) lstB(1) lstC(1) lstOrder(1) lstOrder(1)];
  Ub = [lstA(2) lstB(2) lstC(2) lstOrder(2) lstOrder(2)];
  gx = @(i,A,B,C)A+B*sqrt(i)+C*sqrt(i-1);

PadeShamPoly3(@(x)acos(x),gx,[0:.01:1],Lb,Ub,runNum,"arccos(x)","arccos_9.txt
")

PadeShamPoly3(@(x)asin(x),gx,[0:.01:1],Lb,Ub,runNum,"arcsin(x)","arcsin_9.txt
")

PadeShamPoly3(@(x)atan(x),gx,[0:.01:1],Lb,Ub,runNum,"arctan(x)","arctan_9.txt
")

PadeShamPoly3(@(x)sin(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"sin(x)","sin_9.txt
")
```



```
PadeShamPoly3(@(x)cos(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"cos(x)","cos_9.txt
")
  PadeShamPoly3(@(x)tan(x),gx,[0:.01:1],Lb,Ub,runNum,"tan(x)","tan_9.txt")
  PadeShamPoly3(@(x)sinh(x),gx,[0:.01:5],Lb,Ub,runNum,"sinh(x)","sinh_9.txt")
  PadeShamPoly3(@(x)cosh(x),gx,[0:.01:5],Lb,Ub,runNum,"cosh(x)","cosh_9.txt")
  PadeShamPoly3(@(x)tanh(x),gx,[0:.01:3],Lb,Ub,runNum,"tanh(x)","tanh_9.txt")
  PadeShamPoly3(@(x)erf(x),gx,[0:.01:2.1],Lb,Ub,runNum,"erf(x)","erf_9.txt")
  PadeShamPoly3(@(x)exp(x),gx,[0:.01:2],Lb,Ub,runNum,"exp(x)","exp_9.txt")
  PadeShamPoly3(@(x)log(x),gx,[1:.01:10],Lb,Ub,runNum,"ln(x)","ln_9.txt")
  PadeShamPoly3(@(x)log10(x),gx,[1:.01:10],Lb,Ub,runNum,"log(x)","log_9.txt")
  PadeShamPoly3(@(x)10.^x,gx,[0:.01:1],Lb,Ub,runNum,"10^x","pwr10_9.txt")

PadeShamPoly3(@(x)tinv(0.95,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.95,x)","ti
nv1_9.txt")

PadeShamPoly3(@(x)tinv(0.975,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.975,x)","
tinv2_9.txt")

PadeShamPoly3(@(x)log10(gamma(x)),gx,[2:0.1:100],Lb,Ub,runNum,"log10Gamma(x)"
,"log10Gamma_9.txt")

PadeShamPoly3(@(x)digamma(x),gx,[2:0.1:100],Lb,Ub,runNum,"digamma(x)","digamm
a_9.txt")

PadeShamPoly3(@(x)trigamma(x),gx,[1:0.1:100],Lb,Ub,runNum,"trigamma(x)","trig
amma_9.txt")

PadeShamPoly3(@(x)besselj(0,x),gx,[2:0.1:30],Lb,Ub,runNum,"J0(x)","J0x_9.txt"
)

PadeShamPoly3(@(x)besselj(1,x),gx,[0:0.1:30],Lb,Ub,runNum,"J1(x)","J1x_9.txt"
)

PadeShamPoly3(@(x)besselj(2,x),gx,[0:0.1:30],Lb,Ub,runNum,"J2(x)","J2x_9.txt"
)

PadeShamPoly3(@(x)besselj(3,x),gx,[0:0.1:30],Lb,Ub,runNum,"J3(x)","J3x_9.txt"
)

PadeShamPoly3(@(x)besselj(4,x),gx,[0:0.1:30],Lb,Ub,runNum,"J4(x)","J4x_9.txt"
)

PadeShamPoly3(@(x)besselj(5,x),gx,[0:0.1:30],Lb,Ub,runNum,"J5(x)","J5x_9.txt"
)

PadeShamPoly3(@(x)integral(@(z)sin(z.^2),0,x),gx,[1:0.01:5],Lb,Ub,runNum,"Fre
snelSine(x)","FresnelSine_9.txt")

PadeShamPoly3(@(x)integral(@(z)cos(z.^2),0,x),gx,[0.5:0.01:5],Lb,Ub,runNum,"F
resnelCosine(x)","FresnelCosine_9.txt")

PadeShamPoly3(@(x)integral(@(z)sin(z)./z,0,x),gx,[1.3:0.1:20],Lb,Ub,runNum,"S
i(x)","Si_9.txt")
  PadeShamPoly3(@(x)0.57721566+log(x)-integral(@(z)(1-
cos(z))./z,0,x),gx,[0.5:0.1:20],Lb,Ub,runNum,"Ci(x)","Ci_9.txt")
```



```
PadeShamPoly3(@(x)asinh(x),gx,[0:0.1:100],Lb,Ub,runNum,"asinh(x)","asinh_9.tx
t")

PadeShamPoly3(@(x)acosh(x),gx,[1:0.1:100],Lb,Ub,runNum,"acosh(x)","acosh_9.tx
t")

PadeShamPoly3(@(x)atanh(x),gx,[0:0.001:0.999],Lb,Ub,runNum,"atanh(x)","atanh_
9.txt")
end

if selIdx==0 || selIdx==10
  lstA = [0 2.2];
  lstB = [0.1 3.5];
  lstOrder = [3 7];
  Lb = [lstA(1) lstB(1) lstOrder(1) lstOrder(1)];
  Ub = [lstA(2) lstB(2) lstOrder(2) lstOrder(2)];
  gx = @(i,A,B)A+B*log10(i)^4;

PadeShamPoly2(@(x)acos(x),gx,[0:.01:1],Lb,Ub,runNum,"arccos(x)","arccos_10.tx
t")

PadeShamPoly2(@(x)asin(x),gx,[0:.01:1],Lb,Ub,runNum,"arcsin(x)","arcsin_10.tx
t")

PadeShamPoly2(@(x)atan(x),gx,[0:.01:1],Lb,Ub,runNum,"arctan(x)","arctan_10.tx
t")

PadeShamPoly2(@(x)sin(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"sin(x)","sin_10.tx
t")

PadeShamPoly2(@(x)cos(x),gx,[0:pi/1000:pi/2],Lb,Ub,runNum,"cos(x)","cos_10.tx
t")
  PadeShamPoly2(@(x)tan(x),gx,[0:.01:1],Lb,Ub,runNum,"tan(x)","tan_10.txt")

PadeShamPoly2(@(x)sinh(x),gx,[0:.01:5],Lb,Ub,runNum,"sinh(x)","sinh_10.txt")

PadeShamPoly2(@(x)cosh(x),gx,[0:.01:5],Lb,Ub,runNum,"cosh(x)","cosh_10.txt")

PadeShamPoly2(@(x)tanh(x),gx,[0:.01:3],Lb,Ub,runNum,"tanh(x)","tanh_10.txt")
  PadeShamPoly2(@(x)erf(x),gx,[0:.01:2.1],Lb,Ub,runNum,"erf(x)","erf_10.txt")
  PadeShamPoly2(@(x)exp(x),gx,[0:.01:2],Lb,Ub,runNum,"exp(x)","exp_10.txt")
  PadeShamPoly2(@(x)log(x),gx,[1:.01:10],Lb,Ub,runNum,"ln(x)","ln_10.txt")

PadeShamPoly2(@(x)log10(x),gx,[1:.01:10],Lb,Ub,runNum,"log(x)","log_10.txt")
  PadeShamPoly2(@(x)10.^x,gx,[0:.01:1],Lb,Ub,runNum,"10^x","pwr10_10.txt")

PadeShamPoly2(@(x)tinv(0.95,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.95,x)","ti
nv1_10.txt")

PadeShamPoly2(@(x)tinv(0.975,x),gx,[2:0.1:100],Lb,Ub,runNum,"tinv(0.975,x)","
tinv2_10.txt")

PadeShamPoly2(@(x)log10(gamma(x)),gx,[2:0.1:100],Lb,Ub,runNum,"log10Gamma(x)"
,"log10Gamma_10.txt")
```



```
PadeShamPoly2(@(x)digamma(x),gx,[2:0.1:100],Lb,Ub,runNum,"digamma(x)","digamm
a_10.txt")

PadeShamPoly2(@(x)trigamma(x),gx,[1:0.1:100],Lb,Ub,runNum,"trigamma(x)","trig
amma_10.txt")

PadeShamPoly2(@(x)besselj(0,x),gx,[2:0.1:30],Lb,Ub,runNum,"J0(x)","J0x_10.txt
")

PadeShamPoly2(@(x)besselj(1,x),gx,[0:0.1:30],Lb,Ub,runNum,"J1(x)","J1x_10.txt
")

PadeShamPoly2(@(x)besselj(2,x),gx,[0:0.1:30],Lb,Ub,runNum,"J2(x)","J2x_10.txt
")

PadeShamPoly2(@(x)besselj(3,x),gx,[0:0.1:30],Lb,Ub,runNum,"J3(x)","J3x_10.txt
")

PadeShamPoly2(@(x)besselj(4,x),gx,[0:0.1:30],Lb,Ub,runNum,"J4(x)","J4x_10.txt
")

PadeShamPoly2(@(x)besselj(5,x),gx,[0:0.1:30],Lb,Ub,runNum,"J5(x)","J5x_10.txt
")

PadeShamPoly2(@(x)integral(@(z)sin(z.^2),0,x),gx,[1:0.01:5],Lb,Ub,runNum,"Fre
snelSine(x)","FresnelSine_10.txt")

PadeShamPoly2(@(x)integral(@(z)cos(z.^2),0,x),gx,[0.5:0.01:5],Lb,Ub,runNum,"F
resnelCosine(x)","FresnelCosine_10.txt")

PadeShamPoly2(@(x)integral(@(z)sin(z)./z,0,x),gx,[1.3:0.1:20],Lb,Ub,runNum,"S
i(x)","Si_10.txt")
  PadeShamPoly2(@(x)0.57721566+log(x)-integral(@(z)(1-
cos(z))./z,0,x),gx,[0.5:0.1:20],Lb,Ub,runNum,"Ci(x)","Ci_10.txt")

PadeShamPoly2(@(x)asinh(x),gx,[0:0.1:100],Lb,Ub,runNum,"asinh(x)","asinh_10.t
xt")

PadeShamPoly2(@(x)acosh(x),gx,[1:0.1:100],Lb,Ub,runNum,"acosh(x)","acosh_10.t
xt")

PadeShamPoly2(@(x)atanh(x),gx,[0:0.001:0.999],Lb,Ub,runNum,"atanh(x)","atanh_
10.txt")
end

toc;

% make some sound
playmynotes()

if bShutdown
  system('shutdown -s');
else
  fprintf("\n\nDone!\n\n");
end
```


The above listing has the following global and operational variabls:

- The global variable bUseDiary is a Boolean flag used to tell the functions PadeShamPoly2() and PadeShamPoly3() whether you want to copy the screen output to diary text files.
- The global variable bDeleteIfExists is a Boolean flag used to tell the functions PadeShamPoly2() and PadeShamPoly3() whether you want to delete diary files if they exist.
- The variable selIdx allows you to select calculations for one of the seven groups (when set to the targeted group number) or all of the groups (when set to 0). I am using this scheme to reduce calculation time which can be done by working with a specific set of approximations.
- The Boolean variable bShutdown tells the Matlab script whether to shut down the computer when done.

When the script goAll is done with the calculation it executes the script playmynotes.m to play ten random sinusoidal notes. You can comment out the call to the audible script to keep the script quiet. I wrote the script for playmynotes.m based on the code I found in the fine book *Programming Fundamentals Using MATLAB* written by M. Weeks and published in 2020 by Mercury Learning. The book comes with a CD that containes all the Matlab source code and figures in that book. The source code for playmynotes.m is:

```
n = 10;
frq = 8192;
notes = 220 + fix(220*rand(n,1));

for i =1:n
  znotes(i,1:frq) = 0.5*sin(2*pi*notes(i)*(0:frq-1)/frq);
end

for i = 1:n
    sound(znotes(i,1:frq));
    pause(1)
end
```

You can edit the script in goAll.m to perform Pade-Shammas polynomial fitting for other functions you are interested in or for other Shammas polynomial orders.

General Comments on Results

The next sections show the results of fitting various common functions each with a variety of Pade-Shammas polynomials. There is a summary table for each fitted function showing the Shammas polynomial parameters, the F statistic, the AICc

statistic, the root mean sum of errors squared, and orders of the fitted Pade-Shammas polynomials. I use the the root mean sum of errors squared to select the best Shammas polynomial powers. I have deliberately not included the coefficient of determination and its adjusted value in the table. Why? These values are very close to 1 and we are better served to look at other statistics. I have included the values for the F and AICc statistics, even thought they are infinity and minus infinity.

The quality of the curve fitting varies between very good and (honestly) somewhat disappointing. This variation shows that the process of using the Pade-Shammas polynomials works and does not, by some erroneous calculations, always produce excellent results.

✍ The next sections display the results of the Pade-Shammas polynomials. The regression coefficients for the $Q_n(x)$ and $D_m(x)$ Shammas polynomials appear as one long list labeled x1 and up. The results include the orders for the $Q_n(x)$ and $D_m(x)$ Shammas polynomials which should easily indicate with regression coefficients belong to $Q_n(x)$ Shammas polynomials and which one belong to $D_m(x)$ Shammas polynomials. The results also include the list of powers associated with $Q_n(x)$ and $D_m(x)$ Shammas polynomials. Here is an example. Looking at the first results for the inverse hyperbolic cosine:


```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8

Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______

    (Intercept)      1.03398879205403     0       Inf        0
    x1              -10.9974814959213     0      -Inf        0
    x2               16.7889606896366     0       Inf        0
    x3               -6.8254689103896     0      -Inf        0
    x4               19.7883228436892     0       Inf        0
    x5              -47.7844201615567     0      -Inf        0
    x6               46.1548001917542     0       Inf        0
    x7              -21.0743569471578     0      -Inf        0
    x8               3.91565497948532     0       Inf        0


Number of observations: 991, Error degrees of freedom: 982
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F      pValue
                  ____________________    ___    ____________________    ___    _____

    Total        28.6459377780118     990    0.0289352906848604
    Model        28.6459377780118       8      3.58074222225147    Inf        0
    Residual                    0     982                     0

A = 0.246404, B = 0.120959
order Q(x) = 3.000000, order D(x) = 5.000000
Xmin = 1.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 5.298292
List of powers for Q(x): [0.367363, 0.488322, 0.609281]
List of powers for D(x): [0.367363, 0.488322, 0.609281, 0.730240, 0.851199]
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 2.898804e-05
Corrected MSS of errors squared = 4.099528e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

First look at the orders of the $Q_n(x)$ and $D_m(x)$ Shammas polynomials. They are 3 and 5, respectively. Then look at the list of regression coefficients labeled x1 through x8. Since $Q_n(x)$ has the order of three, its regression coefficients are the

ones labeled x1, x2, and x3. The remaining five regression coefficients are labeled x4 through x8. You can find the list of the $Q_n(x)$ and $D_m(x)$ Shammas polynomials two lines below the line showing the polynomial orders.

RESULTS FOR THE INVERSE HYPERBOLIC COSINE

Using Power A+B* i

```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8

Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)      1.03398879205403     0       Inf       0
    x1              -10.9974814959213     0      -Inf       0
    x2               16.7889606896366     0       Inf       0
    x3               -6.8254689103896     0      -Inf       0
    x4               19.7883228436892     0       Inf       0
    x5              -47.7844201615567     0      -Inf       0
    x6               46.1548001917542     0       Inf       0
    x7              -21.0743569471578     0      -Inf       0
    x8               3.91565497948532     0       Inf       0


Number of observations: 991, Error degrees of freedom: 982
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq           DF            MeanSq            F      pValue
                  _____________________    ____    ____________________    ____    ______

    Total        28.6459377780118     990    0.0289352906848604
    Model        28.6459377780118       8      3.58074222225147    Inf        0
    Residual                    0     982                     0

A = 0.246404, B = 0.120959
order Q(x) = 3.000000, order D(x) = 5.000000
Xmin = 1.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 5.298292
List of powers for Q(x): [0.367363, 0.488322, 0.609281]
List of powers for D(x): [0.367363, 0.488322, 0.609281, 0.730240, 0.851199]
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 2.898804e-05
Corrected MSS of errors squared = 4.099528e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```


Using Power A+B/i

```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      0.98293030406142    0      Inf        0
    x1              -0.033658222843951   0     -Inf        0
    x2                1.75917964518031   0      Inf        0
    x3               -10.7432459096718   0     -Inf        0
    x4                14.815012763153    0      Inf        0
    x5                             0     0      NaN      NaN
    x6                             0     0      NaN      NaN
    x7               -6.78021857943718   0     -Inf        0
    x8               0.0114684453566834  0      Inf        0
    x9               -0.167086685987091  0     -Inf        0
    x10              -4.05562124738676   0     -Inf        0
    x11               26.4326209205479   0      Inf        0
    x12              -37.6701641296323   0     -Inf        0
    x13                            0     0      NaN      NaN
    x14               16.4487826966627   0      Inf        0


Number of observations: 991, Error degrees of freedom: 979
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF            MeanSq            F      pValue
                 ________________    ___    ____________________    ___    ______

    Total        28.6459377780157    990    0.0289352906848644
    Model        28.6459377780157     11      2.60417616163779    Inf       0
    Residual                    0    979                     0

A = 0.000000, B = 2.000000
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 1.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 5.298292
List of powers for Q(x): [2.000000, 1.000000, 0.666667, 0.500000, 0.400000, 0.333333,
0.285714]
List of powers for D(x): [2.000000, 1.000000, 0.666667, 0.500000, 0.400000, 0.333333,
0.285714]
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 1.050563e-05
Corrected MSS of errors squared = 1.485720e-05
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ____________________    __    _____    ______

    (Intercept)      1.7180421979475     0      Inf        0
    x1              -1147.75664945103    0     -Inf        0
    x2               2484.15994178331    0      Inf        0
    x3                             0     0      NaN      NaN
    x4              -1833.41029331599    0     -Inf        0
    x5                             0     0      NaN      NaN
    x6                             0     0      NaN      NaN
    x7                495.28895895737    0      Inf        0
    x8               2147.84937420345    0      Inf        0
    x9              -8684.46428223304    0     -Inf        0
    x10              9949.06816367511    0      Inf        0
    x11                            0     0      NaN      NaN
    x12             -5869.34776620395    0     -Inf        0
    x13              2457.89451038878    0      Inf        0


Number of observations: 991, Error degrees of freedom: 981
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F      pValue
                  _______________    ___    ___________________    ___    ______

    Total       28.6459377786588    990    0.028935290685514
    Model       28.6459377786588      9     3.18288197540654    Inf        0
    Residual                   0    981                    0

A = 0.521873, B = 0.100000
order Q(x) = 7.000000, order D(x) = 6.000000
Xmin = 1.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 5.298292
List of powers for Q(x): [0.621873, 0.663295, 0.695078, 0.721873, 0.745480, 0.766822,
0.786448]
List of powers for D(x): [0.621873, 0.663295, 0.695078, 0.721873, 0.745480, 0.766822]
```



```
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 1.790167e-05
Corrected MSS of errors squared = 2.531679e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                   ________________    ___    _____    ______

    (Intercept)     1.80413208907666    0       Inf        0
    x1              -579.13245792602    0      -Inf        0
    x2              1211.72249922982    0       Inf        0
    x3                             0    0       NaN      NaN
    x4              -976.372394764632   0      -Inf        0
    x5                             0    0       NaN      NaN
    x6              341.978221771909    0       Inf        0
    x7              870.466492858199    0       Inf        0
    x8              -3498.00762344262   0      -Inf        0
    x9              5056.65155173191    0       Inf        0
    x10             -3101.25806575029   0      -Inf        0
    x11             673.147644206675    0       Inf        0


Number of observations: 991, Error degrees of freedom: 981
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _____________    ___    ____________________    ___    ______

    Total        28.6459377779225    990    0.0289352906847702
    Model        28.6459377779225      9      3.18288197532473    Inf       0
    Residual                    0    981                     0

A = 0.478806, B = 0.193610, C = 0.697437
order Q(x) = 6.000000, order D(x) = 5.000000
Xmin = 1.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 5.298292
List of powers for Q(x): [0.731053, 0.796789, 0.851094, 0.898429, 0.940940, 0.979858]
List of powers for D(x): [0.731053, 0.796789, 0.851094, 0.898429, 0.940940]
```



```
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 1.747398e-05
Corrected MSS of errors squared = 3.026582e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8

Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                   ________________    ___    _____    ______

    (Intercept)     -2.5580134172496    0     -Inf       0
    x1               14.6416446057505    0      Inf       0
    x2              -31.0131525244367    0     -Inf       0
    x3               30.7009950493793    0      Inf       0
    x4              -14.1387867057249    0     -Inf       0
    x5               2.36731290955493    0      Inf       0
    x6               3.04233262891973    0      Inf       0
    x7              -3.08487018888863    0     -Inf       0
    x8               1.04253764109165    0      Inf       0


Number of observations: 991, Error degrees of freedom: 982
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _________________    ___    ____________________    ___    ______

    Total        28.6459377780125    990    0.0289352906848611
    Model        28.6459377780125      8      3.58074222225157    Inf       0
    Residual                    0    982                     0

A = 0.000000, B = 0.100000, C = 0.000000
order Q(x) = 5.000000, order D(x) = 3.000000
Xmin = 1.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 5.298292
List of powers for Q(x): [0.100000, 0.200000, 0.300000, 0.400000, 0.500000]
List of powers for D(x): [0.100000, 0.200000, 0.300000]
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 2.819233e-05
Corrected MSS of errors squared = 4.883054e-05
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ____________________    __    _____    ______

    (Intercept)    -0.0780892650326805    0     -Inf        0
    x1                3239.50539682417    0      Inf        0
    x2               -8059.82335268454    0     -Inf        0
    x3                               0    0      NaN      NaN
    x4                15972.1484614865    0      Inf        0
    x5               -15903.5736403197    0     -Inf        0
    x6                4751.82122398565    0      Inf        0
    x7                -835.77719894892    0     -Inf        0
    x8                2632.52645840158    0      Inf        0
    x9               -2773.08648332371    0     -Inf        0
    x10               977.337223844184    0      Inf        0


Number of observations: 991, Error degrees of freedom: 981
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______

    Total        28.6459377780534    990    0.0289352906849025
    Model        28.6459377780534      9      3.18288197533927    Inf       0
    Residual                    0    981                     0

A = 0.614721, B = 0.102271, C = 0.141313
order Q(x) = 6.000000, order D(x) = 4.000000
Xmin = 1.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 5.298292
List of powers for Q(x): [0.716992, 0.677950, 0.638908, 0.599867, 0.560825, 0.521783]
List of powers for D(x): [0.716992, 0.677950, 0.638908, 0.599867]
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 1.147737e-05
Corrected MSS of errors squared = 1.987938e-05
R-Squared =   1.00000000
```



```
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      2.18791446538618    0      Inf       0
    x1              -14.5074031036715    0     -Inf       0
    x2               36.1496269340649    0      Inf       0
    x3              -40.0229054313143    0     -Inf       0
    x4               17.4561644166755    0      Inf       0
    x5              0.715383066098472    0      Inf       0
    x6              -2.15893718260442    0     -Inf       0
    x7              0.180156825262362    0      Inf       0
    x8               7.19384729392718    0      Inf       0
    x9              -19.6998889191768    0     -Inf       0
    x10              25.0520693759506    0      Inf       0
    x11             -15.0738963701672    0     -Inf       0
    x12               3.5278686295482    0      Inf       0


Number of observations: 991, Error degrees of freedom: 978
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq           DF            MeanSq             F      pValue
                  ____________________    ___    ____________________    ___    ______

    Total        28.6459377780091    990    0.0289352906848577
    Model        28.6459377780091     12      2.38716148150076    Inf       0
    Residual                    0    978                     0

A = 0.100965, B = 0.373036, C = 0.063371
order Q(x) = 7.000000, order D(x) = 5.000000
Xmin = 1.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 5.298292
List of powers for Q(x): [0.474000, 0.910406, 1.309691, 1.702868, 2.092884, 2.480879,
2.867440]
List of powers for D(x): [0.474000, 0.910406, 1.309691, 1.702868, 2.092884]
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 4.725419e-06
```



```
Corrected MSS of errors squared = 8.184665e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ________________    ___    _____    ______

    (Intercept)    0.858284564274045     0      Inf        0
    x1             -26.6450102413849     0     -Inf        0
    x2              52.9912729169435     0      Inf        0
    x3             -27.2045510481111     0     -Inf        0
    x4              206.113305483714     0      Inf        0
    x5             -1098.00635652252     0     -Inf        0
    x6              2123.95941138495     0      Inf        0
    x7              -1613.3156435848     0     -Inf        0
    x8                             0     0      NaN      NaN
    x9              602.158480140337     0      Inf        0
    x10            -219.909193146917     0     -Inf        0


Number of observations: 991, Error degrees of freedom: 981
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _________________    ___    ____________________    ___    ______

    Total        28.6459377780516    990    0.0289352906849006
    Model        28.6459377780516      9      3.18288197533907    Inf       0
    Residual                    0    981                     0

A = 0.550000, B = 0.400000, C = 0.000000
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 1.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 5.298292
List of powers for Q(x): [0.950000, 1.115685, 1.242820]
List of powers for D(x): [0.950000, 1.115685, 1.242820, 1.350000, 1.444427, 1.529796,
1.608301]
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 2.207741e-05
```



```
Corrected MSS of errors squared = 3.823920e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ________________    __    _____    ______

    (Intercept)      1.99096140828156    0      Inf       0
    x1              -16.5920129422835    0     -Inf       0
    x2               42.9895294147516    0      Inf       0
    x3              -11.7975021412532    0     -Inf       0
    x4               -52.636216975808    0     -Inf       0
    x5               43.1964144268563    0      Inf       0
    x6              -7.15117054551339    0     -Inf       0
    x7               9.95493632610206    0      Inf       0
    x8              -37.7018438933706    0     -Inf       0
    x9               47.6218211433546    0      Inf       0
    x10             -18.8749162017105    0     -Inf       0


Number of observations: 991, Error degrees of freedom: 980
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _____________    ___    ____________________    ___    ______

    Total        28.6459377780227    990    0.0289352906848714
    Model        28.6459377780227     10      2.86459377780227    Inf       0
    Residual                    0    980                     0

A = 0.252393, B = 0.686068, C = 0.391825
order Q(x) = 6.000000, order D(x) = 4.000000
Xmin = 1.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 5.298292
List of powers for Q(x): [0.938461, 1.614465, 1.994823, 2.303191, 2.570139, 2.809058]
List of powers for D(x): [0.938461, 1.614465, 1.994823, 2.303191]
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 1.323631e-05
Corrected MSS of errors squared = 2.292595e-05
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting acosh(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ____________________    __    _____    ______

    (Intercept)    0.297191442105009     0      Inf        0
    x1              -20.899782428928     0     -Inf        0
    x2                             0     0      NaN      NaN
    x3              31.8264713871992     0      Inf        0
    x4             -11.2238805297548     0     -Inf        0
    x5              799.370424864157     0      Inf        0
    x6                             0     0      NaN      NaN
    x7             -1937.75462873374     0     -Inf        0
    x8              1818.92368244323     0      Inf        0
    x9             -890.750183299469     0     -Inf        0
    x10             238.989395318847     0      Inf        0
    x11            -27.7786904652866     0     -Inf        0


Number of observations: 991, Error degrees of freedom: 981
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq           DF           MeanSq             F      pValue
                   ________________    ___    ____________________    ___    ______

    Total       28.6459377781074     990    0.028935290684957
    Model       28.6459377781074       9     3.18288197534527    Inf        0
    Residual                   0     981                    0

A = 0.493390, B = 0.839397
order Q(x) = 4.000000, order D(x) = 7.000000
Xmin = 1.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 5.298292
List of powers for Q(x): [0.493390, 0.500283, 0.536890, 0.603678]
List of powers for D(x): [0.493390, 0.500283, 0.536890, 0.603678, 0.693746, 0.801158,
0.921540]
Fitting acosh(x) in range (1.000000, 100.000000)
MSS of errors squared = 2.056573e-05
```



```
Corrected MSS of errors squared = 2.908434e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the inverse hyperbolic cosine function. The results are sorted in ascending order using the MSSE values.

Power	*A*	*B*	*C*	*F*	*AICc*	*MSSE*	*Q/D Orders*
A+B*i+C*sqrt(i-1)	0.1	0.373	0.063	Inf	-Inf	4.72541900E-06	7, 5
A+B/i	0	2		Inf	-Inf	1.05056300E-05	7, 7
A+B*i-C*(i-1)	0.614	0.102	0.141	Inf	-Inf	1.14773700E-05	6, 4
A+B*sqrt(i)+C*sqrt(i-1)	0.252	0.686	0.391	Inf	-Inf	1.32363100E-05	6, 4
A+B*sqrt(i+C)	0.478	0.193	0.697	Inf	-Inf	1.74739800E-05	6, 5
A+B*sqrt(i)	0.521	0.1		Inf	-Inf	1.79016700E-05	7, 6
A+B*log10(i)^4	0.493	0.839		Inf	-Inf	2.05657300E-05	4, 7
A+B*sqrt(i)+C*(i-1)	0.55	0.4	0	Inf	-Inf	2.20774100E-05	3, 7
A+B*i+C*(i-1)	0	0.1	0	Inf	-Inf	2.81923300E-05	5, 3
A+B*i	0.246	0.12		Inf	-Inf	2.89880400E-05	3, 5

RESULTS FOR THE ARC COSINE

Using Power A+B* i

```
Fitting arccos(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)       3.03784002286424     0      Inf       0
    x1               -10.4611977403428     0     -Inf       0
    x2                12.3324295955256     0      Inf       0
    x3               -5.88696836399771     0     -Inf       0
    x4                1.02190325966233     0      Inf       0
    x5                2.95750495451565     0      Inf       0
    x6               -2.88570031923183     0     -Inf       0
    x7                0.918581651168425    0      Inf       0
    x8               0.0319037138288525    0      Inf       0
    x9               -0.0442933859139496   0     -Inf       0


Number of observations: 101, Error degrees of freedom: 91
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _________________    ___    ____________________    ___    ______

    Total       6.03050467942534     100    0.0603050467942534
    Model       6.03050467942534       9     0.670056075491704    Inf       0
    Residual                   0      91                     0

A = 0.140433, B = 0.403658
order Q(x) = 4.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.570796
List of powers for Q(x): [0.544091, 0.947748, 1.351406, 1.755064]
List of powers for D(x): [0.544091, 0.947748, 1.351406, 1.755064, 2.158721]
Fitting arccos(x) in range (0.000000, 1.000000)
MSS of errors squared = 8.554848e-06
Corrected MSS of errors squared = 1.209838e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```


Using Power A+B/i

```
Fitting arccos(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B/i

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      3.02798256187323     0      Inf        0
    x1                541.925016222891    0      Inf        0
    x2               -7896.37055713692    0     -Inf        0
    x3                18984.4914419909    0      Inf        0
    x4               -11632.9919271264    0     -Inf        0
    x5               -278.837300424283    0     -Inf        0
    x6                6883.05904634294    0      Inf        0
    x7               -27633.242606474     0     -Inf        0
    x8                30469.4785976229    0      Inf        0
    x9                               0    0      NaN      NaN
    x10              -9439.49871532086    0     -Inf        0


Number of observations: 101, Error degrees of freedom: 91
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                        SumSq          DF            MeanSq            F      pValue
                    ____________________    ___    ____________________    ___    ______

    Total        6.03050467942402     100    0.0603050467942402
    Model        6.03050467942402       9     0.670056075491558    Inf        0
    Residual                    0      91                      0

A = 1.121150, B = 0.220092
order Q(x) = 4.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.570796
List of powers for Q(x): [1.341242, 1.231196, 1.194514, 1.176173]
List of powers for D(x): [1.341242, 1.231196, 1.194514, 1.176173, 1.165168, 1.157832]
Fitting arccos(x) in range (0.000000, 1.000000)
MSS of errors squared = 9.399889e-06
Corrected MSS of errors squared = 1.329345e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)


```
Fitting arccos(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      2.47282370195162     0      Inf       0
    x1               16722.9325730302     0      Inf       0
    x2                -98465.17959028     0     -Inf       0
    x3               196963.416253217     0      Inf       0
    x4              -165381.563730124     0     -Inf       0
    x5               50158.0222908331     0      Inf       0
    x6               142.720822469262     0      Inf       0
    x7              -303.395646886382     0     -Inf       0
    x8               161.624514200649     0      Inf       0


Number of observations: 101, Error degrees of freedom: 92
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______

    Total       6.03050467992211    100    0.0603050467992211
    Model       6.03050467992211      8     0.753813084990263    Inf       0
    Residual                   0     92                     0

A = 0.828535, B = 0.100000
order Q(x) = 5.000000, order D(x) = 3.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.570796
List of powers for Q(x): [0.928535, 0.969956, 1.001740, 1.028535, 1.052141]
List of powers for D(x): [0.928535, 0.969956, 1.001740]
Fitting arccos(x) in range (0.000000, 1.000000)
MSS of errors squared = 2.629221e-05
Corrected MSS of errors squared = 3.718281e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting arccos(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
```


Linear regression model:

y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	3.29089277059122	0	Inf	0
x1	-373.438142175909	0	-Inf	0
x2	824.624154569707	0	Inf	0
x3	0	0	NaN	NaN
x4	-919.275598504709	0	-Inf	0
x5	0	0	NaN	NaN
x6	845.592643719449	0	Inf	0
x7	-380.793751967926	0	-Inf	0
x8	105.737989178379	0	Inf	0
x9	-230.679471154007	0	-Inf	0
x10	0	0	NaN	NaN
x11	251.778919054119	0	Inf	0
x12	0	0	NaN	NaN
x13	-227.31342292172	0	-Inf	0
x14	101.475886637632	0	Inf	0

Number of observations: 101, Error degrees of freedom: 90

R-squared: 1, Adjusted R-Squared: 1

F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	6.03050467943213	100	0.0603050467943213		
Model	6.03050467943213	10	0.603050467943213	Inf	0
Residual	0	90	0		

A = 0.000000, B = 0.191027, C = 2.000000

order Q(x) = 7.000000, order D(x) = 7.000000

Xmin = 0.000000 and Xmax = 1.000000

Ymin = 0.000000 and Ymax = 1.570796

List of powers for Q(x): [0.330868, 0.382054, 0.427149, 0.467918, 0.505409, 0.540305, 0.573080]

List of powers for D(x): [0.330868, 0.382054, 0.427149, 0.467918, 0.505409, 0.540305, 0.573080]

Fitting arccos(x) in range (0.000000, 1.000000)

MSS of errors squared = 4.840964e-05

Corrected MSS of errors squared = 8.384796e-05

R-Squared = 1.00000000

R-Squared Adjusted = 1.00000000

Particle swarm AICc = -1.000000e+99

AIC = -Inf

AICc = -Inf

Using Power A+B*i+C*(i–1)

```
Fitting arccos(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)       2.95724787629783    0      Inf       0
    x1                -7.0815042443696    0     -Inf       0
    x2                6.34283664517569    0      Inf       0
    x3               -1.74259326845846    0     -Inf       0
    x4               -1.26289604468525    0     -Inf       0
    x5                0.973032043842618   0      Inf       0
    x6               -0.184658558815806   0     -Inf       0
    x7                2.41799400425029    0      Inf       0
    x8               -2.38383715245674    0     -Inf       0
    x9                1.19691624054893    0      Inf       0
    x10              -0.231805316840537   0     -Inf       0


Number of observations: 101, Error degrees of freedom: 90
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F     pValue
                  _________________    ___    ____________________    ___    ______

    Total       6.03050467942534     100    0.0603050467942534
    Model       6.03050467942534      10     0.603050467942534     Inf       0
    Residual                   0      90                     0

A = 0.015753, B = 0.156686, C = 0.090108
order Q(x) = 6.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.570796
List of powers for Q(x): [0.172439, 0.419233, 0.666028, 0.912822, 1.159616, 1.406410]
List of powers for D(x): [0.172439, 0.419233, 0.666028, 0.912822]
Fitting arccos(x) in range (0.000000, 1.000000)
MSS of errors squared = 3.424214e-06
Corrected MSS of errors squared = 5.930912e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)


```
Fitting arccos(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      3.27006361207884     0      Inf        0
    x1               -259.52694609767     0     -Inf        0
    x2               781.193049247551     0      Inf        0
    x3              -905.113967766164     0     -Inf        0
    x4               475.008227548912     0      Inf        0
    x5              -94.8285065614856     0     -Inf        0
    x6               53.4308708022762     0      Inf        0
    x7              -108.977702721574     0     -Inf        0
    x8                             0     0      NaN      NaN
    x9               177.571171533318     0      Inf        0
    x10             -189.316436160582     0     -Inf        0
    x11              81.2985835190506     0      Inf        0
    x12             -13.0074469640988     0     -Inf        0


Number of observations: 101, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F      pValue
                   _______________    ___    ____________________    ___    ______

    Total       6.0305046794226     100    0.060305046794226
    Model       6.0305046794226      11    0.548227698129327    Inf       0
    Residual                  0      89                    0

A = 0.360589, B = 0.134825, C = 0.038302
order Q(x) = 5.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.570796
List of powers for Q(x): [0.495414, 0.591937, 0.688460, 0.784983, 0.881507]
List of powers for D(x): [0.495414, 0.591937, 0.688460, 0.784983, 0.881507, 0.978030,
1.074553]
Fitting arccos(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.620218e-06
Corrected MSS of errors squared = 2.806299e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```


Using Power A+B*i+C*sqrt(i–1)

```
Fitting arccos(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)       0.13989823365476    0      Inf       0
    x1               -0.192893340065552   0     -Inf       0
    x2                0.0848834875643482  0      Inf       0
    x3               -0.0316266759571385  0     -Inf       0
    x4                2.11586830936712    0      Inf       0
    x5               -7.64947285681567    0     -Inf       0
    x6                20.2531951412346    0      Inf       0
    x7               -27.6606418443691    0     -Inf       0
    x8                20.9185656089497    0      Inf       0
    x9               -8.36412054425819    0     -Inf       0
    x10               1.38647533328278    0      Inf       0


Number of observations: 101, Error degrees of freedom: 90
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______

    Total       6.03050467942548     100    0.0603050467942548
    Model       6.03050467942548      10     0.603050467942548     Inf       0
    Residual                   0      90                       0

A = 0.000000, B = 0.100000, C = 0.270055
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.570796
List of powers for Q(x): [0.100000, 0.470055, 0.681916]
List of powers for D(x): [0.100000, 0.470055, 0.681916, 0.867749, 1.040111, 1.203862,
1.361498]
Fitting arccos(x) in range (0.000000, 1.000000)
MSS of errors squared = 6.106361e-06
Corrected MSS of errors squared = 1.057653e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```


Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting arccos(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14

Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)       3.11348804621236     0      Inf        0
    x1               -4.88226176143527     0     -Inf        0
    x2                2.99688546228725     0      Inf        0
    x3               -1.4467764570001      0     -Inf        0
    x4                                0     0      NaN      NaN
    x5                0.34537410750996     0      Inf        0
    x6               -0.145577330451211    0     -Inf        0
    x7                0.0194015587787412   0      Inf        0
    x8                1.54143324786512     0      Inf        0
    x9               -0.890327947600004    0     -Inf        0
    x10               0.41439885982932     0      Inf        0
    x11              -0.0295929744434161   0     -Inf        0
    x12              -0.0528945128858675   0     -Inf        0
    x13               0.0181332375568619   0      Inf        0
    x14              -0.0014167232728748   0     -Inf        0


Number of observations: 101, Error degrees of freedom: 87
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF            MeanSq            F      pValue
                ____________________    ___    ____________________    ___    ______

    Total       6.03050467942531     100    0.0603050467942531
    Model       6.03050467942531      13     0.463884975340409     Inf       0
    Residual                   0      87                     0

A = 0.000000, B = 0.112490, C = 0.342160
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.570796
List of powers for Q(x): [0.112490, 0.501245, 0.879159, 1.251460, 1.620176, 1.986343,
2.350581]
List of powers for D(x): [0.112490, 0.501245, 0.879159, 1.251460, 1.620176, 1.986343,
2.350581]
Fitting arccos(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.436180e-07
Corrected MSS of errors squared = 2.487536e-07
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting arccos(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10

Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      2.91879514229108     0      Inf       0
    x1              -7.08398204512842     0     -Inf       0
    x2               12.1585710856902     0      Inf       0
    x3              -11.6241772282578     0     -Inf       0
    x4               3.67082839213572     0      Inf       0
    x5               2.26997393444984     0      Inf       0
    x6              -4.47768159486899     0     -Inf       0
    x7               7.68180433849557     0      Inf       0
    x8              -8.49158830296053     0     -Inf       0
    x9               5.44587463879119     0      Inf       0
    x10             -1.44840068753944     0     -Inf       0


Number of observations: 101, Error degrees of freedom: 90
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq           DF            MeanSq            F      pValue
                  _________________    ___    ____________________    ___    ______

    Total        6.03050467942517     100    0.0603050467942517
    Model        6.03050467942517      10     0.603050467942517    Inf       0
    Residual                    0      90                      0

A = 0.321702, B = 0.174889, C = 0.595616
order Q(x) = 4.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.570796
List of powers for Q(x): [0.496591, 1.164647, 1.466945, 1.703116]
List of powers for D(x): [0.496591, 1.164647, 1.466945, 1.703116, 1.903996, 2.081927]
Fitting arccos(x) in range (0.000000, 1.000000)
MSS of errors squared = 3.740385e-06
Corrected MSS of errors squared = 6.478537e-06
R-Squared =   1.00000000
```



```
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting arccos(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4

Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9

Estimated Coefficients:
                        Estimate          SE     tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      3.11328654073824     0      Inf       0
    x1              -694.537942269767     0     -Inf       0
    x2               829.240223408322     0      Inf       0
    x3              -150.612716465693     0     -Inf       0
    x4               12.8256600213501     0      Inf       0
    x5               167.910840452078     0      Inf       0
    x6              -196.979072673681     0     -Inf       0
    x7               31.8284922541382     0      Inf       0
    x8              -1.64180630203542     0     -Inf       0
    x9              -0.132709347057222    0     -Inf       0


Number of observations: 101, Error degrees of freedom: 91
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______

    Total       6.03050467942492    100    0.0603050467942492
    Model       6.03050467942492      9     0.670056075491658    Inf       0
    Residual                   0     91                     0

A = 0.747315, B = 3.182212
order Q(x) = 4.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.570796
List of powers for Q(x): [0.747315, 0.773447, 0.912224, 1.165423]
List of powers for D(x): [0.747315, 0.773447, 0.912224, 1.165423, 1.506877]
Fitting arccos(x) in range (0.000000, 1.000000)
MSS of errors squared = 9.562543e-06
Corrected MSS of errors squared = 1.352348e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
```



```
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the arc cosine function. The results are sorted in ascending order using the MSSE values.

Power	*A*	*B*	*C*	*F*	*AICc*	*MSSE*	*Q/D Orders*
A+B*i+C*sqrt(i-1)	0.1	0.373	0.063	Inf	-Inf	4.72541900E-06	7, 5
A+B/i	0	2		Inf	-Inf	1.05056300E-05	7, 7
A+B*i-C*(i-1)	0.614	0.102	0.141	Inf	-Inf	1.14773700E-05	6, 4
A+B*sqrt(i)+C*sqrt(i-1)	0.252	0.686	0.391	Inf	-Inf	1.32363100E-05	6, 4
A+B*sqrt(i+C)	0.478	0.193	0.697	Inf	-Inf	1.74739800E-05	6, 5
A+B*sqrt(i)	0.521	0.1		Inf	-Inf	1.79016700E-05	7, 6
A+B*log10(i)^4	0.493	0.839		Inf	-Inf	2.05657300E-05	4, 7
A+B*sqrt(i)+C*(i-1)	0.55	0.4	0	Inf	-Inf	2.20774100E-05	3, 7
A+B*i+C*(i-1)	0	0.1	0	Inf	-Inf	2.81923300E-05	5, 3
A+B*i	0.246	0.12		Inf	-Inf	2.89880400E-05	3, 5

RESULTS FOR THE ARC SINE

Using Power A+B* i

```
Fitting arcsin(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.0376330468209987     0     -Inf       0
    x1                0.82321573878107     0      Inf       0
    x2               -1.63866440497613     0     -Inf       0
    x3                1.13263240001925     0      Inf       0
    x4               -0.263118222329546    0     -Inf       0
    x5                  2.3938239922764    0      Inf       0
    x6               -2.05970305730744     0     -Inf       0
    x7                0.737396395807817    0      Inf       0
    x8               -0.0879497882704047    0     -Inf       0
Number of observations: 101, Error degrees of freedom: 92
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______
    Total       6.03050467942532     100    0.0603050467942532
    Model       6.03050467942532       8     0.753813084928165     Inf       0
    Residual                   0      92                     0
A = 0.000000, B = 0.400500
order Q(x) = 4.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.570796
List of powers for Q(x): [0.400500, 0.801000, 1.201500, 1.602000]
List of powers for D(x): [0.400500, 0.801000, 1.201500, 1.602000]
Fitting arcsin(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.979544e-05
Corrected MSS of errors squared = 2.799498e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting arcsin(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
```



```
                    ____________________     __     _____     ______
    (Intercept)     0.000892935437828605     0       Inf        0
    x1              -0.00034479020085787     0      -Inf        0
    x2               0.00314074362587796     0       Inf        0
    x3              -0.00368857866332642     0      -Inf        0
    x4                0.0242385513735792     0       Inf        0
    x5                  -1.6988726977572     0      -Inf        0
    x6                  12.0986913693835     0       Inf        0
    x7                 -19.6442202695078     0      -Inf        0
    x8                                 0     0       NaN      NaN
    x9                  10.2201627363086     0       Inf        0
Number of observations: 101, Error degrees of freedom: 92
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq           DF            MeanSq            F     pValue
                   _______________    ___    ____________________    ___    ______
    Total       6.03050467942531     100    0.0603050467942531
    Model       6.03050467942531       8     0.753813084928164    Inf       0
    Residual                   0      92                     0
A = 0.000000, B = 0.256543
order Q(x) = 3.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.570796
List of powers for Q(x): [0.256543, 0.128271, 0.085514]
List of powers for D(x): [0.256543, 0.128271, 0.085514, 0.064136, 0.051309, 0.042757]
Fitting arcsin(x) in range (0.000000, 1.000000)
MSS of errors squared = 2.711992e-05
Corrected MSS of errors squared = 3.835337e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting arcsin(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________     __     _____     ______
    (Intercept)     -0.024693258900676     0      -Inf        0
    x1                22.5267094916245     0       Inf        0
    x2               -152.663615460133     0      -Inf        0
    x3                443.985415551138     0       Inf        0
    x4               -708.119514061442     0      -Inf        0
    x5                 647.43123419989     0       Inf        0
    x6               -318.223774949055     0      -Inf        0
    x7                65.1121558232508     0       Inf        0
```



```
    x8               11.7724029225372    0      Inf       0
    x9              -27.0899011760407    0     -Inf       0
    x10              23.1617549218766    0      Inf       0
    x11              -6.8681740048525    0     -Inf       0
Number of observations: 101, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ___________________    ___    ______
    Total       6.03050467942173    100    0.0603050467942173
    Model       6.03050467942173     11     0.548227698129248    Inf       0
    Residual                   0     89                     0
A = 0.213578, B = 0.731042
order Q(x) = 7.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.570796
List of powers for Q(x): [0.944619, 1.247427, 1.479779, 1.675661, 1.848236, 2.004256,
2.147732]
List of powers for D(x): [0.944619, 1.247427, 1.479779, 1.675661]
Fitting arcsin(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.386508e-06
Corrected MSS of errors squared = 1.960819e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting arcsin(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                      Estimate          SE    tStat    pValue
                  _____________________    __    _____    ______
    (Intercept)    0.188583554674248    0      Inf       0
    x1              71.2031768040498    0      Inf       0
    x2             -303.733214981523    0     -Inf       0
    x3              459.021563535543    0      Inf       0
    x4             -265.367333704493    0     -Inf       0
    x5                             0    0      NaN      NaN
    x6              49.8880907918461    0      Inf       0
    x7             -11.1159849337835    0     -Inf       0
    x8             -27.5039667547335    0     -Inf       0
    x9              260.142900604627    0      Inf       0
    x10            -691.368068394659    0     -Inf       0
    x11             815.525109992632    0      Inf       0
    x12            -452.81814991979     0     -Inf       0
    x13             96.9372934057594    0      Inf       0
```



```
Number of observations: 101, Error degrees of freedom: 88
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ________________    ___    ________________    ___    ______
    Total       6.03050467942107    100    0.0603050467942107
    Model       6.03050467942107     12     0.502542056618423    Inf       0
    Residual                   0     88                     0
A = 0.612969, B = 0.810906, C = 0.843594
order Q(x) = 7.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.570796
List of powers for Q(x): [1.714009, 1.980396, 2.202757, 2.397624, 2.573214, 2.734320,
2.884026]
List of powers for D(x): [1.714009, 1.980396, 2.202757, 2.397624, 2.573214, 2.734320]
Fitting arcsin(x) in range (0.000000, 1.000000)
MSS of errors squared = 3.840227e-07
Corrected MSS of errors squared = 6.651468e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting arcsin(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.203231048934298     0     -Inf       0
    x1               2.32338696618412     0      Inf       0
    x2              -6.01762437802004     0     -Inf       0
    x3               6.66860365244131     0      Inf       0
    x4              -3.46370276024682     0     -Inf       0
    x5              0.693018961817793     0      Inf       0
    x6               5.93555055969945     0      Inf       0
    x7              -10.9436299457341     0     -Inf       0
    x8               9.05184965966931     0      Inf       0
    x9              -3.60576438483573     0     -Inf       0
    x10             0.561542717952278     0      Inf       0
Number of observations: 101, Error degrees of freedom: 90
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ________________    ___    ________________    ___    ______
    Total       6.03050467942536    100    0.0603050467942536
    Model       6.03050467942536     10     0.603050467942536    Inf       0
```



```
    Residual                   0     90                     0
A = 0.151696, B = 0.133426, C = 0.066972
order Q(x) = 5.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.570796
List of powers for Q(x): [0.285122, 0.485519, 0.685916, 0.886313, 1.086710]
List of powers for D(x): [0.285122, 0.485519, 0.685916, 0.886313, 1.086710]
Fitting arcsin(x) in range (0.000000, 1.000000)
MSS of errors squared = 2.900888e-05
Corrected MSS of errors squared = 5.024486e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting arcsin(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    0.00870555481914905     0      Inf        0
    x1              -0.173303518948062     0     -Inf        0
    x2                0.40515330559981     0      Inf        0
    x3              -0.240529331118603     0     -Inf        0
    x4                18.2173399993901     0      Inf        0
    x5               -71.4568458042624     0     -Inf        0
    x6                 85.482633754588     0      Inf        0
    x7                               0     0      NaN      NaN
    x8                -65.700905203736     0     -Inf        0
    x9                34.4577512436911     0      Inf        0
Number of observations: 101, Error degrees of freedom: 92
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______
    Total       6.03050467942506     100    0.0603050467942506
    Model       6.03050467942506       8     0.753813084928133    Inf       0
    Residual                   0      92                     0
A = 0.170758, B = 0.167844, C = 0.205481
order Q(x) = 3.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.570796
List of powers for Q(x): [0.338602, 0.300964, 0.263327]
List of powers for D(x): [0.338602, 0.300964, 0.263327, 0.225689, 0.188052, 0.150415]
Fitting arcsin(x) in range (0.000000, 1.000000)
MSS of errors squared = 2.693376e-05
```



```
Corrected MSS of errors squared = 4.665064e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting arcsin(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    0.00593218615712107    0      Inf       0
    x1               0.795677827071942    0      Inf       0
    x2               -1.34148297501202    0     -Inf       0
    x3               -1.79696318989545    0     -Inf       0
    x4                5.14360313463095    0      Inf       0
    x5               -3.64293038859668    0     -Inf       0
    x6               0.843745257924713    0      Inf       0
    x7                3.96239772202819    0      Inf       0
    x8               -6.24876424051023    0     -Inf       0
    x9                 4.2689594160925    0      Inf       0
    x10             -0.990174749787121    0     -Inf       0
Number of observations: 101, Error degrees of freedom: 90
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______
    Total       6.03050467942537     100    0.0603050467942537
    Model       6.03050467942537      10     0.603050467942537     Inf       0
    Residual                   0      90                       0
A = 0.259784, B = 0.214353, C = 0.152625
order Q(x) = 6.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.570796
List of powers for Q(x): [0.474137, 0.841115, 1.118687, 1.381550, 1.636799, 1.887182]
List of powers for D(x): [0.474137, 0.841115, 1.118687, 1.381550]
Fitting arcsin(x) in range (0.000000, 1.000000)
MSS of errors squared = 3.183136e-06
Corrected MSS of errors squared = 5.513354e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting arcsin(x) in range (0.000000, 1.000000)
```



```
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ___________________    __    _____    ______
    (Intercept)     -0.149011071227731     0     -Inf       0
    x1                1.16348352790417     0      Inf       0
    x2               -2.91168971077042     0     -Inf       0
    x3                3.26453049057563     0      Inf       0
    x4               -1.70881838767843     0     -Inf       0
    x5               0.342211436279263     0      Inf       0
    x6                3.93421198670097     0      Inf       0
    x7               -6.43039326016932     0     -Inf       0
    x8                5.44322768369137     0      Inf       0
    x9               -2.44770483391472     0     -Inf       0
    x10              0.546609609373658     0      Inf       0
    x11             -0.0466574707664484    0     -Inf       0
Number of observations: 101, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ___________________    ___    ______
    Total       6.03050467942533     100    0.0603050467942533
    Model       6.03050467942533      11     0.548227698129575     Inf       0
    Residual                   0      89                     0
A = 0.130931, B = 0.133516, C = 0.205320
order Q(x) = 5.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.570796
List of powers for Q(x): [0.264447, 0.525072, 0.772829, 1.013924, 1.250764]
List of powers for D(x): [0.264447, 0.525072, 0.772829, 1.013924, 1.250764, 1.484579]
Fitting arcsin(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.393137e-06
Corrected MSS of errors squared = 2.412984e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting arcsin(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ___________________    __    _____    ______
    (Intercept)     0.0657802427262099     0      Inf       0
```

x1	1.24716550253573	0	Inf	0
x2	-5.54165774518965	0	-Inf	0
x3	3.79038505433492	0	Inf	0
x4	6.53541242608584	0	Inf	0
x5	-9.12570969772333	0	-Inf	0
x6	3.05259749521646	0	Inf	0
x7	3.42546712580821	0	Inf	0
x8	-6.62194672588868	0	-Inf	0
x9	5.96723674116648	0	Inf	0
x10	-1.79473041859177	0	-Inf	0

Number of observations: 101, Error degrees of freedom: 90
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	6.03050467942548	100	0.0603050467942548		
Model	6.03050467942548	10	0.603050467942548	Inf	0
Residual	0	90	0		

A = 0.259255, B = 0.430264, C = 0.395667
order Q(x) = 6.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.570796
List of powers for Q(x): [0.689519, 1.263407, 1.564052, 1.805098, 2.012688, 2.197920]
List of powers for D(x): [0.689519, 1.263407, 1.564052, 1.805098]
Fitting arcsin(x) in range (0.000000, 1.000000)
MSS of errors squared = 3.223691e-06
Corrected MSS of errors squared = 5.583597e-06
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*log10(i)^4

Fitting arcsin(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8
Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	0	0	NaN	NaN
x1	-1.36692031857759e-08	0	-Inf	0
x2	1.59812192686054e-08	0	Inf	0
x3	-2.3228062992083e-09	0	-Inf	0
x4	1.0000002249191	0	Inf	0
x5	-2.87333384803348e-07	0	-Inf	0
x6	7.53193299310423e-08	0	Inf	0
x7	-1.41697922404295e-08	0	-Inf	0
x8	1.27553774195209e-09	0	Inf	0


```
Number of observations: 101, Error degrees of freedom: 93
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ___________________    ___    ___________________    ___    ______

    Total       6.03050467942531    100    0.0603050467942531
    Model       6.03050467942531      7      0.86150066848933    Inf       0
    Residual                   0     93                     0
A = 0.000000, B = 2.478300
order Q(x) = 3.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.570796
List of powers for Q(x): [0.000000, 0.020351, 0.128431]
List of powers for D(x): [0.000000, 0.020351, 0.128431, 0.325621, 0.591545]
Fitting arcsin(x) in range (0.000000, 1.000000)
MSS of errors squared = 6.742726e-04
Corrected MSS of errors squared = 9.535655e-04
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the arc sine function. The results are sorted in ascending order using the MSSE values.

Power	*A*	*B*	*C*	*F*	*AICc*	*MSSE*	*Q/D Orders*
A+B*sqrt(i+C)	0.612	0.81	0.843	Inf	-Inf	3.84022700E-07	7, 6
A+B*sqrt(i)	0.213	0.731		Inf	-Inf	1.38650800E-06	7, 4
A+B*sqrt(i)+C*(i-1)	0.13	0.133	0.205	Inf	-Inf	1.39313700E-06	5, 6
A+B*i+C*sqrt(i-1)	0.259	0.214	0.152	Inf	-Inf	3.18313600E-06	6, 4
A+B*sqrt(i)+C*sqrt(i-1)	0.259	0.43	0.395	Inf	-Inf	3.22369100E-06	6, 4
A+B*i	0	0.4		Inf	-Inf	1.97954400E-05	4, 4
A+B*i-C*(i-1)	0.17	0.167	0.205	Inf	-Inf	2.69337600E-05	3, 6
A+B/i	0	0.256		Inf	-Inf	2.71199200E-05	3, 6
A+B*i+C*(i-1)	0.151	0.133	0.066	Inf	-Inf	2.90088800E-05	5, 5
A+B*log10(i)^4	0	2.478		Inf	-Inf	6.74272600E-04	3, 5

RESULTS FOR THE ARC TANGENT

Using Power A+B* i

```
Fitting arctan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)    -0.220436735053578     0     -Inf       0
    x1               3.12801566968573     0      Inf       0
    x2              -9.54212240955542     0     -Inf       0
    x3               13.8392740300967     0      Inf       0
    x4              -11.0348611464651     0     -Inf       0
    x5               4.66083151301807     0      Inf       0
    x6              -0.806439329617796    0     -Inf       0
    x7               6.13753288448597     0      Inf       0
    x8              -11.6378533169045     0     -Inf       0
    x9               9.98122129269713     0      Inf       0
    x10             -4.21090162859268     0     -Inf       0
    x11             0.705739176204544     0      Inf       0
Number of observations: 101, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                        SumSq          DF           MeanSq            F     pValue
                    ____________________    ___    ____________________    ___    _______
    Total          8.80077198802871     100    0.0880077198802871
    Model          8.80077198802871      11     0.800070180729883    Inf       0
    Residual                      0      89                      0
A = 0.200313, B = 0.343512
order Q(x) = 6.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 0.785398
List of powers for Q(x): [0.543825, 0.887336, 1.230848, 1.574359, 1.917871, 2.261382]
List of powers for D(x): [0.543825, 0.887336, 1.230848, 1.574359, 1.917871]
Fitting arctan(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.578001e-12
Corrected MSS of errors squared = 2.231631e-12
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting arctan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
```



```
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    0.0460236230606327    0      Inf        0
    x1               3.28818608758809    0      Inf        0
    x2              -17.9550233582728    0     -Inf        0
    x3               20.2909029720603    0      Inf        0
    x4                              0    0      NaN      NaN
    x5              -5.4426637675014     0     -Inf        0
    x6               19.4115447451539    0      Inf        0
    x7              -993.640589548405    0     -Inf        0
    x8               5327.15270751652    0      Inf        0
    x9              -6896.05019852564    0     -Inf        0
    x10                             0    0      NaN      NaN
    x11              2543.89911025598    0      Inf        0
Number of observations: 101, Error degrees of freedom: 91
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _________________    ___    ____________________    ___    ______
    Total        8.80077198802807    100    0.0880077198802807
    Model        8.80077198802807      9     0.977863554225341    Inf       0
    Residual                    0     91                      0
A = 1.405741, B = 0.606119
order Q(x) = 5.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 0.785398
List of powers for Q(x): [2.011859, 1.708800, 1.607780, 1.557270, 1.526964]
List of powers for D(x): [2.011859, 1.708800, 1.607780, 1.557270, 1.526964, 1.506760]
Fitting arctan(x) in range (0.000000, 1.000000)
MSS of errors squared = 8.285699e-11
Corrected MSS of errors squared = 1.171775e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting arctan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.074507098432849    0     -Inf        0
    x1               169.562082508553    0      Inf        0
    x2              -745.669809349309    0     -Inf        0
```



```
    x3                913.289327603507     0      Inf        0
    x4                               0     0      NaN      NaN
    x5               -471.939844103125     0     -Inf        0
    x6                               0     0      NaN      NaN
    x7                135.916536085994     0      Inf        0
    x8               -19.8972581261361     0     -Inf        0
    x9                92.5764591765006     0      Inf        0
    x10                              0     0      NaN      NaN
    x11              -385.143962916407     0     -Inf        0
    x12               502.875364485545     0      Inf        0
    x13              -190.494388267016     0     -Inf        0
Number of observations: 101, Error degrees of freedom: 90
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                 _______________    ___    ____________________    ___    _______
    Total        8.80077198799969    100    0.0880077198799969
    Model        8.80077198799969     10     0.880077198799969    Inf        0
    Residual                    0     90                     0
A = 1.400000, B = 0.587229
order Q(x) = 7.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 0.785398
List of powers for Q(x): [1.987229, 2.230467, 2.417110, 2.574457, 2.713083, 2.838410,
2.953661]
List of powers for D(x): [1.987229, 2.230467, 2.417110, 2.574457, 2.713083, 2.838410]
Fitting arctan(x) in range (0.000000, 1.000000)
MSS of errors squared = 8.506483e-12
Corrected MSS of errors squared = 1.202998e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting arctan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    _________________    ___    _____    _______
    (Intercept)    -0.171736675603109     0     -Inf        0
    x1               438.709444160812     0      Inf        0
    x2              -1624.69655276922     0     -Inf        0
    x3               1752.03170265258     0      Inf        0
    x4                               0     0      NaN      NaN
    x5              -758.277990615604     0     -Inf        0
    x6                               0     0      NaN      NaN
```



```
    x7               193.023969276099    0      Inf        0
    x8               11.4042749195355    0      Inf        0
    x9               11.2604588409469    0      Inf        0
    x10                             0    0      NaN      NaN
    x11             -93.9457214858052    0     -Inf        0
    x12                             0    0      NaN      NaN
    x13              164.07737550878     0      Inf        0
    x14             -92.4152238126815    0     -Inf        0
Number of observations: 101, Error degrees of freedom: 90
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ________________    ___    ___________________    ___    ______
    Total       8.80077198805517    100    0.0880077198805517
    Model       8.80077198805517     10     0.880077198805517    Inf       0
    Residual                   0     90                      0
A = 0.681029, B = 0.581358, C = 2.000000
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 0.785398
List of powers for Q(x): [1.687970, 1.843744, 1.980984, 2.105059, 2.219157, 2.325357,
2.425102]
List of powers for D(x): [1.687970, 1.843744, 1.980984, 2.105059, 2.219157, 2.325357,
2.425102]
Fitting arctan(x) in range (0.000000, 1.000000)
MSS of errors squared = 8.476487e-12
Corrected MSS of errors squared = 1.468171e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting arctan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ________________    ___    _____    ______
    (Intercept)    -0.106916549036105    0     -Inf       0
    x1               1.52614984006267    0      Inf       0
    x2              -4.00163792700684    0     -Inf       0
    x3               4.72374834901435    0      Inf       0
    x4              -2.80684198755633    0     -Inf       0
    x5              0.702134119276317    0      Inf       0
    x6               6.30107299109611    0      Inf       0
    x7              -12.473004430762     0     -Inf       0
    x8               11.7087479338376    0      Inf       0
```



```
    x9              -6.06429374295832    0     -Inf       0
    x10              1.71773103814769    0      Inf       0
    x11             -0.226889634116008   0     -Inf       0
Number of observations: 101, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ___________________    ___    ______
    Total       8.80077198802859    100    0.0880077198802859
    Model       8.80077198802859     11     0.800070180729872    Inf       0
    Residual                   0     89                     0
A = 0.474044, B = 0.119571, C = 0.244208
order Q(x) = 5.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 0.785398
List of powers for Q(x): [0.593615, 0.957394, 1.321172, 1.684950, 2.048729]
List of powers for D(x): [0.593615, 0.957394, 1.321172, 1.684950, 2.048729, 2.412507]
Fitting arctan(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.080037e-12
Corrected MSS of errors squared = 1.870679e-12
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting arctan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                    Estimate          SE    tStat    pValue
                ________________    ___    _____    ______
    (Intercept)  0.976555620416784    0      Inf       0
    x1             826.87166410153    0      Inf       0
    x2            -1520.24233566136    0     -Inf       0
    x3                           0    0      NaN      NaN
    x4             886.39928450308    0      Inf       0
    x5                           0    0      NaN      NaN
    x6                           0    0      NaN      NaN
    x7             -193.94834055482    0     -Inf       0
    x8            -919.625290100979    0     -Inf       0
    x9             2564.70734810423    0      Inf       0
    x10           -2017.50616638214    0     -Inf       0
    x11                          0    0      NaN      NaN
    x12              373.3672803714    0      Inf       0
Number of observations: 101, Error degrees of freedom: 92
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```


	SumSq	DF	MeanSq	F	pValue
Total	8.80077198803699	100	0.0880077198803699		
Model	8.80077198803699	8	1.10009649850462	Inf	0
Residual	0	92	0		

A = 0.652132, B = 0.185313, C = 0.218886
order Q(x) = 7.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 0.785398
List of powers for Q(x): [0.837445, 0.803873, 0.770300, 0.736727, 0.703155, 0.669582, 0.636010]
List of powers for D(x): [0.837445, 0.803873, 0.770300, 0.736727, 0.703155]
Fitting arctan(x) in range (0.000000, 1.000000)
MSS of errors squared = 7.795869e-10
Corrected MSS of errors squared = 1.350284e-09
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*i+C*sqrt(i–1)

Fitting arctan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8
Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	0.610308166888669	0	Inf	0
x1	-3.20508802843187	0	-Inf	0
x2	8.15390869567702	0	Inf	0
x3	-8.51891410846731	0	-Inf	0
x4	3.0490237731112	0	Inf	0
x5	2.99990948519988	0	Inf	0
x6	-4.91787780858186	0	-Inf	0
x7	4.10929873362083	0	Inf	0
x8	-1.28056890689772	0	-Inf	0

Number of observations: 101, Error degrees of freedom: 92
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	8.80077198802867	100	0.0880077198802867		
Model	8.80077198802867	8	1.10009649850358	Inf	0
Residual	0	92	0		

A = 0.191166, B = 0.240585, C = 0.194052
order Q(x) = 4.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 0.785398


```
List of powers for Q(x): [0.431750, 0.866387, 1.187350, 1.489611]
List of powers for D(x): [0.431750, 0.866387, 1.187350, 1.489611]
Fitting arctan(x) in range (0.000000, 1.000000)
MSS of errors squared = 7.288135e-10
Corrected MSS of errors squared = 1.262342e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting arctan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    0.0549906470537258    0      Inf       0
    x1             -0.445204285856624    0     -Inf       0
    x2               1.53167079617756    0      Inf       0
    x3              -1.8234506664154     0     -Inf       0
    x4              0.745641926174658    0      Inf       0
    x5               5.50811860179076    0      Inf       0
    x6              -10.4648508247947    0     -Inf       0
    x7               9.06137494720829    0      Inf       0
    x8              -3.73619023345456    0     -Inf       0
    x9              0.567899092254967    0      Inf       0
Number of observations: 101, Error degrees of freedom: 91
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______
    Total       8.80077198802876    100    0.0880077198802876
    Model       8.80077198802876      9     0.977863554225417    Inf       0
    Residual                   0     91                     0
A = 0.271311, B = 0.338158, C = 0.225927
order Q(x) = 4.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 0.785398
List of powers for Q(x): [0.609469, 0.975465, 1.308871, 1.625407]
List of powers for D(x): [0.609469, 0.975465, 1.308871, 1.625407, 1.931162]
Fitting arctan(x) in range (0.000000, 1.000000)
MSS of errors squared = 8.449025e-11
Corrected MSS of errors squared = 1.463414e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
```



```
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting arctan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.0116013836905448     0     -Inf       0
    x1               0.188025630664285     0      Inf       0
    x2              -0.774017606986211     0     -Inf       0
    x3               0.990219141260193     0      Inf       0
    x4              -0.379911215484229     0     -Inf       0
    x5                5.42405757680897     0      Inf       0
    x6               -36.4663910273376     0     -Inf       0
    x7                93.0440519082063     0      Inf       0
    x8               -108.069771635189     0     -Inf       0
    x9                60.0609776445031     0      Inf       0
    x10              -13.0056390327524     0     -Inf       0
Number of observations: 101, Error degrees of freedom: 90
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F     pValue
                  _________________    ___    ____________________    ___    ______
    Total        8.80077198802642     100    0.0880077198802642
    Model        8.80077198802642      10     0.880077198802642     Inf       0
    Residual                    0      90                       0
A = 0.415799, B = 0.100000, C = 0.512098
order Q(x) = 4.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 0.785398
List of powers for Q(x): [0.515799, 1.069319, 1.313220, 1.502779]
List of powers for D(x): [0.515799, 1.069319, 1.313220, 1.502779, 1.663602, 1.805834]
Fitting arctan(x) in range (0.000000, 1.000000)
MSS of errors squared = 1.533663e-11
Corrected MSS of errors squared = 2.656383e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting arctan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
```


(Intercept)	0	0	NaN	NaN
x1	9.15440863197722e-06	0	Inf	0
x2	0	0	NaN	NaN
x3	-2.36592931975378e-05	0	-Inf	0
x4	2.36328412886639e-05	0	Inf	0
x5	-1.14711404597697e-05	0	-Inf	0
x6	2.35820461579769e-06	0	Inf	0
x7	0.999992136360254	0	Inf	0
x8	0	0	NaN	NaN
x9	1.81704769960712e-05	0	Inf	0
x10	-1.5542182564951e-05	0	-Inf	0
x11	6.32909763014318e-06	0	Inf	0
x12	-1.10877319204414e-06	0	-Inf	0

```
Number of observations: 101, Error degrees of freedom: 91
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```

	SumSq	DF	MeanSq	F	pValue
Total	8.80077198802877	100	0.0880077198802877		
Model	8.80077198802877	9	0.977863554225419	Inf	0
Residual	0	91	0		

```
A = 0.000000, B = 2.048444
order Q(x) = 6.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 0.785398
List of powers for Q(x): [0.000000, 0.016821, 0.106155, 0.269143, 0.488943, 0.751070]
List of powers for D(x): [0.000000, 0.016821, 0.106155, 0.269143, 0.488943, 0.751070]
Fitting arctan(x) in range (0.000000, 1.000000)
MSS of errors squared = 3.436739e-09
Corrected MSS of errors squared = 4.860284e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the arc tangent function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i+C*(i-1)	0.474	0.119	0.244	Inf	-Inf	1.08003700E-12	5, 6
A+B*i	0.2	0.343		Inf	-Inf	1.57800100E-12	6, 5
A+B*sqrt(i+C)	0.681	0.581	2	Inf	-Inf	8.47648700E-12	7, 7
A+B*sqrt(i)	1.4	0.587		Inf	-Inf	8.50648300E-12	7, 6
A+B*sqrt(i)+C*sqrt(i-1)	0.415	0.1	0.512	Inf	-Inf	1.53366300E-11	4, 6
A+B/i	1.405	0.606		Inf	-Inf	8.28569900E-11	5, 6
A+B*sqrt(i)+C*(i-1)	0.271	0.338	0.225	Inf	-Inf	8.44902500E-11	4, 5
A+B*i+C*sqrt(i-1)	0.191	0.24	0.194	Inf	-Inf	7.28813500E-10	4, 4
A+B*i-C*(i-1)	0.652	0.185	0.218	Inf	-Inf	7.79586900E-10	7, 5
A+B*log10(i)^4	0	2.048		Inf	-Inf	3.43673900E-09	6, 6

RESULTS FOR INEVRSE HYPERBOLIC SINE

Using Power A+B* i

```
Fitting asinh(x) in range (0.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                    ________________    ___    _____    _______
    (Intercept)    0.907542193116162     0      Inf         0
    x1             -10.1612018177425     0     -Inf         0
    x2              14.9016625528469     0      Inf         0
    x3                             0     0      NaN       NaN
    x4             -9.96927248413274     0     -Inf         0
    x5              4.32126959811881     0      Inf         0
    x6              24.9733666900933     0      Inf         0
    x7             -80.6567609695715     0     -Inf         0
    x8              120.743688916647     0      Inf         0
    x9             -105.812502734981     0     -Inf         0
    x10             56.8915280141135     0      Inf         0
    x11            -17.3656138982489     0     -Inf         0
    x12             2.22629393984063     0      Inf         0
Number of observations: 1001, Error degrees of freedom: 989
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq            DF           MeanSq            F      pValue
                    ________________    ____    ____________________    ___    _______
    Total          32.9032466460337    1000    0.0329032466460337
    Model          32.9032466460337      11      2.99120424054852    Inf         0
    Residual                      0     989                     0
A = 0.294847, B = 0.195129
order Q(x) = 5.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 5.298342
```



```
List of powers for Q(x): [0.489977, 0.685106, 0.880235, 1.075364, 1.270493]
List of powers for D(x): [0.489977, 0.685106, 0.880235, 1.075364, 1.270493, 1.465623,
1.660752]
Fitting asinh(x) in range (0.000000, 100.000000)
MSS of errors squared = 2.911493e-06
Corrected MSS of errors squared = 4.117473e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting asinh(x) in range (0.000000, 100.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      0.0507146368053394     0      Inf       0
    x1              -0.0389289384538173     0     -Inf       0
    x2               0.284412196636556      0      Inf       0
    x3              -0.296197885560234      0     -Inf       0
    x4               0.0893190022073432     0      Inf       0
    x5              -5.63975104832233       0     -Inf       0
    x6               42.1156743804635       0      Inf       0
    x7              -85.6909032334495       0     -Inf       0
    x8               50.1256608896443       0      Inf       0
Number of observations: 1001, Error degrees of freedom: 992
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq            DF            MeanSq            F      pValue
                  _________________    ____    ____________________    ___    ______
    Total        32.9032466460575     1000    0.0329032466460575
    Model        32.9032466460575        8      4.11290583075719    Inf       0
    Residual                    0      992                     0
A = 0.028672, B = 0.320026
order Q(x) = 3.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 5.298342
List of powers for Q(x): [0.348698, 0.188685, 0.135347]
List of powers for D(x): [0.348698, 0.188685, 0.135347, 0.108678, 0.092677]
Fitting asinh(x) in range (0.000000, 100.000000)
MSS of errors squared = 4.965441e-05
Corrected MSS of errors squared = 7.022193e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
```



```
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting asinh(x) in range (0.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    ________________    __    _____    ______
    (Intercept)      2.07213538808539    0      Inf        0
    x1              -398.828660758807    0     -Inf        0
    x2               887.930652012261    0      Inf        0
    x3                              0    0      NaN      NaN
    x4              -780.726315248146    0     -Inf        0
    x5                              0    0      NaN      NaN
    x6               289.552182834068    0      Inf        0
    x7               366.745177557046    0      Inf        0
    x8              -1421.31607063944    0     -Inf        0
    x9               1880.84336996768    0      Inf        0
    x10             -943.689640228558    0     -Inf        0
    x11                             0    0      NaN      NaN
    x12              217.112415498951    0      Inf        0
    x13             -98.6952462501634    0     -Inf        0
Number of observations: 1001, Error degrees of freedom: 990
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq            DF            MeanSq            F      pValue
                  _________________    ____    ____________________    ___    ______
    Total        32.9032466456852     1000    0.0329032466456852
    Model        32.9032466456852       10      3.29032466456852    Inf       0
    Residual                    0      990                     0
A = 1.400000, B = 0.618432
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 5.298342
List of powers for Q(x): [2.018432, 2.274595, 2.471156, 2.636864, 2.782856, 2.914843]
List of powers for D(x): [2.018432, 2.274595, 2.471156, 2.636864, 2.782856, 2.914843,
3.036217]
Fitting asinh(x) in range (0.000000, 100.000000)
MSS of errors squared = 3.031572e-05
Corrected MSS of errors squared = 4.287290e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting asinh(x) in range (0.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
```



```
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ________________    ___    _____    ______
    (Intercept)      0.31219922006108    0      Inf        0
    x1              -7.91774120329202    0     -Inf        0
    x2               11.1817842543916    0      Inf        0
    x3                             0    0      NaN      NaN
    x4              -3.57624218053191    0     -Inf        0
    x5               176.626845917017    0      Inf        0
    x6              -620.494991520643    0     -Inf        0
    x7               658.351035042677    0      Inf        0
    x8                             0    0      NaN      NaN
    x9              -369.674277316921    0     -Inf        0
    x10              156.191387787014    0      Inf        0
Number of observations: 1001, Error degrees of freedom: 992
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq            DF            MeanSq             F      pValue
                  _________________    ____    ____________________    ___    ______
    Total        32.9032466460598     1000    0.0329032466460598
    Model        32.9032466460598        8      4.11290583075747     Inf       0
    Residual                    0      992                     0
A = 0.062365, B = 0.100000, C = 0.943374
order Q(x) = 4.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 5.298342
List of powers for Q(x): [0.201770, 0.233928, 0.260945, 0.284702]
List of powers for D(x): [0.201770, 0.233928, 0.260945, 0.284702, 0.306156, 0.325868]
Fitting asinh(x) in range (0.000000, 100.000000)
MSS of errors squared = 5.317823e-05
Corrected MSS of errors squared = 9.210739e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting asinh(x) in range (0.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ________________    ___    _____    ______
    (Intercept)      1.87003489962366    0      Inf        0
    x1              -15.4687006675401    0     -Inf        0
    x2               26.0625489395688    0      Inf        0
```



```
    x3                              0     0      NaN      NaN
    x4                -35.501215886953    0     -Inf        0
    x5                 31.8786331142985   0      Inf        0
    x6                -8.84130059611631   0     -Inf        0
    x7                 9.91753576187329   0      Inf        0
    x8                -22.3221664085872   0     -Inf        0
    x9                 17.0006546318603   0      Inf        0
    x10                1.8729995278266    0      Inf        0
    x11               -8.54692585013954   0     -Inf        0
    x12                3.1109169800934    0      Inf        0
    x13             -0.0330144459237978   0     -Inf        0
Number of observations: 1001, Error degrees of freedom: 988
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq            DF            MeanSq            F     pValue
                ________________    _____    ____________________    ___    _______
    Total       32.9032466460596    1000    0.0329032466460596
    Model       32.9032466460596      12      2.74193722050496    Inf        0
    Residual                   0     988                     0
A = 0.409865, B = 0.226378, C = 0.188619
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 5.298342
List of powers for Q(x): [0.636243, 1.051240, 1.466237, 1.881233, 2.296230, 2.711227]
List of powers for D(x): [0.636243, 1.051240, 1.466237, 1.881233, 2.296230, 2.711227,
3.126223]
Fitting asinh(x) in range (0.000000, 100.000000)
MSS of errors squared = 7.835248e-07
Corrected MSS of errors squared = 1.357105e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting asinh(x) in range (0.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                    Estimate         SE    tStat    pValue
                ________________    ___    _____    _______
    (Intercept)  1.02441219076902    0      Inf        0
    x1          -9.37737886775207    0     -Inf        0
    x2           13.2103947391848    0      Inf        0
    x3                          0    0      NaN      NaN
    x4          -8.49586510031827    0     -Inf        0
    x5           3.63843703994922    0      Inf        0
    x6           15.9373368346991    0      Inf        0
```



```
    x7             -41.9109933187652    0     -Inf        0
    x8              47.8620499290362    0      Inf        0
    x9             -29.0315294932353    0     -Inf        0
    x10             9.65649527077884    0      Inf        0
    x11            -1.51335922430548    0     -Inf        0
Number of observations: 1001, Error degrees of freedom: 990
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq           DF           MeanSq           F     pValue
                ________________    ____    ________________    ___    ______
    Total       32.9032466460603    1000    0.0329032466460603
    Model       32.9032466460603      10      3.29032466460603    Inf       0
    Residual                   0     990                     0
A = 0.043123, B = 0.226632, C = 0.105246
order Q(x) = 5.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 5.298342
List of powers for Q(x): [0.269755, 0.391141, 0.512528, 0.633914, 0.755301]
List of powers for D(x): [0.269755, 0.391141, 0.512528, 0.633914, 0.755301, 0.876687]
Fitting asinh(x) in range (0.000000, 100.000000)
MSS of errors squared = 2.341769e-05
Corrected MSS of errors squared = 4.056063e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting asinh(x) in range (0.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8
Estimated Coefficients:
                    Estimate         SE    tStat    pValue
                ________________    ___    _____    ______
    (Intercept)  7.49029815473679    0      Inf       0
    x1          -16.2114002005062    0     -Inf       0
    x2           27.2843029960006    0      Inf       0
    x3          -28.2989868347399    0     -Inf       0
    x4           9.73578244270755    0      Inf       0
    x5           2.84607601064504    0      Inf       0
    x6          -7.01696345732329    0     -Inf       0
    x7           8.24245040483111    0      Inf       0
    x8          -3.07155948532202    0     -Inf       0
Number of observations: 1001, Error degrees of freedom: 992
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq           DF           MeanSq           F     pValue
                ________________    ____    ________________    ___    ______
```



```
    Total       32.9032466460577    1000    0.0329032466460577
    Model       32.9032466460577       8      4.11290583075722    Inf       0
    Residual                   0     992                     0
A = 0.000000, B = 0.100000, C = 0.184061
order Q(x) = 4.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 5.298342
List of powers for Q(x): [0.100000, 0.384061, 0.560301, 0.718803]
List of powers for D(x): [0.100000, 0.384061, 0.560301, 0.718803]
Fitting asinh(x) in range (0.000000, 100.000000)
MSS of errors squared = 2.543930e-05
Corrected MSS of errors squared = 4.406216e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting asinh(x) in range (0.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    _________________    ___    _____    ______
    (Intercept)      1.78748288048109    0       Inf       0
    x1              -32.3865531148384    0      -Inf       0
    x2               76.4202930719394    0       Inf       0
    x3              -65.0433379803026    0      -Inf       0
    x4               19.2221151148617    0       Inf       0
    x5               22.1174295113391    0       Inf       0
    x6              -60.8830551795649    0      -Inf       0
    x7               67.8800817514062    0       Inf       0
    x8              -36.7096196702336    0      -Inf       0
    x9               9.92749391362109    0       Inf       0
    x10             -1.33233029670388    0      -Inf       0
Number of observations: 1001, Error degrees of freedom: 990
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                        SumSq           DF           MeanSq           F      pValue
                    _________________    ____    _________________    ___    ______
    Total       32.9032466460494    1000    0.0329032466460494
    Model       32.9032466460494      10      3.29032466460494    Inf       0
    Residual                   0     990                     0
A = 0.550000, B = 0.100000, C = 0.208951
order Q(x) = 4.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 5.298342
List of powers for Q(x): [0.650000, 0.900372, 1.141107, 1.376853]
```



```
List of powers for D(x): [0.650000, 0.900372, 1.141107, 1.376853, 1.609411, 1.839704]
Fitting asinh(x) in range (0.000000, 100.000000)
MSS of errors squared = 6.782565e-05
Corrected MSS of errors squared = 1.174775e-04
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting asinh(x) in range (0.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                    ________________    ___    _____    _______
    (Intercept)      2.27868266002486    0       Inf        0
    x1               -11.790609742777    0      -Inf        0
    x2               69.4051257437675    0       Inf        0
    x3              -137.520995774265    0      -Inf        0
    x4               91.7822549085485    0       Inf        0
    x5                              0    0       NaN      NaN
    x6              -14.1544582296851    0      -Inf        0
    x7               5.69004231015647    0       Inf        0
    x8              -40.7078221519441    0      -Inf        0
    x9               99.3853040705725    0       Inf        0
    x10             -103.276697733141    0      -Inf        0
    x11              49.7210586803983    0       Inf        0
    x12             -12.2694954756918    0      -Inf        0
    x13              2.45761073375542    0       Inf        0
Number of observations: 1001, Error degrees of freedom: 988
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq           DF           MeanSq            F      pValue
                  _____________    ____    ____________________    ___    _______
    Total        32.9032466460211    1000    0.0329032466460211
    Model        32.9032466460211      12      2.74193722050176    Inf        0
    Residual                    0     988                     0
A = 0.189900, B = 0.387109, C = 0.700000
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 5.298342
List of powers for Q(x): [0.577010, 1.437355, 1.850343, 2.176554, 2.455503, 2.703368]
List of powers for D(x): [0.577010, 1.437355, 1.850343, 2.176554, 2.455503, 2.703368,
2.928738]
Fitting asinh(x) in range (0.000000, 100.000000)
MSS of errors squared = 8.622467e-07
Corrected MSS of errors squared = 1.493455e-06
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting asinh(x) in range (0.000000, 100.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                      ____________________    __    _____    ______
    (Intercept)                          0     0      NaN      NaN
    x1             -3.03791868526194e-06     0     -Inf        0
    x2                                   0     0      NaN      NaN
    x3                                   0     0      NaN      NaN
    x4              6.76702500001886e-06     0      Inf        0
    x5             -3.72910767401094e-06     0     -Inf        0
    x6                  1.00064840834633     0      Inf        0
    x7                                   0     0      NaN      NaN
    x8               -0.0010655879933374     0     -Inf        0
    x9                                   0     0      NaN      NaN
    x10             0.000936305008333148     0      Inf        0
    x11            -0.000673044729420512     0     -Inf        0
    x12             0.000153919369478858     0      Inf        0
Number of observations: 1001, Error degrees of freedom: 993
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq            DF           MeanSq            F     pValue
                  _________________    ____    ____________________    ___    ______
    Total        32.9032466460572    1000    0.0329032466460572
    Model        32.9032466460572       7      4.70046380657959    Inf       0
    Residual                    0     993                     0
A = 0.000000, B = 0.100000
order Q(x) = 5.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 5.298342
List of powers for Q(x): [0.000000, 0.000821, 0.005182, 0.013139, 0.023869]
List of powers for D(x): [0.000000, 0.000821, 0.005182, 0.013139, 0.023869, 0.036665,
0.051007]
Fitting asinh(x) in range (0.000000, 100.000000)
MSS of errors squared = 1.882828e-05
Corrected MSS of errors squared = 2.662721e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```


Comments

The following table shows the summary results for the ten models that fit the inverse hyperbolic sine function. The results are sorted in ascending order using the MSSE values.

Power	*A*	*B*	*C*	*F*	*AICc*	*MSSE*	*Q/D Orders*
A+B*i+C*(i-1)	0.409	0.226	0.188	Inf	-Inf	7.83524800E-07	6, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.189	0.387	0.7	Inf	-Inf	8.62246700E-07	6, 7
A+B*i	0.294	0.195		Inf	-Inf	2.91149300E-06	5, 7
A+B*log10(i)^4	0	0.1		Inf	-Inf	1.88282800E-05	5, 7
A+B*i-C*(i-1)	0.043	0.226	0.105	Inf	-Inf	2.34176900E-05	5, 6
A+B*i+C*sqrt(i-1)	0	0.1	0.184	Inf	-Inf	2.54393000E-05	4, 4
A+B*sqrt(i)	1.4	0.618		Inf	-Inf	3.03157200E-05	6, 7
A+B/i	0.028	0.32		Inf	-Inf	4.96544100E-05	3, 5
A+B*sqrt(i+C)	0.062	0.1	0.943	Inf	-Inf	5.31782300E-05	4, 6
A+B*sqrt(i)+C*(i-1)	0.55	0.1	0.208	Inf	-Inf	6.78256500E-05	4, 6

RESULTS FOR INVERSE HYPERBOLIC TANGENT

Using Power A+B* i

```
Fitting atanh(x) in range (0.000000, 0.999000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)      0.216863886366257      0      Inf         0
    x1              -0.771706460988133      0     -Inf         0
    x2               1.02869965924752       0      Inf         0
    x3              -0.608768997010245      0     -Inf         0
    x4               0.134934887027411      0      Inf         0
    x5               4.42046568308383       0      Inf         0
    x6              -7.49324440811497       0     -Inf         0
    x7               5.3359785533005        0      Inf         0
    x8                                      0      NaN        NaN
    x9              -2.41040601483196       0     -Inf         0
    x10              1.41923710436496       0      Inf         0
    x11             -0.272053892444036      0     -Inf         0
Number of observations: 1000, Error degrees of freedom: 989
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF            MeanSq            F      pValue
                   ____________________    ____    ____________________    ____    _______
    Total          23.1321964724073        999     0.0231553518242315
    Model          23.1321964724073         10       2.31321964724073     Inf        0
    Residual                       0        989                      0
A = 0.000000, B = 0.113425
order Q(x) = 4.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 0.999000
Ymin = 0.000000 and Ymax = 3.800201
List of powers for Q(x): [0.113425, 0.226851, 0.340276, 0.453702]
List of powers for D(x): [0.113425, 0.226851, 0.340276, 0.453702, 0.567127, 0.680552,
0.793978]
Fitting atanh(x) in range (0.000000, 0.999000)
MSS of errors squared = 1.454923e-05
Corrected MSS of errors squared = 2.057571e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting atanh(x) in range (0.000000, 0.999000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
```

```
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      0.0242666176446117     0      Inf        0
    x1              -0.0112291326747082     0     -Inf        0
    x2               0.0933804896512938     0      Inf        0
    x3              -0.106417681376289      0     -Inf        0
    x4              -0.103078366820526      0     -Inf        0
    x5                2.42574211412618      0      Inf        0
    x6               -6.33935747706094      0     -Inf        0
    x7                              0      0      NaN      NaN
    x8                5.01669343651155     0      Inf        0
Number of observations: 1000, Error degrees of freedom: 992
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq           F     pValue
                  _________________    ___    ___________________    ___    ______
    Total       23.1321964724077     999    0.023155351824232
    Model       23.1321964724077       7     3.30459949605825    Inf       0
    Residual                   0     992                    0
A = 0.000000, B = 0.100000
order Q(x) = 3.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 0.999000
Ymin = 0.000000 and Ymax = 3.800201
List of powers for Q(x): [0.100000, 0.050000, 0.033333]
List of powers for D(x): [0.100000, 0.050000, 0.033333, 0.025000, 0.020000]
Fitting atanh(x) in range (0.000000, 0.999000)
MSS of errors squared = 8.720928e-04
Corrected MSS of errors squared = 1.233325e-03
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting atanh(x) in range (0.000000, 0.999000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)     0.683803918905653       0      Inf       0
    x1                -3.614886467897       0     -Inf       0
    x2               5.29231513602108       0      Inf       0
    x3              -2.14623983465899       0     -Inf       0
    x4               71.8548104704286       0      Inf       0
    x5              -465.463763270284       0     -Inf       0
```



```
    x6              1269.62258887601    0      Inf       0
    x7              -1826.5156943727    0     -Inf       0
    x8              1457.90581643265    0      Inf       0
    x9             -612.750669891129    0     -Inf       0
    x10             106.131919051446    0      Inf       0
Number of observations: 1000, Error degrees of freedom: 989
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ____________________    ___    ______
    Total       23.1321964722297    999    0.0231553518240537
    Model       23.1321964722297     10      2.31321964722297    Inf       0
    Residual                   0    989                     0
A = 1.337285, B = 0.939214
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 0.999000
Ymin = 0.000000 and Ymax = 3.800201
List of powers for Q(x): [2.276498, 2.665533, 2.964050]
List of powers for D(x): [2.276498, 2.665533, 2.964050, 3.215712, 3.437430, 3.637879,
3.822210]
Fitting atanh(x) in range (0.000000, 0.999000)
MSS of errors squared = 1.510340e-05
Corrected MSS of errors squared = 2.135944e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting atanh(x) in range (0.000000, 0.999000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                    Estimate          SE    tStat    pValue
                ________________    ___    _____    ______
    (Intercept)    0.510330262183141    0      Inf       0
    x1             -253.406892803987    0     -Inf       0
    x2              741.698026136295    0      Inf       0
    x3             -604.508743988744    0     -Inf       0
    x4                             0    0      NaN      NaN
    x5              115.704955027396    0      Inf       0
    x6              882.565645992241    0      Inf       0
    x7             -2245.93106220606    0     -Inf       0
    x8                             0    0      NaN      NaN
    x9              3023.6584493277     0      Inf       0
    x10                            0    0      NaN      NaN
    x11            -3139.74258176266    0     -Inf       0
    x12             1480.45187401807    0      Inf       0
```



```
Number of observations: 1000, Error degrees of freedom: 990
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ___________________    ___    ____________________    ___    ______
    Total       23.1321964725679    999    0.0231553518243923
    Model       23.1321964725679      9      2.57024405250755    Inf       0
    Residual                   0    990                     0
A = 0.511062, B = 0.155714, C = 0.715562
order Q(x) = 5.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 0.999000
Ymin = 0.000000 and Ymax = 3.800201
List of powers for Q(x): [0.715014, 0.767661, 0.811212, 0.849199, 0.883330]
List of powers for D(x): [0.715014, 0.767661, 0.811212, 0.849199, 0.883330, 0.914584,
0.943586]
Fitting atanh(x) in range (0.000000, 0.999000)
MSS of errors squared = 2.260707e-05
Corrected MSS of errors squared = 3.915660e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting atanh(x) in range (0.000000, 0.999000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      0.331608600179186    0      Inf       0
    x1                1.07125518844268    0      Inf       0
    x2               -6.96268726418524    0     -Inf       0
    x3                 11.224503264395    0      Inf       0
    x4               -7.94038647691965    0     -Inf       0
    x5                2.60330269947609    0      Inf       0
    x6              -0.318792931174084    0     -Inf       0
    x7                5.43866219867965    0      Inf       0
    x8               -8.10661718431605    0     -Inf       0
    x9                4.5826753708017    0      Inf       0
    x10             -0.923523464895733    0     -Inf       0
Number of observations: 1000, Error degrees of freedom: 989
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ___________________    ___    ____________________    ___    ______
    Total       23.1321964724065    999    0.0231553518242307
    Model       23.1321964724065     10      2.31321964724065    Inf       0
```



```
    Residual                  0    989                     0
A = 0.407559, B = 0.139388, C = 0.156564
order Q(x) = 6.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 0.999000
Ymin = 0.000000 and Ymax = 3.800201
List of powers for Q(x): [0.546947, 0.842899, 1.138851, 1.434803, 1.730755, 2.026707]
List of powers for D(x): [0.546947, 0.842899, 1.138851, 1.434803]
Fitting atanh(x) in range (0.000000, 0.999000)
MSS of errors squared = 1.359788e-05
Corrected MSS of errors squared = 2.355223e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting atanh(x) in range (0.000000, 0.999000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ___________________    __    _____    ______
    (Intercept)    0.495053802142238     0      Inf       0
    x1             -675.183418511045     0     -Inf       0
    x2              1895.83509514809     0      Inf       0
    x3             -1777.94548257169     0     -Inf       0
    x4              556.796389407101     0      Inf       0
    x5              5060.02237277158     0      Inf       0
    x6             -18346.8230470892     0     -Inf       0
    x7              24969.4239221692     0      Inf       0
    x8             -15113.7015936995     0     -Inf       0
    x9              3432.08070857654     0      Inf       0
Number of observations: 1000, Error degrees of freedom: 990
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF          MeanSq            F     pValue
                   ___________________    ___    ___________________    ___    ______
    Total          23.1321964755057     999    0.023155351827333
    Model          23.1321964755057       9     2.57024405283397    Inf       0
    Residual                      0     990                     0
A = 0.605822, B = 0.208985, C = 0.169455
order Q(x) = 4.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 0.999000
Ymin = 0.000000 and Ymax = 3.800201
List of powers for Q(x): [0.814807, 0.854337, 0.893867, 0.933397]
List of powers for D(x): [0.814807, 0.854337, 0.893867, 0.933397, 0.972927]
Fitting atanh(x) in range (0.000000, 0.999000)
MSS of errors squared = 2.276065e-05
```



```
Corrected MSS of errors squared = 3.942259e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting atanh(x) in range (0.000000, 0.999000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ________________    __    _____    ______
    (Intercept)      0.35643470321705    0      Inf       0
    x1              0.548296982667772    0      Inf       0
    x2              -3.79944903496209    0     -Inf       0
    x3                4.3539047231362    0      Inf       0
    x4              -1.44007444443886    0     -Inf       0
    x5               3.20508128133585    0      Inf       0
    x6              -3.14656145499076    0     -Inf       0
    x7              0.742825234544216    0      Inf       0
    x8              0.179542032878403    0      Inf       0
Number of observations: 1000, Error degrees of freedom: 991
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ___________________    ___    ______
    Total       23.1321964724076     999    0.0231553518242319
    Model       23.1321964724076       8      2.89152455905095    Inf       0
    Residual                   0     991                     0
A = 0.268190, B = 0.184889, C = 0.097741
order Q(x) = 4.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 0.999000
Ymin = 0.000000 and Ymax = 3.800201
List of powers for Q(x): [0.453079, 0.735710, 0.961085, 1.177040]
List of powers for D(x): [0.453079, 0.735710, 0.961085, 1.177040]
Fitting atanh(x) in range (0.000000, 0.999000)
MSS of errors squared = 3.487119e-05
Corrected MSS of errors squared = 6.039868e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting atanh(x) in range (0.000000, 0.999000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
```

```
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                   _________________    __    _____    ______
    (Intercept)    0.406994501631289    0      Inf        0
    x1             -2.94558030815932    0     -Inf        0
    x2              4.01961897448567    0      Inf        0
    x3             -1.47047794519758    0     -Inf        0
    x4              18.4908775945772    0      Inf        0
    x5             -36.2992135073765    0     -Inf        0
    x6                             0    0      NaN      NaN
    x7              65.5507681141733    0      Inf        0
    x8             -77.8917788935115    0     -Inf        0
    x9              38.2716140509933    0      Inf        0
    x10            -7.13282258234949    0     -Inf        0
Number of observations: 1000, Error degrees of freedom: 990
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F      pValue
                 _______________    ___    ____________________    ___    ______
    Total        23.132196472427    999    0.0231553518242512
    Model        23.132196472427      9      2.57024405249189    Inf        0
    Residual                   0    990                     0
A = 0.550000, B = 0.100000, C = 0.151590
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 0.999000
Ymin = 0.000000 and Ymax = 3.800201
List of powers for Q(x): [0.650000, 0.843011, 1.026385]
List of powers for D(x): [0.650000, 0.843011, 1.026385, 1.204770, 1.379967, 1.552899,
1.724116]
Fitting atanh(x) in range (0.000000, 0.999000)
MSS of errors squared = 2.036820e-05
Corrected MSS of errors squared = 3.527876e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting atanh(x) in range (0.000000, 0.999000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                   _________________    __    _____    ______
    (Intercept)    0.479643474507123    0      Inf        0
    x1             -0.912288612374286    0     -Inf        0
    x2              -60.8784914200302    0     -Inf        0
```



```
    x3               246.152848811181    0      Inf       0
    x4              -370.177332376525    0     -Inf       0
    x5               247.104822305122    0      Inf       0
    x6              -61.7798740458327    0     -Inf       0
    x7               41.7838567867761    0      Inf       0
    x8              -165.275396763684    0     -Inf       0
    x9                253.18233421241    0      Inf       0
    x10             -172.781907811574    0     -Inf       0
    x11              44.1017854404277    0      Inf       0
Number of observations: 1000, Error degrees of freedom: 988
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ____________________    ___    ______
    Total       23.1321964723653    999    0.0231553518241895
    Model       23.1321964723653     11      2.10292695203321    Inf       0
    Residual                   0    988                     0
A = 0.229080, B = 0.700000, C = 0.000000
order Q(x) = 6.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 0.999000
Ymin = 0.000000 and Ymax = 3.800201
List of powers for Q(x): [0.929080, 1.219030, 1.441516, 1.629080, 1.794328, 1.943723]
List of powers for D(x): [0.929080, 1.219030, 1.441516, 1.629080, 1.794328]
Fitting atanh(x) in range (0.000000, 0.999000)
MSS of errors squared = 7.596813e-06
Corrected MSS of errors squared = 1.315807e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting atanh(x) in range (0.000000, 0.999000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ________________    ___    _____    ______
    (Intercept)     0.276445783836262    0      Inf        0
    x1               0.13878054342383    0      Inf        0
    x2                              0    0      NaN      NaN
    x3              -3.70090420453072    0     -Inf        0
    x4               5.79577990997132    0      Inf        0
    x5              -3.22350945479142    0     -Inf        0
    x6              0.786708910988819    0      Inf        0
    x7             -0.070890001912352    0     -Inf        0
    x8                206.83563971078    0      Inf        0
    x9              -247.336641498399    0     -Inf        0
```



```
    x10              45.5290004428729    0      Inf        0
    x11             -4.03041014211164    0     -Inf        0
Number of observations: 1000, Error degrees of freedom: 989
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq           F     pValue
                 _________________    ___    _________________    ___    ______
    Total        23.1321964724188    999    0.023155351824243
    Model        23.1321964724188     10     2.31321964724188    Inf       0
    Residual                    0    989                     0
A = 0.521296, B = 2.533307
order Q(x) = 7.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 0.999000
Ymin = 0.000000 and Ymax = 3.800201
List of powers for Q(x): [0.521296, 0.542099, 0.652578, 0.854145, 1.125971, 1.450143,
1.813456]
List of powers for D(x): [0.521296, 0.542099, 0.652578, 0.854145]
Fitting atanh(x) in range (0.000000, 0.999000)
MSS of errors squared = 1.381262e-05
Corrected MSS of errors squared = 1.953400e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the inverse hyperbolic tangent function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i)+C*sqrt(i-1)	0.229	0.7	0	Inf	-Inf	7.59681300E-06	6, 5
A+B*i+C*(i-1)	0.407	0.139	0.156	Inf	-Inf	1.35978800E-05	6, 4
A+B*log10(i)^4	0.521	2.533		Inf	-Inf	1.38126200E-05	7, 4
A+B*i	0	0.113		Inf	-Inf	1.45492300E-05	4, 7
A+B*sqrt(i)	1.337	0.939		Inf	-Inf	1.51034000E-05	3, 7
A+B*sqrt(i)+C*(i-1)	0.55	0.1	0.151	Inf	-Inf	2.03682000E-05	3, 7
A+B*sqrt(i+C)	0.511	0.155	0.715	Inf	-Inf	2.26070700E-05	5, 7
A+B*i-C*(i-1)	0.605	0.208	0.169	Inf	-Inf	2.27606500E-05	4, 5
A+B*i+C*sqrt(i-1)	0.268	0.184	0.097	Inf	-Inf	3.48711900E-05	4, 4
A+B/i	0	0.1		Inf	-Inf	8.72092800E-04	3, 5

RESULTS FOR THE COSINE INTEGRAL

Using Power A+B* i

```
Fitting Ci(x) in range (0.500000, 20.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______
    (Intercept)    0.000150218252099034     0      Inf       0
    x1             -0.00094208899275477     0     -Inf       0
    x2              0.00162339474599667     0      Inf       0
    x3             -0.00110282263748233     0     -Inf       0
    x4             0.000271309968726464     0      Inf       0
    x5                 13.7938548919374     0      Inf       0
    x6                -46.3999112098688     0     -Inf       0
    x7                 76.8837903795216     0      Inf       0
    x8                -73.7910933545573     0     -Inf       0
    x9                 41.8485705771054     0      Inf       0
    x10               -13.0781917263155     0     -Inf       0
    x11                1.74298043225331     0      Inf       0
Number of observations: 196, Error degrees of freedom: 184
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ________________    ___    ____________________    ___    ______
    Total       8.9194231798362     195    0.0457406316914677
    Model       8.9194231798362      11     0.810856652712382     Inf       0
    Residual                  0     184                     0
A = 0.118878, B = 0.218363
order Q(x) = 4.000000, order D(x) = 7.000000
Xmin = 0.500000 and Xmax = 20.000000
Ymin = -0.198391 and Ymax = 0.471733
```



```
List of powers for Q(x): [0.337241, 0.555604, 0.773967, 0.992329]
List of powers for D(x): [0.337241, 0.555604, 0.773967, 0.992329, 1.210692, 1.429055,
1.647418]
Fitting Ci(x) in range (0.500000, 20.000000)
MSS of errors squared = 8.633458e-03
Corrected MSS of errors squared = 1.220955e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting Ci(x) in range (0.500000, 20.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                          Estimate          SE    tStat    pValue
                      ____________________    __    _____    ______
    (Intercept)         1.38309863839841      0      Inf       0
    x1                0.00652622220119813     0      Inf       0
    x2                  -1.39338224015397     0     -Inf       0
    x3                   23.2554771066816     0      Inf       0
    x4                  -102.696336114863     0     -Inf       0
    x5                   161.149610275301     0      Inf       0
    x6                  -81.7049938843445     0     -Inf       0
    x7                -0.00473217641538629    0     -Inf       0
    x8                   1.00889720472947     0      Inf       0
    x9                  -16.8303251503698     0     -Inf       0
    x10                  74.3048670069423     0      Inf       0
    x11                 -116.58120623039      0     -Inf       0
    x12                  59.1024993429784     0      Inf       0
Number of observations: 196, Error degrees of freedom: 183
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F      pValue
                  _____________    ___    ____________________    ___    ______
    Total       8.9194231798359    195    0.0457406316914662
    Model       8.9194231798359     12     0.743285264986325    Inf       0
    Residual                  0    183                      0
A = 0.000000, B = 1.740681
order Q(x) = 6.000000, order D(x) = 6.000000
Xmin = 0.500000 and Xmax = 20.000000
Ymin = -0.198391 and Ymax = 0.471733
List of powers for Q(x): [1.740681, 0.870340, 0.580227, 0.435170, 0.348136, 0.290113]
List of powers for D(x): [1.740681, 0.870340, 0.580227, 0.435170, 0.348136, 0.290113]
Fitting Ci(x) in range (0.500000, 20.000000)
MSS of errors squared = 1.679149e-01
Corrected MSS of errors squared = 2.374675e-01
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting Ci(x) in range (0.500000, 20.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.0122254020931007     0     -Inf       0
    x1                3.86385155870494     0      Inf       0
    x2               -18.8700312549148     0     -Inf       0
    x3                33.2522832711216     0      Inf       0
    x4               -25.3255200252363     0     -Inf       0
    x5                7.09164206742573     0      Inf       0
    x6                2305.90622428303     0      Inf       0
    x7               -20875.7133277798     0     -Inf       0
    x8                71037.3247400908     0      Inf       0
    x9               -121444.581236307     0     -Inf       0
    x10               112283.277721702     0      Inf       0
    x11              -53835.0490324662     0     -Inf       0
    x12               10529.8349102751     0      Inf       0
Number of observations: 196, Error degrees of freedom: 183
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _____________    ___    ____________________    ___    ______
    Total        8.91942318092911    195    0.0457406316970723
    Model        8.91942318092911     12     0.743285265077426    Inf       0
    Residual                    0    183                     0
A = 0.708254, B = 0.404335
order Q(x) = 5.000000, order D(x) = 7.000000
Xmin = 0.500000 and Xmax = 20.000000
Ymin = -0.198391 and Ymax = 0.471733
List of powers for Q(x): [1.112589, 1.280070, 1.408583, 1.516924, 1.612375]
List of powers for D(x): [1.112589, 1.280070, 1.408583, 1.516924, 1.612375, 1.698669,
1.778024]
Fitting Ci(x) in range (0.500000, 20.000000)
MSS of errors squared = 1.592395e-01
Corrected MSS of errors squared = 2.251987e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```


Using Power A+B*sqrt(i+C)

```
Fitting Ci(x) in range (0.500000, 20.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ________________    __    _____    ______
    (Intercept)      1.28731121410294    0      Inf        0
    x1              -43.1261201527084    0     -Inf        0
    x2               147.071231315354    0      Inf        0
    x3              -157.244026230579    0     -Inf        0
    x4                             0    0      NaN      NaN
    x5               91.0773100488044    0      Inf        0
    x6              -39.0657061944411    0     -Inf        0
    x7               33.4991196670267    0      Inf        0
    x8              -114.238462122167    0     -Inf        0
    x9               122.138361679946    0      Inf        0
    x10                            0    0      NaN      NaN
    x11             -70.7419864351301    0     -Inf        0
    x12              30.3429672099033    0      Inf        0
Number of observations: 196, Error degrees of freedom: 185
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq          DF            MeanSq             F      pValue
                    _______________    ___    ___________________    ___    ______
    Total       8.9194231798369    195    0.0457406316914713
    Model       8.9194231798369     10      0.89194231798369    Inf       0
    Residual                  0    185                     0
A = 0.000000, B = 0.100000, C = 0.214800
order Q(x) = 6.000000, order D(x) = 6.000000
Xmin = 0.500000 and Xmax = 20.000000
Ymin = -0.198391 and Ymax = 0.471733
List of powers for Q(x): [0.110218, 0.148822, 0.179299, 0.205300, 0.228359, 0.249295]
List of powers for D(x): [0.110218, 0.148822, 0.179299, 0.205300, 0.228359, 0.249295]
Fitting Ci(x) in range (0.500000, 20.000000)
MSS of errors squared = 1.231862e-01
Corrected MSS of errors squared = 2.133648e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting Ci(x) in range (0.500000, 20.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
```


	Estimate	SE	tStat	pValue
(Intercept)	0.00028579837617683	0	Inf	0
x1	-0.000580512370304626	0	-Inf	0
x2	0.000416187715608287	0	Inf	0
x3	-0.000121392062080807	0	-Inf	0
x4	3.07805172570759	0	Inf	0
x5	-5.26859053482203	0	-Inf	0
x6	5.86207980074628	0	Inf	0
x7	-3.81822587568989	0	-Inf	0
x8	1.34552073171401	0	Inf	0
x9	-0.19883591858522	0	-Inf	0

Number of observations: 196, Error degrees of freedom: 186
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	8.91942317983624	195	0.0457406316914679		
Model	8.91942317983624	9	0.991047019981804	Inf	0
Residual	0	186	0		

A = 0.057589, B = 0.121910, C = 0.171768
order Q(x) = 3.000000, order D(x) = 6.000000
Xmin = 0.500000 and Xmax = 20.000000
Ymin = -0.198391 and Ymax = 0.471733
List of powers for Q(x): [0.179499, 0.473177, 0.766855]
List of powers for D(x): [0.179499, 0.473177, 0.766855, 1.060533, 1.354211, 1.647889]
Fitting Ci(x) in range (0.500000, 20.000000)
MSS of errors squared = 7.661730e-02
Corrected MSS of errors squared = 1.327051e-01
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*i–C*(i–1)

Fitting Ci(x) in range (0.500000, 20.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	-0.0103454531078278	0	-Inf	0
x1	1.92535486250924	0	Inf	0
x2	-6.37639768563944	0	-Inf	0
x3	8.0696090833262	0	Inf	0
x4	-4.60508967500876	0	-Inf	0
x5	0.996868932579321	0	Inf	0
x6	821.243350990723	0	Inf	0


```
    x7               -4049.50518647566    0      -Inf       0
    x8                 8477.6651662924    0       Inf       0
    x9               -9603.13691915613    0      -Inf       0
    x10                6189.0931782741    0       Inf       0
    x11              -2147.08929590105    0      -Inf       0
    x12               312.729705915223    0       Inf       0
Number of observations: 196, Error degrees of freedom: 183
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                 _______________    ___    ____________________    ___    _______
    Total        8.91942317983288    195    0.0457406316914507
    Model        8.91942317983288     12     0.743285264986073    Inf        0
    Residual                    0    183                     0
A = 0.704922, B = 0.144502, C = 0.009268
order Q(x) = 5.000000, order D(x) = 7.000000
Xmin = 0.500000 and Xmax = 20.000000
Ymin = -0.198391 and Ymax = 0.471733
List of powers for Q(x): [0.849424, 0.984658, 1.119892, 1.255126, 1.390360]
List of powers for D(x): [0.849424, 0.984658, 1.119892, 1.255126, 1.390360, 1.525594,
1.660828]
Fitting Ci(x) in range (0.500000, 20.000000)
MSS of errors squared = 8.325487e-02
Corrected MSS of errors squared = 1.442017e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting Ci(x) in range (0.500000, 20.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    0.000214139735753486    0      Inf       0
    x1            -0.000925850737963251    0     -Inf       0
    x2             0.00157720375378835     0      Inf       0
    x3            -0.00116882317173229     0     -Inf       0
    x4             0.000303350967565301    0      Inf       0
    x5                8.41679770031617     0      Inf       0
    x6               -31.9903864712254     0     -Inf       0
    x7                59.8198675456542     0      Inf       0
    x8               -61.8673010244787     0     -Inf       0
    x9                37.0119082658171     0      Inf       0
    x10              -12.050446343041      0     -Inf       0
    x11               1.65956030898903     0      Inf       0
```



```
Number of observations: 196, Error degrees of freedom: 184
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ________________    ___    ________________    ___    ______
    Total       8.91942317983559    195    0.0457406316914645
    Model       8.91942317983559     11     0.810856652712326    Inf       0
    Residual                   0    184                     0
A = 0.109107, B = 0.214923, C = 0.082519
order Q(x) = 4.000000, order D(x) = 7.000000
Xmin = 0.500000 and Xmax = 20.000000
Ymin = -0.198391 and Ymax = 0.471733
List of powers for Q(x): [0.324030, 0.621472, 0.870576, 1.111726]
List of powers for D(x): [0.324030, 0.621472, 0.870576, 1.111726, 1.348760, 1.583163,
1.815698]
Fitting Ci(x) in range (0.500000, 20.000000)
MSS of errors squared = 9.559900e-03
Corrected MSS of errors squared = 1.655823e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting Ci(x) in range (0.500000, 20.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.0134574694176418     0     -Inf       0
    x1               0.466565705059716     0      Inf       0
    x2              -1.62486673722216      0     -Inf       0
    x3                2.2310944101527      0      Inf       0
    x4              -1.38332293618847      0     -Inf       0
    x5               0.323987122307274     0      Inf       0
    x6                95.5025838841527     0      Inf       0
    x7              -553.000217359779      0     -Inf       0
    x8               1352.06832084044      0      Inf       0
    x9              -1759.78397243024      0     -Inf       0
    x10              1284.00948386644      0      Inf       0
    x11             -498.088272510347      0     -Inf       0
    x12              80.2920736208645      0      Inf       0
Number of observations: 196, Error degrees of freedom: 183
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ________________    ___    ________________    ___    ______
```



```
    Total       8.91942317982896    195    0.0457406316914305
    Model       8.91942317982896     12     0.743285264985746    Inf       0
    Residual                   0    183                      0
A = 0.366817, B = 0.374101, C = 0.090243
order Q(x) = 5.000000, order D(x) = 7.000000
Xmin = 0.500000 and Xmax = 20.000000
Ymin = -0.198391 and Ymax = 0.471733
List of powers for Q(x): [0.740918, 0.986119, 1.195266, 1.385749, 1.564306]
List of powers for D(x): [0.740918, 0.986119, 1.195266, 1.385749, 1.564306, 1.734390,
1.898055]
Fitting Ci(x) in range (0.500000, 20.000000)
MSS of errors squared = 5.136898e-02
Corrected MSS of errors squared = 8.897368e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting Ci(x) in range (0.500000, 20.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      0.00118107458592587    0      Inf       0
    x1              -0.00273335610466033    0     -Inf       0
    x2               0.0048190363972401     0      Inf       0
    x3              -0.00493841369947316    0     -Inf       0
    x4               0.0016723682932547     0      Inf       0
    x5                 3.92311535062814     0      Inf       0
    x6                -30.3756500739512     0     -Inf       0
    x7                 102.884507528547     0      Inf       0
    x8                -169.695318085211     0     -Inf       0
    x9                 150.284525166131     0      Inf       0
    x10               -68.9200718693471     0     -Inf       0
    x11                12.8988913674971     0      Inf       0
Number of observations: 196, Error degrees of freedom: 184
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                ____________________    ___    ____________________    ___    ______
    Total       8.91942317983461    195    0.0457406316914595
    Model       8.91942317983461     11     0.810856652712237    Inf       0
    Residual                   0    184                      0
A = 0.000000, B = 0.431981, C = 0.659036
order Q(x) = 4.000000, order D(x) = 7.000000
Xmin = 0.500000 and Xmax = 20.000000
```



```
Ymin = -0.198391 and Ymax = 0.471733
List of powers for Q(x): [0.431981, 1.269949, 1.680231, 2.005446]
List of powers for D(x): [0.431981, 1.269949, 1.680231, 2.005446, 2.284011, 2.531782,
2.757216]
Fitting Ci(x) in range (0.500000, 20.000000)
MSS of errors squared = 1.849966e-01
Corrected MSS of errors squared = 3.204235e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting Ci(x) in range (0.500000, 20.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)                        0    0      NaN      NaN
    x1              -8.14383640314868e-06    0     -Inf        0
    x2               1.1016305309411e-05     0      Inf        0
    x3              -3.96496021347462e-06    0     -Inf        0
    x4               1.43228433197774e-06    0      Inf        0
    x5              -4.09255689705583e-07    0     -Inf        0
    x6               7.61646676118055e-08    0      Inf        0
    x7                 -6.70200222171e-09    0     -Inf        0
    x8                  1.00000489788838     0      Inf        0
    x9               -6.6254485185833e-06    0     -Inf        0
    x10              2.38460317490174e-06    0      Inf        0
    x11             -8.61402335353293e-07    0     -Inf        0
    x12              2.46137401468081e-07    0      Inf        0
    x13             -4.58093454646286e-08    0     -Inf        0
    x14              4.03124228705485e-09    0      Inf        0
Number of observations: 196, Error degrees of freedom: 182
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                 _______________    ___    ___________________    ___    ______
    Total        8.91942317983627    195    0.0457406316914681
    Model        8.91942317983627     13     0.686109475372021    Inf        0
    Residual                    0    182                     0
A = 0.000000, B = 3.148377
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 0.500000 and Xmax = 20.000000
Ymin = -0.198391 and Ymax = 0.471733
List of powers for Q(x): [0.000000, 0.025854, 0.163156, 0.413662, 0.751486, 1.154364,
1.605887]
```



```
List of powers for D(x): [0.000000, 0.025854, 0.163156, 0.413662, 0.751486, 1.154364,
1.605887]
Fitting Ci(x) in range (0.500000, 20.000000)
MSS of errors squared = 4.168351e-01
Corrected MSS of errors squared = 5.894938e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the cosine integral. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i	0.118	0.218		Inf	-Inf	8.63345800E-03	4, 7
A+B*i+C*sqrt(i-1)	0.109	0.214	0.082	Inf	-Inf	9.55990000E-03	4, 7
A+B*sqrt(i)+C*(i-1)	0.366	0.374	0.09	Inf	-Inf	5.13689800E-02	5, 7
A+B*i+C*(i-1)	0.057	0.121	0.171	Inf	-Inf	7.66173000E-02	3, 6
A+B*i-C*(i-1)	0.704	0.144	0.009	Inf	-Inf	8.32548700E-02	5, 7
A+B*sqrt(i+C)	0	0.1	0.214	Inf	-Inf	1.23186200E-01	6, 6
A+B*sqrt(i)	0.708	0.404		Inf	-Inf	1.59239500E-01	5, 7
A+B/i	0	1.74		Inf	-Inf	1.67914900E-01	6, 6
A+B*sqrt(i)+C*sqrt(i-1)	0	0.431	0.659	Inf	-Inf	1.84996600E-01	4, 7
A+B*log10(i)^4	0	3.148		Inf	-Inf	4.16835100E-01	7, 7

RESULTS FOR THE COSINE FUNCTION

Using Power A+B* i

```
Fitting cos(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ___________________    __    _____    _______
    (Intercept)     0.982963740408589      0      Inf        0
    x1              -2.45155466476948      0     -Inf        0
    x2                2.5302607723644      0      Inf        0
    x3                             0       0      NaN       NaN
    x4              -2.45593681592334      0     -Inf        0
    x5               1.95597263622848      0      Inf        0
    x6              -0.469257322895362     0     -Inf        0
    x7               2.54621224006473      0      Inf        0
    x8              -2.08270216634506      0     -Inf        0
    x9                             0       0      NaN       NaN
    x10             0.891290130481046      0      Inf        0
    x11             -0.487797172269106     0     -Inf        0
    x12             0.0867727953626479     0      Inf        0
Number of observations: 501, Error degrees of freedom: 490
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF          MeanSq            F      pValue
                   _______________    ___    _________________    ___    _______
    Total          47.6265936125815    500    0.095253187225163
    Model          47.6265936125815     10     4.76265936125815    Inf        0
    Residual                      0    490                    0
A = 0.070436, B = 0.366945
order Q(x) = 6.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 1.570796
Ymin = 0.000000 and Ymax = 1.000000
```



```
List of powers for Q(x): [0.437380, 0.804325, 1.171270, 1.538214, 1.905159, 2.272103]
List of powers for D(x): [0.437380, 0.804325, 1.171270, 1.538214, 1.905159, 2.272103]
Fitting cos(x) in range (0.000000, 1.570796)
MSS of errors squared = 1.227704e-12
Corrected MSS of errors squared = 1.736236e-12
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting cos(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)       1.18753885264133    0      Inf       0
    x1               -0.563180425551127   0     -Inf       0
    x2                 32.5867273115904    0      Inf       0
    x3                -107.525226236019    0     -Inf       0
    x4                 76.0985676639761    0      Inf       0
    x5                0.107574530021214    0      Inf       0
    x6               -0.992682338802699    0     -Inf       0
    x7                0.99289436299892     0      Inf       0
Number of observations: 501, Error degrees of freedom: 493
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _________________    ___    _________________    ___    ______
    Total        47.6265936125891     500    0.0952531872251783
    Model        47.6265936125891       7      6.80379908751273    Inf       0
    Residual                   0     493                     0
A = 1.824661, B = 1.232817
order Q(x) = 4.000000, order D(x) = 3.000000
Xmin = 0.000000 and Xmax = 1.570796
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [3.057478, 2.441070, 2.235600, 2.132866]
List of powers for D(x): [3.057478, 2.441070, 2.235600]
Fitting cos(x) in range (0.000000, 1.570796)
MSS of errors squared = 3.375548e-09
Corrected MSS of errors squared = 4.773745e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)


```
Fitting cos(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    _________________    __    _____    ______
    (Intercept)      1.04467973323715    0      Inf       0
    x1               5.73264998409857    0      Inf       0
    x2              -11.6527192938996    0     -Inf       0
    x3               5.81145332739985    0      Inf       0
    x4               27.1379518298833    0      Inf       0
    x5              -105.766171069479    0     -Inf       0
    x6               166.636947828539    0      Inf       0
    x7              -128.259811389618    0     -Inf       0
    x8               47.2646966219516    0      Inf       0
    x9              -6.48164569655013    0     -Inf       0
Number of observations: 501, Error degrees of freedom: 491
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    _________________    ___    ______
    Total        47.6265936125826    500    0.0952531872251653
    Model        47.6265936125826      9       5.2918437347314    Inf       0
    Residual                    0    491                     0
A = 1.047673, B = 0.579109
order Q(x) = 3.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 1.570796
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [1.626782, 1.866657, 2.050720]
List of powers for D(x): [1.626782, 1.866657, 2.050720, 2.205891, 2.342601, 2.466195]
Fitting cos(x) in range (0.000000, 1.570796)
MSS of errors squared = 8.477255e-12
Corrected MSS of errors squared = 1.198865e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting cos(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    _________________    __    _____    ______
    (Intercept)      1.10347167169334    0      Inf       0
    x1              -55.2668717993673    0     -Inf       0
```

```
    x2                96.591090395466    0      Inf        0
    x3                              0    0      NaN      NaN
    x4               -53.6274605244725    0     -Inf        0
    x5                              0    0      NaN      NaN
    x6                              0    0      NaN      NaN
    x7                11.2020878408351    0      Inf        0
    x8                77.9552526824511    0      Inf        0
    x9               -222.843492840111    0     -Inf        0
    x10               217.510215896272    0      Inf        0
    x11              -71.6231345306879    0     -Inf        0
Number of observations: 501, Error degrees of freedom: 492
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F     pValue
                  _________________    ___    ____________________    ___    _______
    Total        47.6265936126003    500    0.0952531872252007
    Model        47.6265936126003      8      5.95332420157504    Inf        0
    Residual                    0    492                     0
A = 0.020303, B = 0.116675, C = 1.544045
order Q(x) = 7.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 1.570796
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.206401, 0.239952, 0.269017, 0.295024, 0.318774, 0.340768,
0.361347]
List of powers for D(x): [0.206401, 0.239952, 0.269017, 0.295024]
Fitting cos(x) in range (0.000000, 1.570796)
MSS of errors squared = 2.031367e-11
Corrected MSS of errors squared = 3.518431e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting cos(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                   _____________________    __    _____    _______
    (Intercept)     0.847278728369514    0      Inf        0
    x1              -2.11136200724792    0     -Inf        0
    x2                              0    0      NaN      NaN
    x3               3.18911567748149    0      Inf        0
    x4                              0    0      NaN      NaN
    x5              -5.16653766427774    0     -Inf        0
    x6                  4.38004980051    0      Inf        0
    x7              -1.11389558236601    0     -Inf        0
```


x8	3.4729624106838	0	Inf	0
x9	-3.33054469077486	0	-Inf	0
x10	0	0	NaN	NaN
x11	1.1380796835872	0	Inf	0
x12	0	0	NaN	NaN
x13	-0.420036470844297	0	-Inf	0
x14	0.127214591113475	0	Inf	0

Number of observations: 501, Error degrees of freedom: 490
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	47.6265936125817	500	0.0952531872251634		
Model	47.6265936125817	10	4.76265936125817	Inf	0
Residual	0	490	0		

A = 0.139832, B = 0.148746, C = 0.062133
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 1.570796
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.288577, 0.499456, 0.710334, 0.921213, 1.132092, 1.342970, 1.553849]
List of powers for D(x): [0.288577, 0.499456, 0.710334, 0.921213, 1.132092, 1.342970, 1.553849]
Fitting cos(x) in range (0.000000, 1.570796)
MSS of errors squared = 1.184416e-13
Corrected MSS of errors squared = 2.051469e-13
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*i–C*(i–1)

Fitting cos(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	0.00896661353596707	0	Inf	0
x1	0.000948451517537524	0	Inf	0
x2	-0.0063121728903084	0	-Inf	0
x3	-0.00360358535710986	0	-Inf	0
x4	-0.0304273686419809	0	-Inf	0
x5	0.427204918281692	0	Inf	0
x6	0.760158454451709	0	Inf	0
x7	-0.174472057177324	0	-Inf	0
x8	0	0	NaN	NaN
x9	0.0238366162598088	0	Inf	0


```
    x10            -0.00630021657694628    0     -Inf        0
Number of observations: 501, Error degrees of freedom: 491
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ________________    ___    ____________________    ___    ______
    Total       47.626593612582    500    0.0952531872251639
    Model       47.626593612582      9      5.29184373473133    Inf       0
    Residual                  0    491                        0
A = 0.055193, B = 0.113710, C = 0.225902
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 1.570796
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.168903, 0.056711, -0.055482]
List of powers for D(x): [0.168903, 0.056711, -0.055482, -0.167674, -0.279867, -
0.392059, -0.504251]
Fitting cos(x) in range (0.000000, 1.570796)
MSS of errors squared = 2.776221e-09
Corrected MSS of errors squared = 4.808555e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting cos(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______
    (Intercept)      0.961651712344707    0      Inf        0
    x1               -1.84115966208247    0     -Inf        0
    x2                2.65377864482299    0      Inf        0
    x3               -2.33855032460169    0     -Inf        0
    x4                               0    0      NaN      NaN
    x5               0.954846482705962    0      Inf        0
    x6              -0.341639558187287    0     -Inf        0
    x7                1.97258533722741    0      Inf        0
    x8               -2.51230549089329    0     -Inf        0
    x9                2.41302503109903    0      Inf        0
    x10              -1.09075575834588    0     -Inf        0
    x11              0.192987233411695    0      Inf        0
Number of observations: 501, Error degrees of freedom: 490
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ________________    ___    ____________________    ___    ______
```



```
    Total       47.6265936125817    500    0.0952531872251634
    Model       47.6265936125817     10      4.76265936125817    Inf       0
    Residual                   0    490                     0
A = 0.102882, B = 0.184528, C = 0.344362
order Q(x) = 6.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 1.570796
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.287410, 0.816300, 1.143467, 1.437445, 1.714244, 1.980065]
List of powers for D(x): [0.287410, 0.816300, 1.143467, 1.437445, 1.714244]
Fitting cos(x) in range (0.000000, 1.570796)
MSS of errors squared = 2.434740e-13
Corrected MSS of errors squared = 4.217093e-13
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting cos(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)    -0.150593115263403    0     -Inf       0
    x1              -3.62977338050728    0     -Inf       0
    x2               4.28352365059376    0      Inf       0
    x3              -1.27898623352147    0     -Inf       0
    x4               10.4462579412638    0      Inf       0
    x5              -27.6045087792201    0     -Inf       0
    x6               37.3665151755566    0      Inf       0
    x7              -29.4452281810684    0     -Inf       0
    x8               13.8489256741935    0      Inf       0
    x9              -3.63885996115397    0     -Inf       0
    x10             0.414812669733466    0      Inf       0
Number of observations: 501, Error degrees of freedom: 490
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F     pValue
                    ____________________    ___    ____________________    ___    _____
    Total       47.6265936125808    500    0.0952531872251617
    Model       47.6265936125808     10      4.76265936125808    Inf       0
    Residual                   0    490                     0
A = 0.066206, B = 0.370518, C = 0.389512
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 1.570796
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.436724, 0.979709, 1.486985]
```



```
List of powers for D(x): [0.436724, 0.979709, 1.486985, 1.975776, 2.452755, 2.921343,
3.383573]
Fitting cos(x) in range (0.000000, 1.570796)
MSS of errors squared = 2.424161e-12
Corrected MSS of errors squared = 4.198770e-12
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting cos(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)       1.04304276356295      0      Inf         0
    x1               -3.53363925472476      0     -Inf         0
    x2                5.95248104898399      0      Inf         0
    x3                              0      0      NaN       NaN
    x4               -7.07088411706922      0     -Inf         0
    x5                3.8049163450253       0      Inf         0
    x6                2.94067866457116      0      Inf         0
    x7               -3.11038049631723      0     -Inf         0
    x8              -0.705423015651399      0     -Inf         0
    x9                3.06672102285882      0      Inf         0
    x10              -1.28955456833357      0     -Inf         0
Number of observations: 501, Error degrees of freedom: 491
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F      pValue
                  _________________    ___    ____________________    ___    _______
    Total       47.6265936125815     500    0.0952531872251629
    Model       47.6265936125815       9      5.29184373473128     Inf        0
    Residual                    0     491                     0
A = 0.006607, B = 0.486387, C = 0.136795
order Q(x) = 5.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 1.570796
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.492993, 0.831256, 1.042510, 1.216316, 1.367790]
List of powers for D(x): [0.492993, 0.831256, 1.042510, 1.216316, 1.367790]
Fitting cos(x) in range (0.000000, 1.570796)
MSS of errors squared = 6.681315e-12
Corrected MSS of errors squared = 1.157238e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
```



```
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting cos(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)     0.313027594702882    0      Inf        0
    x1              -10.7061222772281    0     -Inf        0
    x2                             0     0      NaN      NaN
    x3               15.2346209266164    0      Inf        0
    x4              -4.87200754587634    0     -Inf        0
    x5               138.813918260507    0      Inf        0
    x6                             0     0      NaN      NaN
    x7              -287.363033088405    0     -Inf        0
    x8               208.179077628992    0      Inf        0
    x9              -67.3823967765561    0     -Inf        0
    x10              8.76767462603658    0      Inf        0
Number of observations: 501, Error degrees of freedom: 492
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F      pValue
                   _________________    ___    ____________________    ___    ______
    Total       47.6265936125785    500    0.0952531872251569
    Model       47.6265936125785      8      5.95332420157231    Inf        0
    Residual                   0    492                      0
A = 0.591517, B = 1.496362
order Q(x) = 4.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 1.570796
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.591517, 0.603805, 0.669062, 0.788123]
List of powers for D(x): [0.591517, 0.603805, 0.669062, 0.788123, 0.948684, 1.140164]
Fitting cos(x) in range (0.000000, 1.570796)
MSS of errors squared = 5.363906e-10
Corrected MSS of errors squared = 7.585708e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the cosine function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i+C*(i-1)	0.139	0.148	0.062	Inf	-Inf	1.18441600E-13	7, 7
A+B*i+C*sqrt(i-1)	0.102	0.184	0.344	Inf	-Inf	2.43474000E-13	6, 5
A+B*i	0.07	0.366		Inf	-Inf	1.22770400E-12	6, 6
A+B*sqrt(i)+C*(i-1)	0.066	0.37	0.389	Inf	-Inf	2.42416100E-12	3, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.006	0.486	0.136	Inf	-Inf	6.68131500E-12	5, 5
A+B*sqrt(i)	1.047	0.579		Inf	-Inf	8.47725500E-12	3, 6
A+B*sqrt(i+C)	0.02	0.116	1.544	Inf	-Inf	2.03136700E-11	7, 4
A+B*log10(i)^4	0.591	1.496		Inf	-Inf	5.36390600E-10	4, 6
A+B*i-C*(i-1)	0.055	0.113	0.225	Inf	-Inf	2.77622100E-09	3, 7
A+B/i	1.824	1.232		Inf	-Inf	3.37554800E-09	4, 3

RESULTS FOR THE HYPERBOLIC COSINE FUNCTION

Using Power A+B* i

```
Fitting cosh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)    -0.188071377737151       0     -Inf        0
    x1              0.637841600037003       0      Inf        0
    x2             -0.401246136070954       0     -Inf        0
    x3                             0       0      NaN       NaN
    x4                             0       0      NaN       NaN
    x5             -0.547788656101327       0     -Inf        0
    x6              0.754639332744976       0      Inf        0
    x7             -0.260031374645091       0     -Inf        0
    x8               29.0273384159283       0      Inf        0
    x9              -95.3131660170789       0     -Inf        0
    x10              135.283898598952       0      Inf        0
    x11             -100.213619087238       0     -Inf        0
    x12              38.1322798469663       0      Inf        0
    x13             -5.91207514576711       0     -Inf        0
Number of observations: 501, Error degrees of freedom: 489
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq            DF            MeanSq             F      pValue
                ________________    ____    ____________________    ___    _______
    Total       31.2187326156319     500    0.0624374652312638
    Model       31.2187326156319      11      2.83806660142108    Inf        0
    Residual                   0     489                     0
A = 0.307886, B = 0.227368
order Q(x) = 7.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 5.000000
Ymin = 1.000000 and Ymax = 74.209949
```



```
List of powers for Q(x): [0.535254, 0.762621, 0.989989, 1.217357, 1.444725, 1.672093,
1.899461]
List of powers for D(x): [0.535254, 0.762621, 0.989989, 1.217357, 1.444725, 1.672093]
Fitting cosh(x) in range (0.000000, 5.000000)
MSS of errors squared = 1.113335e-10
Corrected MSS of errors squared = 1.574493e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting cosh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ________________    ___    _____    ______
    (Intercept)    0.620427827617133    0       Inf        0
    x1              664.718272593866    0       Inf        0
    x2             -7799.9311272593     0      -Inf        0
    x3              13596.9077235905    0       Inf        0
    x4                             0    0       NaN      NaN
    x5             -6462.33107830643    0      -Inf        0
    x6             -1101.38389533779    0      -Inf        0
    x7              12826.4767580983    0       Inf        0
    x8             -22306.5823778855    0      -Inf        0
    x9                             0    0       NaN      NaN
    x10             10582.5052966789    0       Inf        0
Number of observations: 501, Error degrees of freedom: 492
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _______________    ___    ___________________    ___    ______
    Total        31.2187326157495    500    0.0624374652314989
    Model        31.2187326157495      8      3.90234157696868    Inf       0
    Residual                    0    492                     0
A = 0.680463, B = 0.100000
order Q(x) = 5.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 5.000000
Ymin = 1.000000 and Ymax = 74.209949
List of powers for Q(x): [0.780463, 0.730463, 0.713796, 0.705463, 0.700463]
List of powers for D(x): [0.780463, 0.730463, 0.713796, 0.705463, 0.700463]
Fitting cosh(x) in range (0.000000, 5.000000)
MSS of errors squared = 1.060293e-08
Corrected MSS of errors squared = 1.499481e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
```



```
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting cosh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)       3.20264697181036      0      Inf        0
    x1               -59.2788291138941      0     -Inf        0
    x2                230.876550261597      0      Inf        0
    x3               -305.96670466606       0     -Inf        0
    x4                             0        0      NaN       NaN
    x5                354.650756166849      0      Inf        0
    x6               -304.811551305649      0     -Inf        0
    x7                81.6037347803421      0      Inf        0
    x8               0.400424593777266      0      Inf        0
    x9                 1.8168519269391      0      Inf        0
    x10                            0        0      NaN       NaN
    x11              -2.91511019561726      0     -Inf        0
    x12                            0        0      NaN       NaN
    x13                2.4126174049082      0      Inf        0
    x14              -0.991386824810566     0     -Inf        0
Number of observations: 501, Error degrees of freedom: 489
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F     pValue
                  _________________    ___    ___________________    ___    _______
    Total        31.2187326156875     500    0.0624374652313751
    Model        31.2187326156875      11      2.83806660142614    Inf        0
    Residual                    0     489                     0
A = 0.000000, B = 1.000000
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 5.000000
Ymin = 1.000000 and Ymax = 74.209949
List of powers for Q(x): [1.000000, 1.414214, 1.732051, 2.000000, 2.236068, 2.449490,
2.645751]
List of powers for D(x): [1.000000, 1.414214, 1.732051, 2.000000, 2.236068, 2.449490,
2.645751]
Fitting cosh(x) in range (0.000000, 5.000000)
MSS of errors squared = 2.337495e-11
Corrected MSS of errors squared = 3.305718e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
```



```
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting cosh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    ________________    ___    _____    ______
    (Intercept)    0.735544580513032    0       Inf        0
    x1             -17.6068338004418    0      -Inf        0
    x2               42.364066523423    0       Inf        0
    x3                             0    0       NaN      NaN
    x4             -87.7405154579984    0      -Inf        0
    x5              89.6758597105484    0       Inf        0
    x6             -27.4303644578877    0      -Inf        0
    x7                21.45819531631    0       Inf        0
    x8             -46.7103685355117    0      -Inf        0
    x9                             0    0       NaN      NaN
    x10             54.8760061893888    0       Inf        0
    x11                            0    0       NaN      NaN
    x12            -52.8438904428021    0      -Inf        0
    x13             24.2223003743756    0       Inf        0
Number of observations: 501, Error degrees of freedom: 490
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F      pValue
                  _______________    ___    ___________________    ___    ______
    Total        31.2187326156187    500    0.0624374652312374
    Model        31.2187326156187     10      3.12187326156187    Inf       0
    Residual                    0    490                     0
A = 0.054612, B = 0.437102, C = 0.036009
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 5.000000
Ymin = 1.000000 and Ymax = 74.209949
List of powers for Q(x): [0.499515, 0.678308, 0.816226, 0.932743, 1.035516, 1.128498]
List of powers for D(x): [0.499515, 0.678308, 0.816226, 0.932743, 1.035516, 1.128498,
1.214047]
Fitting cosh(x) in range (0.000000, 5.000000)
MSS of errors squared = 2.996038e-07
Corrected MSS of errors squared = 5.189289e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting cosh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
```



```
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.0882021510957782     0     -Inf       0
    x1               0.156910601598169     0      Inf       0
    x2                             0     0      NaN      NaN
    x3              0.0876980616373646     0      Inf       0
    x4                             0     0      NaN      NaN
    x5              -0.716925590774788     0     -Inf       0
    x6               0.813581684018114     0      Inf       0
    x7              -0.254219652080519     0     -Inf       0
    x8                14.6151122620733     0      Inf       0
    x9               -36.4623866843871     0     -Inf       0
    x10               31.8519504827159     0      Inf       0
    x11                            0     0      NaN      NaN
    x12              -18.1192649954024     0     -Inf       0
    x13               11.3909390345753     0      Inf       0
    x14              -2.27519305288026     0     -Inf       0
Number of observations: 501, Error degrees of freedom: 489
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _________________    ___    ____________________    ___    ______
    Total        31.2187326156295     500    0.0624374652312589
    Model        31.2187326156295      11      2.83806660142086    Inf       0
    Residual                    0     489                     0
A = 0.231395, B = 0.131040, C = 0.058420
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 5.000000
Ymin = 1.000000 and Ymax = 74.209949
List of powers for Q(x): [0.362435, 0.551894, 0.741353, 0.930813, 1.120272, 1.309731,
1.499191]
List of powers for D(x): [0.362435, 0.551894, 0.741353, 0.930813, 1.120272, 1.309731,
1.499191]
Fitting cosh(x) in range (0.000000, 5.000000)
MSS of errors squared = 1.268798e-10
Corrected MSS of errors squared = 2.197623e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting cosh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
```



```
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ________________    __    _____    ______
    (Intercept)    0.641458569354219    0      Inf        0
    x1             -1066.35883592321    0     -Inf        0
    x2              1912.04691449359    0      Inf        0
    x3                             0    0      NaN      NaN
    x4             -1169.10290890926    0     -Inf        0
    x5                             0    0      NaN      NaN
    x6              322.768254253796    0      Inf        0
    x7              1461.62559250896    0      Inf        0
    x8             -2521.82991778886    0     -Inf        0
    x9                             0    0      NaN      NaN
    x10             1291.34279367693    0      Inf        0
    x11                            0    0      NaN      NaN
    x12                            0    0      NaN      NaN
    x13            -230.133350881289    0     -Inf        0
Number of observations: 501, Error degrees of freedom: 492
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ________________    ___    ______
    Total       31.2187326155167    500    0.0624374652310333
    Model       31.2187326155167      8      3.90234157693958    Inf       0
    Residual                   0    492                      0
A = 0.375966, B = 0.128686, C = 0.110632
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 5.000000
Ymin = 1.000000 and Ymax = 74.209949
List of powers for Q(x): [0.504651, 0.522705, 0.540759, 0.558813, 0.576867, 0.594920]
List of powers for D(x): [0.504651, 0.522705, 0.540759, 0.558813, 0.576867, 0.594920,
0.612974]
Fitting cosh(x) in range (0.000000, 5.000000)
MSS of errors squared = 1.064603e-08
Corrected MSS of errors squared = 1.843946e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting cosh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ________________    __    _____    ______
    (Intercept)    6.74014590001155    0      Inf        0
```



```
    x1              -40.6791982936159    0     -Inf        0
    x2               141.127604618088    0      Inf        0
    x3              -172.028527830987    0     -Inf        0
    x4                             0    0      NaN      NaN
    x5               151.933623934136    0      Inf        0
    x6              -113.689452750444    0     -Inf        0
    x7               26.6205151191206    0      Inf        0
    x8                2.6626332160977    0      Inf        0
    x9              -3.48332454070738    0     -Inf        0
    x10              2.29636956757893    0      Inf        0
    x11            -0.500388939256283    0     -Inf        0
Number of observations: 501, Error degrees of freedom: 490
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ________________    ___    _______
    Total       31.2187326156179    500    0.0624374652312358
    Model       31.2187326156179     10      3.12187326156179    Inf        0
    Residual                   0    490                      0
A = 0.241103, B = 0.130994, C = 0.232878
order Q(x) = 7.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 5.000000
Ymin = 1.000000 and Ymax = 74.209949
List of powers for Q(x): [0.372098, 0.735970, 0.963426, 1.168437, 1.361831, 1.547800,
1.728496]
List of powers for D(x): [0.372098, 0.735970, 0.963426, 1.168437]
Fitting cosh(x) in range (0.000000, 5.000000)
MSS of errors squared = 2.263905e-11
Corrected MSS of errors squared = 3.921198e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting cosh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate         SE     tStat     pValue
                    ________________    __    ______    _______
    (Intercept)    -0.425376361413491    0      -Inf        0
    x1              0.691404581254471    0       Inf        0
    x2             -0.354261344744162    0      -Inf        0
    x3             0.0671994497669736    0       Inf        0
    x4               5.10297953313833    0       Inf        0
    x5              -12.4366689662844    0      -Inf        0
    x6               17.3714607318283    0       Inf        0
```



```
    x7              -14.5163317881283    0     -Inf       0
    x8               7.30455486436604    0      Inf       0
    x9              -2.05329499268709    0     -Inf       0
    x10             0.248334292693611    0      Inf       0
Number of observations: 501, Error degrees of freedom: 490
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ____________________    ___    ______
    Total       31.2187326156271    500    0.0624374652312542
    Model       31.2187326156271     10      3.12187326156271    Inf       0
    Residual                   0    490                     0
A = 0.122735, B = 0.278438, C = 0.400000
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 5.000000
Ymin = 1.000000 and Ymax = 74.209949
List of powers for Q(x): [0.401172, 0.916505, 1.405003]
List of powers for D(x): [0.401172, 0.916505, 1.405003, 1.879610, 2.345340, 2.804765,
3.259411]
Fitting cosh(x) in range (0.000000, 5.000000)
MSS of errors squared = 5.643534e-10
Corrected MSS of errors squared = 9.774888e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting cosh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    _________________    ___    _____    ______
    (Intercept)      3.29068990193716    0       Inf        0
    x1              -26.3076127775267    0      -Inf        0
    x2               163.634967204027    0       Inf        0
    x3              -281.609978823802    0      -Inf        0
    x4                             0    0       NaN       NaN
    x5               420.249012288967    0       Inf        0
    x6              -390.049767757015    0      -Inf        0
    x7               110.953500742384    0       Inf        0
    x8               1.73692078155191    0       Inf        0
    x9              -1.48970860149094    0      -Inf        0
    x10             0.591977040074753    0       Inf        0
Number of observations: 501, Error degrees of freedom: 491
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```


	SumSq	DF	MeanSq	F	pValue
Total	31.2187326155959	500	0.0624374652311919		
Model	31.2187326155959	9	3.46874806839955	Inf	0
Residual	0	491	0		

```
A = 0.458506, B = 0.302326, C = 0.478432
order Q(x) = 7.000000, order D(x) = 3.000000
Xmin = 0.000000 and Xmax = 5.000000
Ymin = 1.000000 and Ymax = 74.209949
List of powers for Q(x): [0.760832, 1.364492, 1.658755, 1.891827, 2.091392, 2.268857,
2.430300]
List of powers for D(x): [0.760832, 1.364492, 1.658755]
Fitting cosh(x) in range (0.000000, 5.000000)
MSS of errors squared = 1.760385e-10
Corrected MSS of errors squared = 3.049076e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting cosh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	0	0	NaN	NaN
x1	-2.40840119262373e-05	0	-Inf	0
x2	0	0	NaN	NaN
x3	6.30942798175262e-05	0	Inf	0
x4	-6.60421159792958e-05	0	-Inf	0
x5	3.69000215838212e-05	0	Inf	0
x6	-1.14009108570694e-05	0	-Inf	0
x7	1.53213780765967e-06	0	Inf	0
x8	1.0000015612473	0	Inf	0
x9	0	0	NaN	NaN
x10	-3.10315335487829e-06	0	-Inf	0
x11	1.94603929355913e-06	0	Inf	0
x12	-4.03533683775662e-07	0	-Inf	0

```
Number of observations: 501, Error degrees of freedom: 491
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```

	SumSq	DF	MeanSq	F	pValue
Total	31.218732615627	500	0.062437465231254		
Model	31.218732615627	9	3.468748068403	Inf	0
Residual	0	491	0		


```
A = 0.000000, B = 1.193260
order Q(x) = 7.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 5.000000
Ymin = 1.000000 and Ymax = 74.209949
List of powers for Q(x): [0.000000, 0.009799, 0.061837, 0.156781, 0.284819, 0.437513,
0.608644]
List of powers for D(x): [0.000000, 0.009799, 0.061837, 0.156781, 0.284819]
Fitting cosh(x) in range (0.000000, 5.000000)
MSS of errors squared = 5.335536e-07
Corrected MSS of errors squared = 7.545587e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the hyperbolic cosine function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i+C*sqrt(i-1)	0.241	0.13	0.232	Inf	-Inf	2.26390500E-11	7, 4
A+B*sqrt(i)	0	1		Inf	-Inf	2.33749500E-11	7, 7
A+B*i	0.307	0.227		Inf	-Inf	1.11333500E-10	7, 6
A+B*i+C*(i-1)	0.231	0.131	0.058	Inf	-Inf	1.26879800E-10	7, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.458	0.302	0.478	Inf	-Inf	1.76038500E-10	7, 3
A+B*sqrt(i)+C*(i-1)	0.122	0.278	0.4	Inf	-Inf	5.64353400E-10	3, 7
A+B/i	0.68	0.1		Inf	-Inf	1.06029300E-08	5, 5
A+B*i-C*(i-1)	0.375	0.128	0.11	Inf	-Inf	1.06460300E-08	6, 7
A+B*sqrt(i+C)	0.054	0.437	0.036	Inf	-Inf	2.99603800E-07	6, 7
A+B*log10(i)^4	0	1.193		Inf	-Inf	5.33553600E-07	7, 5

RESULTS FOR THE DIGAMMA FUNCTION

Using Power A+B*i

```
Fitting digamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                    ________________    ___    _____    _______
    (Intercept)    0.752534010585533     0      Inf        0
    x1              9.88208887393292     0      Inf        0
    x2             -56.5001949697839     0     -Inf        0
    x3              100.417929836941     0      Inf        0
    x4             -74.0979425938916     0     -Inf        0
    x5              18.6119508543059     0      Inf        0
    x6             0.933695533601937     0      Inf        0
    x7              2.33002151567002     0      Inf        0
    x8              9.05690224908849     0      Inf        0
    x9             -29.0223025043996     0     -Inf        0
    x10             27.0865224427804     0      Inf        0
    x11            -8.45120524800222     0     -Inf        0
Number of observations: 981, Error degrees of freedom: 969
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq          DF           MeanSq           F      pValue
                    ________________    ___    ____________________    ___    _______
    Total          41.0603113009487    980    0.0418982768377028
    Model          41.0603113009487     11      3.73275557281352     Inf        0
    Residual                      0    969                     0
A = 0.229406, B = 0.438398
order Q(x) = 6.000000, order D(x) = 5.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 0.422784 and Ymax = 4.600162
List of powers for Q(x): [0.667804, 1.106201, 1.544599, 1.982996, 2.421394, 2.859791]
```



```
List of powers for D(x): [0.667804, 1.106201, 1.544599, 1.982996, 2.421394]
Fitting digamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 1.921799e-08
Corrected MSS of errors squared = 2.717834e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting digamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ___________________    ___    _____    _______
    (Intercept)      1.98003957367144      0       Inf         0
    x1               35.4868395187364      0       Inf         0
    x2              -407.497180639997      0      -Inf         0
    x3               651.979892744744      0       Inf         0
    x4                             0      0       NaN       NaN
    x5                             0      0       NaN       NaN
    x6              -281.949478677264      0      -Inf         0
    x7              -7.49694766103224      0      -Inf         0
    x8              -108.191615474284      0      -Inf         0
    x9               1082.05481829244      0       Inf         0
    x10             -1670.97242475817      0      -Inf         0
    x11                            0      0       NaN       NaN
    x12              705.606057090551      0       Inf         0
Number of observations: 981, Error degrees of freedom: 971
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F      pValue
                  _______________    ___    ____________________    ___    _______
    Total         41.0603113011254    980    0.0418982768378831
    Model         41.0603113011254      9      4.56225681123616    Inf        0
    Residual                     0    971                     0
A = 0.884970, B = 0.849143
order Q(x) = 6.000000, order D(x) = 6.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 0.422784 and Ymax = 4.600162
List of powers for Q(x): [1.734113, 1.309542, 1.168018, 1.097256, 1.054799, 1.026494]
List of powers for D(x): [1.734113, 1.309542, 1.168018, 1.097256, 1.054799, 1.026494]
Fitting digamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 1.223240e-07
Corrected MSS of errors squared = 1.729923e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
```



```
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
Using Power A+B*sqrt(i)
Fitting digamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    _________________    __    _____    ______
    (Intercept)      1.72934622423982    0      Inf        0
    x1              -24.5460145141477    0     -Inf        0
    x2               44.9366243445756    0      Inf        0
    x3                             0     0      NaN      NaN
    x4              -33.7747493109203    0     -Inf        0
    x5                             0     0      NaN      NaN
    x6               11.6548180027882    0      Inf        0
    x7               21.6727718799194    0      Inf        0
    x8              -72.1021722885888    0     -Inf        0
    x9               97.0962446949652    0      Inf        0
    x10             -67.9369659643932    0     -Inf        0
    x11              30.7337879848432    0      Inf        0
    x12             -8.46369105245563    0     -Inf        0
Number of observations: 981, Error degrees of freedom: 970
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _________________    ___    _________________    ___    ______
    Total        41.0603113009282     980    0.0418982768376818
    Model        41.0603113009282      10      4.10603113009282    Inf       0
    Residual                    0     970                     0
A = 0.000000, B = 0.690408
order Q(x) = 6.000000, order D(x) = 6.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 0.422784 and Ymax = 4.600162
List of powers for Q(x): [0.690408, 0.976385, 1.195822, 1.380816, 1.543800, 1.691148]
List of powers for D(x): [0.690408, 0.976385, 1.195822, 1.380816, 1.543800, 1.691148]
Fitting digamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 4.852753e-08
Corrected MSS of errors squared = 6.862829e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting digamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
```



```
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ________________    ___    _____    ______
    (Intercept)     2.15486934252935    0       Inf        0
    x1             -377.080361758566    0      -Inf        0
    x2              1096.40278901576    0       Inf        0
    x3                             0    0       NaN      NaN
    x4             -2828.73266837102    0      -Inf        0
    x5              3144.59415154069    0       Inf        0
    x6             -1037.33874017853    0      -Inf        0
    x7              317.057178184938    0       Inf        0
    x8             -1525.49267328144    0      -Inf        0
    x9              2682.29495711222    0       Inf        0
    x10            -2051.40412673586    0      -Inf        0
    x11             578.544625129399    0       Inf        0
Number of observations: 99, Error degrees of freedom: 88
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq           F      pValue
                  _________________    ___    ________________    ___    ______
    Total       4.41829479310191     98     0.0450846407459379
    Model       4.41829479310191     10      0.441829479310191    Inf        0
    Residual                   0     88                       0
A = 0.454672, B = 0.546888, C = 0.408588
order Q(x) = 6.000000, order D(x) = 5.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 0.422784 and Ymax = 4.600162
List of powers for Q(x): [1.103740, 1.303422, 1.464356, 1.602953, 1.726535, 1.839129]
List of powers for D(x): [1.103740, 1.303422, 1.464356, 1.602953, 1.726535]
Fitting digamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 1.322433e-07
Corrected MSS of errors squared = 2.290521e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting digamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8
Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ________________    ___    _____    ______
    (Intercept)     0.44762041054863    0       Inf        0
    x1              10.7637501839932    0       Inf        0
```



```
    x2              -38.5106787296702    0      -Inf        0
    x3               42.6135008255663    0       Inf        0
    x4              -15.3143770657438    0      -Inf        0
    x5               2.53211952454437    0       Inf        0
    x6               2.87848184794496    0       Inf        0
    x7              -8.55685517050969    0      -Inf        0
    x8               4.14643811091191    0       Inf        0
Number of observations: 981, Error degrees of freedom: 972
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq           DF            MeanSq            F      pValue
                 _________________    ___    ____________________    ___    _______
    Total        41.0603113009338    980    0.0418982768376876
    Model        41.0603113009338      8      5.13253891261673    Inf        0
    Residual                    0    972                      0
A = 0.331200, B = 0.130879, C = 0.121390
order Q(x) = 4.000000, order D(x) = 4.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 0.422784 and Ymax = 4.600162
List of powers for Q(x): [0.462079, 0.714348, 0.966617, 1.218886]
List of powers for D(x): [0.462079, 0.714348, 0.966617, 1.218886]
Fitting digamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 5.177987e-07
Corrected MSS of errors squared = 8.968536e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting digamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    _____________________    ___    _____    _______
    (Intercept)      1.51216614751866    0      Inf         0
    x1              -14.6893092802894    0     -Inf         0
    x2               22.6571567470674    0      Inf         0
    x3                             0    0      NaN       NaN
    x4              -21.2327728841013    0     -Inf         0
    x5                 14.61819374686    0      Inf         0
    x6              -2.86543208894058    0     -Inf         0
    x7               13.8662456819886    0      Inf         0
    x8              -33.7832153068569    0     -Inf         0
    x9               33.3830236247352    0      Inf         0
    x10             -14.8282691246735    0     -Inf         0
    x11               2.3622127367689    0      Inf         0
```



```
Number of observations: 981, Error degrees of freedom: 970
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq            F      pValue
                ___________________    ___    ___________________    ___    ______
    Total       41.0603113009333    980    0.041898276837687
    Model       41.0603113009333     10     4.10603113009333    Inf       0
    Residual                   0    970                     0
A = 0.152838, B = 0.241201, C = 0.049206
order Q(x) = 6.000000, order D(x) = 5.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 0.422784 and Ymax = 4.600162
List of powers for Q(x): [0.394039, 0.586034, 0.778029, 0.970024, 1.162019, 1.354013]
List of powers for D(x): [0.394039, 0.586034, 0.778029, 0.970024, 1.162019]
Fitting digamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 4.721355e-08
Corrected MSS of errors squared = 8.177627e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting digamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)       1.7480529224224     0      Inf       0
    x1               -3.32644312889994    0     -Inf       0
    x2                2.84624997622728    0      Inf       0
    x3               -1.26792282549394    0     -Inf       0
    x4                2.03712608882679    0      Inf       0
    x5               -2.31474125771371    0     -Inf       0
    x6                1.74304500311698    0      Inf       0
    x7              -0.557235842902366    0     -Inf       0
    x8               0.0918690387393371    0      Inf       0
Number of observations: 981, Error degrees of freedom: 972
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ___________________    ___    ___________________    ___    ______
    Total       41.0603113009336    980    0.0418982768376873
    Model       41.0603113009336      8       5.1325389126167    Inf       0
    Residual                   0    972                     0
A = 0.012571, B = 0.219916, C = 0.317366
order Q(x) = 3.000000, order D(x) = 5.000000
```



```
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 0.422784 and Ymax = 4.600162
List of powers for Q(x): [0.232487, 0.769769, 1.121142]
List of powers for D(x): [0.232487, 0.769769, 1.121142, 1.441929, 1.746883]
Fitting digamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 6.337455e-07
Corrected MSS of errors squared = 1.097679e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting digamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                    Estimate         SE    tStat    pValue
                  _____________________    __    _____    ______

    (Intercept)     1.21499198091468    0      Inf        0
    x1             -14.1865039621249    0     -Inf        0
    x2                23.46233100353    0      Inf        0
    x3                             0    0      NaN      NaN
    x4             -13.5685218314944    0     -Inf        0
    x5                             0    0      NaN      NaN
    x6                             0    0      NaN      NaN
    x7              3.07770287844842    0      Inf        0
    x8              18.9690684156634    0      Inf        0
    x9             -55.0670777420227    0     -Inf        0
    x10             50.0715159707752    0      Inf        0
    x11                            0    0      NaN      NaN
    x12            -15.8800073224134    0     -Inf        0
    x13                            0    0      NaN      NaN
    x14             2.90650060872933    0      Inf        0
Number of observations: 981, Error degrees of freedom: 971
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq          F     pValue
                  _____________________    ___    _____________________    ___    ______

    Total       41.0603113009362    980    0.04189827683769
    Model       41.0603113009362      9    4.56225681121514    Inf        0
    Residual                   0    971                   0
A = 0.000000, B = 0.141963, C = 0.000000
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 0.422784 and Ymax = 4.600162
List of powers for Q(x): [0.141963, 0.200766, 0.245887, 0.283925, 0.317438, 0.347736,
0.375598]
```



```
List of powers for D(x): [0.141963, 0.200766, 0.245887, 0.283925, 0.317438, 0.347736,
0.375598]
Fitting digamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 1.201296e-07
Corrected MSS of errors squared = 2.080705e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting digamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      1.42528705920906     0      Inf        0
    x1              -7.61025757559444     0     -Inf        0
    x2               12.6921175110551     0      Inf        0
    x3                             0     0      NaN      NaN
    x4               -8.7211640000003     0     -Inf        0
    x5                             0     0      NaN      NaN
    x6                             0     0      NaN      NaN
    x7               2.21402077948702     0      Inf        0
    x8               7.51676739157173     0      Inf        0
    x9              -29.2682770407239     0     -Inf        0
    x10              50.7428277815509     0      Inf        0
    x11             -46.7248779632511     0     -Inf        0
    x12              25.3317284681819     0      Inf        0
    x13             -6.59817241137062     0     -Inf        0
Number of observations: 981, Error degrees of freedom: 970
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______
    Total        41.0603113009299    980    0.0418982768376836
    Model        41.0603113009299     10      4.10603113009299    Inf       0
    Residual                    0    970                     0
A = 0.000000, B = 0.370261, C = 0.153313
order Q(x) = 7.000000, order D(x) = 6.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 0.422784 and Ymax = 4.600162
List of powers for Q(x): [0.370261, 0.676941, 0.858128, 1.006068, 1.134555, 1.249769,
1.355157]
List of powers for D(x): [0.370261, 0.676941, 0.858128, 1.006068, 1.134555, 1.249769]
Fitting digamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 4.456212e-08
```



```
Corrected MSS of errors squared = 7.718386e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting digamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                         Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)                        0    0      NaN      NaN
    x1             -0.000152748839127335    0     -Inf        0
    x2                                 0    0      NaN      NaN
    x3              0.000324289211230445    0      Inf        0
    x4             -0.000224974484622544    0     -Inf        0
    x5              5.34341133859465e-05    0      Inf        0
    x6                  1.00000391003903    0      Inf        0
    x7                                 0    0      NaN      NaN
    x8                                 0    0      NaN      NaN
    x9             -1.41600626491027e-05    0     -Inf        0
    x10             1.48321385716844e-05    0      Inf        0
    x11            -4.58211580930519e-06    0     -Inf        0
Number of observations: 981, Error degrees of freedom: 973
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _________________    ___    ___________________    ___    ______
    Total        41.0603113009344     980    0.0418982768376882
    Model        41.0603113009344       7      5.86575875727635    Inf       0
    Residual                    0     973                      0
A = 0.000000, B = 0.100000
order Q(x) = 5.000000, order D(x) = 6.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 0.422784 and Ymax = 4.600162
List of powers for Q(x): [0.000000, 0.000821, 0.005182, 0.013139, 0.023869]
List of powers for D(x): [0.000000, 0.000821, 0.005182, 0.013139, 0.023869, 0.036665]
Fitting digamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 1.916255e-05
Corrected MSS of errors squared = 2.709994e-05
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the digamma function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i	0.229	0.438		Inf	-Inf	1.92179900E-08	6, 5
A+B*sqrt(i)+C*sqrt(i-1)	0	0.37	0.153	Inf	-Inf	4.45621200E-08	7, 6
A+B*i-C*(i-1)	0.152	0.241	0.049	Inf	-Inf	4.72135500E-08	6, 5
A+B*sqrt(i)	0	0.69		Inf	-Inf	4.85275300E-08	6, 6
A+B*sqrt(i)+C*(i-1)	0	0.141	0	Inf	-Inf	1.20129600E-07	7, 7
A+B/i	0.884	0.849		Inf	-Inf	1.22324000E-07	6, 6
A+B*sqrt(i+C)	0.454	0.546	0.408	Inf	-Inf	1.32243300E-07	6, 5
A+B*i+C*(i-1)	0.331	0.13	0.121	Inf	-Inf	5.17798700E-07	4, 4
A+B*i+C*sqrt(i-1)	0.012	0.219	0.317	Inf	-Inf	6.33745500E-07	3, 5
A+B*log10(i)^4	0	0.1		Inf	-Inf	1.91625500E-05	5, 6

RESULTS FOR THE ERROR FUNCTION

Using Power A+B* i

```
Fitting erf(x) in range (0.000000, 2.100000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                    ___________________    __    _____    ______

    (Intercept)    0.241056117589965      0      Inf       0
    x1             -7.39307050666147      0     -Inf       0
    x2               21.865334153228      0      Inf       0
    x3             -25.9317541002993      0     -Inf       0
    x4              14.2224756243664      0      Inf       0
    x5             -3.00225240404221      0     -Inf       0
    x6              24.5427730090039      0      Inf       0
    x7             -70.9775438983882      0     -Inf       0
    x8              87.0904400636269      0      Inf       0
    x9             -54.7799842211152      0     -Inf       0
    x10             17.2364584888909      0      Inf       0
    x11            -2.1139323261724      0     -Inf       0
Number of observations: 211, Error degrees of freedom: 199
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq          DF           MeanSq           F      pValue
                    ___________________    ___    ___________________    ___    ______

    Total          18.3371112829929      210    0.0873195775380616
    Model          18.3371112829929       11      1.66701011663572     Inf       0
    Residual                      0      199                     0
A = 0.439057, B = 0.245268
order Q(x) = 5.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 2.100000
Ymin = 0.000000 and Ymax = 0.997021
List of powers for Q(x): [0.684325, 0.929593, 1.174861, 1.420130, 1.665398]
```



```
List of powers for D(x): [0.684325, 0.929593, 1.174861, 1.420130, 1.665398, 1.910666]
Fitting erf(x) in range (0.000000, 2.100000)
MSS of errors squared = 2.871920e-07
Corrected MSS of errors squared = 4.061508e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting erf(x) in range (0.000000, 2.100000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    _________________    ___    _____    ______
    (Intercept)      0.58144445292812    0       Inf        0
    x1               484.954140427716    0       Inf        0
    x2              -6282.95950190231    0      -Inf        0
    x3               14523.9449307221    0       Inf        0
    x4              -8726.51895610159    0      -Inf        0
    x5              -496.539233646676    0      -Inf        0
    x6               5277.08669922243    0       Inf        0
    x7              -7689.14936355801    0      -Inf        0
    x8                             0    0       NaN      NaN
    x9                             0    0       NaN      NaN
    x10                            0    0       NaN      NaN
    x11              2909.59984038516    0       Inf        0
Number of observations: 211, Error degrees of freedom: 202
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______
    Total        18.3371112829274    210    0.0873195775377495
    Model        18.3371112829274      8      2.29213891036593    Inf       0
    Residual                    0    202                     0
A = 0.544211, B = 0.100000
order Q(x) = 4.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 2.100000
Ymin = 0.000000 and Ymax = 0.997021
List of powers for Q(x): [0.644211, 0.594211, 0.577544, 0.569211]
List of powers for D(x): [0.644211, 0.594211, 0.577544, 0.569211, 0.564211, 0.560878,
0.558497]
Fitting erf(x) in range (0.000000, 2.100000)
MSS of errors squared = 2.310075e-08
Corrected MSS of errors squared = 3.266939e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
```



```
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting erf(x) in range (0.000000, 2.100000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ___________________    __    _____    ______
    (Intercept)    0.131635228211962     0      Inf        0
    x1             -1.74998116368292     0     -Inf        0
    x2              3.05445338807184     0      Inf        0
    x3                             0     0      NaN      NaN
    x4              -2.127705211248     0     -Inf        0
    x5                             0     0      NaN      NaN
    x6             0.691655424287549     0      Inf        0
    x7              21.9114441069555     0      Inf        0
    x8             -68.8006390493136     0     -Inf        0
    x9              69.269958061009     0      Inf        0
    x10                            0     0      NaN      NaN
    x11            -36.1559161185757     0     -Inf        0
    x12             14.775095334282     0      Inf        0
Number of observations: 211, Error degrees of freedom: 201
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq          DF           MeanSq           F     pValue
                    ___________________    ___    ____________________    ___    ______
    Total          18.3371112829995     210    0.0873195775380931
    Model          18.3371112829995       9      2.03745680922217    Inf       0
    Residual                      0     201                      0
A = 0.012695, B = 0.196350
order Q(x) = 6.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 2.100000
Ymin = 0.000000 and Ymax = 0.997021
List of powers for Q(x): [0.209045, 0.290376, 0.352783, 0.405395, 0.451747, 0.493652]
List of powers for D(x): [0.209045, 0.290376, 0.352783, 0.405395, 0.451747, 0.493652]
Fitting erf(x) in range (0.000000, 2.100000)
MSS of errors squared = 3.059180e-09
Corrected MSS of errors squared = 4.326334e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting erf(x) in range (0.000000, 2.100000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
```



```
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)    -0.027770080703484     0      -Inf        0
    x1               17.7815011307604     0       Inf        0
    x2              -37.308397977607      0      -Inf        0
    x3               19.5571934523302     0       Inf        0
    x4               29922.8182289682     0       Inf        0
    x5              -118425.593437661     0      -Inf        0
    x6               136990.340690398     0       Inf        0
    x7                              0     0       NaN      NaN
    x8              -86938.4011796371     0      -Inf        0
    x9               38451.833171397      0       Inf        0
Number of observations: 211, Error degrees of freedom: 202
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF            MeanSq            F     pValue
                  _________________    ___    ____________________    ___    _______
    Total        18.3371112864965     210    0.0873195775547451
    Model        18.3371112864965       8      2.29213891081206    Inf        0
    Residual                    0     202                     0
A = 0.818479, B = 0.100000, C = 1.258617
order Q(x) = 3.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 2.100000
Ymin = 0.000000 and Ymax = 0.997021
List of powers for Q(x): [0.968766, 0.998996, 1.024843]
List of powers for D(x): [0.968766, 0.998996, 1.024843, 1.047796, 1.068651, 1.087897]
Fitting erf(x) in range (0.000000, 2.100000)
MSS of errors squared = 1.496596e-07
Corrected MSS of errors squared = 2.592180e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting erf(x) in range (0.000000, 2.100000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)    -0.0220828787165694    0      -Inf        0
    x1              0.0519981350750535    0       Inf        0
    x2             -0.0434185976641765    0      -Inf        0
    x3              0.0135518170593897    0       Inf        0
```



```
    x4                3.11717951725034    0      Inf       0
    x5               -4.57058528408878    0     -Inf       0
    x6                3.78862577059506    0      Inf       0
    x7               -1.61214576054369    0     -Inf       0
    x8               0.276877280824993    0      Inf       0
Number of observations: 211, Error degrees of freedom: 202
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ___________________    ___    ______
    Total       18.3371112829989    210    0.0873195775380901
    Model       18.3371112829989      8      2.29213891037487    Inf       0
    Residual                   0    202                     0
A = 0.000000, B = 0.144303, C = 0.055235
order Q(x) = 3.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 2.100000
Ymin = 0.000000 and Ymax = 0.997021
List of powers for Q(x): [0.144303, 0.343841, 0.543378]
List of powers for D(x): [0.144303, 0.343841, 0.543378, 0.742916, 0.942454]
Fitting erf(x) in range (0.000000, 2.100000)
MSS of errors squared = 2.061742e-07
Corrected MSS of errors squared = 3.571042e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting erf(x) in range (0.000000, 2.100000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                    Estimate          SE    tStat    pValue
                ________________    ___    _____    ______
    (Intercept)    0.289425187314055    0      Inf       0
    x1             -4.65323422223439    0     -Inf       0
    x2              10.2519216251167    0      Inf       0
    x3             -8.15672621829611    0     -Inf       0
    x4              2.26900684283037    0      Inf       0
    x5              12.3179230420101    0      Inf       0
    x6             -19.6587503289055    0     -Inf       0
    x7                             0    0      NaN     NaN
    x8              19.5158684261179    0      Inf       0
    x9             -14.4179820920108    0     -Inf       0
    x10             3.24254773804661    0      Inf       0
Number of observations: 211, Error degrees of freedom: 201
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```


	SumSq	DF	MeanSq	F	pValue
Total	18.3371112829993	210	0.087319577538092		
Model	18.3371112829993	9	2.03745680922215	Inf	0
Residual	0	201	0		

```
A = 0.068688, B = 0.223064, C = 0.109971
order Q(x) = 4.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 2.100000
Ymin = 0.000000 and Ymax = 0.997021
List of powers for Q(x): [0.291752, 0.404845, 0.517938, 0.631031]
List of powers for D(x): [0.291752, 0.404845, 0.517938, 0.631031, 0.744125, 0.857218]
Fitting erf(x) in range (0.000000, 2.100000)
MSS of errors squared = 2.743291e-09
Corrected MSS of errors squared = 4.751519e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting erf(x) in range (0.000000, 2.100000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	0.241012458498957	0	Inf	0
x1	-2.86709589579024	0	-Inf	0
x2	8.13002184053809	0	Inf	0
x3	-9.66137648595456	0	-Inf	0
x4	5.26064122094703	0	Inf	0
x5	-1.09856622377689	0	-Inf	0
x6	8.535910991442	0	Inf	0
x7	-21.0418427206718	0	-Inf	0
x8	23.495061666168	0	Inf	0
x9	-13.2969334838113	0	-Inf	0
x10	3.67470109108568	0	Inf	0
x11	-0.371534458596075	0	-Inf	0

```
Number of observations: 211, Error degrees of freedom: 199
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```

	SumSq	DF	MeanSq	F	pValue
Total	18.3371112829988	210	0.0873195775380894		
Model	18.3371112829988	11	1.66701011663625	Inf	0
Residual	0	199	0		

```
A = 0.336932, B = 0.348332, C = 0.065631
order Q(x) = 5.000000, order D(x) = 6.000000
```



```
Xmin = 0.000000 and Xmax = 2.100000
Ymin = 0.000000 and Ymax = 0.997021
List of powers for Q(x): [0.685264, 1.099228, 1.474746, 1.843938, 2.209856]
List of powers for D(x): [0.685264, 1.099228, 1.474746, 1.843938, 2.209856, 2.573682]
Fitting erf(x) in range (0.000000, 2.100000)
MSS of errors squared = 1.964813e-07
Corrected MSS of errors squared = 3.403157e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting erf(x) in range (0.000000, 2.100000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                   ________________    __    _____    ______
    (Intercept)    0.230265682385997    0      Inf       0
    x1             -2.06979345518339    0     -Inf       0
    x2              4.64892304031921    0      Inf       0
    x3             -4.03500815518149    0     -Inf       0
    x4              1.22819926020301    0      Inf       0
    x5              7.99260529803595    0      Inf       0
    x6             -18.3942228630008    0     -Inf       0
    x7              18.5410761333552    0      Inf       0
    x8             -8.65922216767543    0     -Inf       0
    x9              1.51717722663353    0      Inf       0
Number of observations: 211, Error degrees of freedom: 201
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _________________    ___    ___________________    ___    ______
    Total         18.3371112829991    210    0.0873195775380911
    Model         18.3371112829991      9      2.03745680922213    Inf       0
    Residual                     0    201                     0
A = 0.090153, B = 0.339690, C = 0.115625
order Q(x) = 4.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 2.100000
Ymin = 0.000000 and Ymax = 0.997021
List of powers for Q(x): [0.429843, 0.686173, 0.909764, 1.116408]
List of powers for D(x): [0.429843, 0.686173, 0.909764, 1.116408, 1.312223]
Fitting erf(x) in range (0.000000, 2.100000)
MSS of errors squared = 2.854241e-09
Corrected MSS of errors squared = 4.943691e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
```



```
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting erf(x) in range (0.000000, 2.100000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______
    (Intercept)      39.0401817455958     0      Inf       0
    x1              -1588.14680219185     0     -Inf       0
    x2               8500.29154599159     0      Inf       0
    x3              -18781.6930468481     0     -Inf       0
    x4               20645.0474818383     0      Inf       0
    x5              -11239.2877565587     0     -Inf       0
    x6               2424.75410799583     0      Inf       0
    x7               6.34562197494661     0      Inf       0
    x8              -9.48447614988881     0     -Inf       0
    x9               4.13314229471771     0      Inf       0
Number of observations: 211, Error degrees of freedom: 201
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F      pValue
                   _________________    ___    ____________________    ___    ______
    Total       18.3371112831706     210    0.0873195775389074
    Model       18.3371112831706       9      2.03745680924117    Inf       0
    Residual                   0     201                        0
A = 0.000000, B = 0.700000, C = 0.000000
order Q(x) = 6.000000, order D(x) = 3.000000
Xmin = 0.000000 and Xmax = 2.100000
Ymin = 0.000000 and Ymax = 0.997021
List of powers for Q(x): [0.700000, 0.989949, 1.212436, 1.400000, 1.565248, 1.714643]
List of powers for D(x): [0.700000, 0.989949, 1.212436]
Fitting erf(x) in range (0.000000, 2.100000)
MSS of errors squared = 3.175235e-06
Corrected MSS of errors squared = 5.499668e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting erf(x) in range (0.000000, 2.100000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
```


	Estimate	SE	tStat	pValue
(Intercept)	0.0111107111258976	0	Inf	0
x1	0	0	NaN	NaN
x2	-0.0196058463808357	0	-Inf	0
x3	0.0112022951925902	0	Inf	0
x4	-0.00312907611018683	0	-Inf	0
x5	0.00042267965532159	0	Inf	0
x6	2.82634542918986	0	Inf	0
x7	-1.96301638576017	0	-Inf	0
x8	0.147358672268931	0	Inf	0
x9	-0.0111171107145148	0	-Inf	0
x10	0.000428631533073841	0	Inf	0

Number of observations: 211, Error degrees of freedom: 201
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	18.3371112829989	210	0.0873195775380898		
Model	18.3371112829989	9	2.03745680922209	Inf	0
Residual	0	201	0		

A = 0.025893, B = 2.630714
order Q(x) = 5.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 2.100000
Ymin = 0.000000 and Ymax = 0.997021
List of powers for Q(x): [0.025893, 0.047496, 0.162222, 0.371540, 0.653818]
List of powers for D(x): [0.025893, 0.047496, 0.162222, 0.371540, 0.653818]
Fitting erf(x) in range (0.000000, 2.100000)
MSS of errors squared = 3.244011e-09
Corrected MSS of errors squared = 4.587724e-09
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Comments

The following table shows the summary results for the ten models that fit the error function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i-C*(i-1)	0.068	0.223	0.109	Inf	-Inf	2.74329100E-09	4, 6
A+B*sqrt(i)+C*(i-1)	0.09	0.339	0.115	Inf	-Inf	2.85424100E-09	4, 5
A+B*sqrt(i)	0.012	0.196		Inf	-Inf	3.05918000E-09	6, 6
A+B*log10(i)^4	0.025	2.63		Inf	-Inf	3.24401100E-09	5, 5
A+B/i	0.544	0.1		Inf	-Inf	2.31007500E-08	4, 7
A+B*sqrt(i+C)	0.818	0.1	1.258	Inf	-Inf	1.49659600E-07	3, 6
A+B*i+C*sqrt(i-1)	0.336	0.348	0.065	Inf	-Inf	1.96481300E-07	5, 6
A+B*i+C*(i-1)	0	0.144	0.055	Inf	-Inf	2.06174200E-07	3, 5
A+B*i	0.439	0.245		Inf	-Inf	2.87192000E-07	5, 6
A+B*sqrt(i)+C*sqrt(i-1)	0	0.7	0	Inf	-Inf	3.17523500E-06	6, 3

RESULTS FOR THE EXPONENTIAL FUNCTION

Using Power A+B* i

```
Fitting exp(x) in range (0.000000, 2.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ___________________    __    _____    ______
    (Intercept)     0.874126661378952      0      Inf        0
    x1               -10.780408695217      0     -Inf        0
    x2               13.0027809167161      0      Inf        0
    x3                             0      0      NaN      NaN
    x4                             0      0      NaN      NaN
    x5               -6.2673150146967      0     -Inf        0
    x6               3.47337150314461      0      Inf        0
    x7              -0.278568050446949     0     -Inf        0
    x8               9.13336501151758      0      Inf        0
    x9                             0      0      NaN      NaN
    x10             -28.4501401194154      0     -Inf        0
    x11              28.4185901696258      0      Inf        0
    x12             -8.1258023826073       0     -Inf        0
Number of observations: 201, Error degrees of freedom: 191
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F      pValue
                   _______________    ___    ___________________    ___    ______
    Total          15.9274939123249    200    0.0796374695616246
    Model          15.9274939123249      9      1.76972154581388    Inf       0
    Residual                      0    191                     0
A = 0.371774, B = 0.100000
order Q(x) = 7.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 2.000000
Ymin = 1.000000 and Ymax = 7.389056
```



```
List of powers for Q(x): [0.471774, 0.571774, 0.671774, 0.771774, 0.871774, 0.971774,
1.071774]
List of powers for D(x): [0.471774, 0.571774, 0.671774, 0.771774, 0.871774]
Fitting exp(x) in range (0.000000, 2.000000)
MSS of errors squared = 9.295456e-13
Corrected MSS of errors squared = 1.314576e-12
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting exp(x) in range (0.000000, 2.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    _________________    __    _____    ______
    (Intercept)      0.865209628938214    0      Inf        0
    x1              0.0735217505690566    0      Inf        0
    x2               0.127878140338742    0      Inf        0
    x3              -0.283198528484445    0     -Inf        0
    x4               0.240422442292122    0      Inf        0
    x5               -9.34707191117625    0     -Inf        0
    x6                45.0306801121811    0      Inf        0
    x7               -55.9904526740928    0     -Inf        0
    x8                               0    0      NaN      NaN
    x9                20.2830110384081    0      Inf        0
Number of observations: 201, Error degrees of freedom: 192
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    _____________________    ___    ______
    Total        15.9274939123265    200    0.0796374695616324
    Model        15.9274939123265      8      1.99093673904081    Inf       0
    Residual                    0    192                     0
A = 1.789020, B = 1.336117
order Q(x) = 3.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 2.000000
Ymin = 1.000000 and Ymax = 7.389056
List of powers for Q(x): [3.125137, 2.457078, 2.234392]
List of powers for D(x): [3.125137, 2.457078, 2.234392, 2.123049, 2.056243, 2.011706]
Fitting exp(x) in range (0.000000, 2.000000)
MSS of errors squared = 3.949041e-11
Corrected MSS of errors squared = 5.584787e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
```



```
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting exp(x) in range (0.000000, 2.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    ________________    ___    _____    ______
    (Intercept)    0.865200143053698    0      Inf       0
    x1             -7.02138716910023    0     -Inf       0
    x2              14.6437301075832    0      Inf       0
    x3             -9.75690666962471    0     -Inf       0
    x4             0.594843806370177    0      Inf       0
    x5              1.02480753750204    0      Inf       0
    x6              8.9272877784676     0      Inf       0
    x7             -20.0713424324975    0     -Inf       0
    x8              16.470199252937     0      Inf       0
    x9             -4.67643235469403    0     -Inf       0
Number of observations: 201, Error degrees of freedom: 191
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq          DF           MeanSq            F      pValue
                   _______________    ___    ____________________    ___    ______
    Total          15.9274939123246    200    0.0796374695616231
    Model          15.9274939123246      9      1.76972154581385    Inf       0
    Residual                      0    191                     0
A = 0.878003, B = 0.631679
order Q(x) = 5.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 2.000000
Ymin = 1.000000 and Ymax = 7.389056
List of powers for Q(x): [1.509681, 1.771331, 1.972102, 2.141360, 2.290479]
List of powers for D(x): [1.509681, 1.771331, 1.972102, 2.141360]
Fitting exp(x) in range (0.000000, 2.000000)
MSS of errors squared = 4.010337e-13
Corrected MSS of errors squared = 5.671473e-13
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting exp(x) in range (0.000000, 2.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
```


(Intercept)	0.842711832959753	0	Inf	0
x1	-37.9663401360121	0	-Inf	0
x2	0	0	NaN	NaN
x3	116.565976443803	0	Inf	0
x4	0	0	NaN	NaN
x5	-166.732241751488	0	-Inf	0
x6	87.3864970953465	0	Inf	0
x7	145.079809105753	0	Inf	0
x8	-407.369088090249	0	-Inf	0
x9	385.686295491939	0	Inf	0
x10	-122.493619992134	0	-Inf	0

Number of observations: 201, Error degrees of freedom: 192
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	15.9274939123243	200	0.0796374695616214		
Model	15.9274939123243	8	1.99093673904053	Inf	0
Residual	0	192	0		

A = 0.458343, B = 0.223453, C = 1.902225
order Q(x) = 6.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 2.000000
Ymin = 1.000000 and Ymax = 7.389056
List of powers for Q(x): [0.839015, 0.899753, 0.953089, 1.001211, 1.045400, 1.086489]
List of powers for D(x): [0.839015, 0.899753, 0.953089, 1.001211]
Fitting exp(x) in range (0.000000, 2.000000)
MSS of errors squared = 3.840143e-11
Corrected MSS of errors squared = 6.651323e-11
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*i+C*(i–1)

Fitting exp(x) in range (0.000000, 2.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	0.879858692867062	0	Inf	0
x1	-0.158749536244191	0	-Inf	0
x2	-1.01828719590238	0	-Inf	0
x3	-0.764350836622808	0	-Inf	0
x4	2.44207177098711	0	Inf	0
x5	-1.49968704177465	0	-Inf	0
x6	0.305194030920071	0	Inf	0


```
    x7                              0     0      NaN      NaN
    x8                1.7725567766497     0      Inf        0
    x9                              0     0      NaN      NaN
    x10              -2.03325369143062    0     -Inf        0
    x11               1.33039818621025    0      Inf        0
    x12              -0.255751155659115   0     -Inf        0
Number of observations: 201, Error degrees of freedom: 190
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF            MeanSq            F      pValue
                  _______________    ___    ____________________    ___    _______
    Total         15.927493912325    200    0.0796374695616248
    Model         15.927493912325     10       1.5927493912325    Inf        0
    Residual                    0    190                         0
A = 0.230969, B = 0.193169, C = 0.136692
order Q(x) = 6.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 2.000000
Ymin = 1.000000 and Ymax = 7.389056
List of powers for Q(x): [0.424138, 0.753999, 1.083861, 1.413722, 1.743584, 2.073445]
List of powers for D(x): [0.424138, 0.753999, 1.083861, 1.413722, 1.743584, 2.073445]
Fitting exp(x) in range (0.000000, 2.000000)
MSS of errors squared = 1.134956e-13
Corrected MSS of errors squared = 1.965802e-13
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting exp(x) in range (0.000000, 2.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                   ___________________    ___    _____    _______
    (Intercept)    0.858871283470805      0       Inf        0
    x1            -6481.03845589839       0      -Inf        0
    x2                                    0       NaN       NaN
    x3             19222.9768328446       0       Inf        0
    x4            -12742.6639449782       0      -Inf        0
    x5             3180.00042173778       0       Inf        0
    x6                                    0       NaN       NaN
    x7                                    0       NaN       NaN
    x8             -7820.0402689538       0      -Inf        0
    x9                                    0       NaN       NaN
    x10            4640.90654394434       0       Inf        0
Number of observations: 201, Error degrees of freedom: 194
R-squared: 1,  Adjusted R-Squared: 1
```



```
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ________________    ___    ______
    Total       15.9274939122917    200    0.0796374695614584
    Model       15.9274939122917      6      2.65458231871528    Inf       0
    Residual                   0    194                     0
A = 0.719905, B = 0.197183, C = 0.194852
order Q(x) = 4.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 2.000000
Ymin = 1.000000 and Ymax = 7.389056
List of powers for Q(x): [0.917088, 0.919419, 0.921749, 0.924080]
List of powers for D(x): [0.917088, 0.919419, 0.921749, 0.924080, 0.926411, 0.928742]
Fitting exp(x) in range (0.000000, 2.000000)
MSS of errors squared = 8.527592e-09
Corrected MSS of errors squared = 1.477022e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting exp(x) in range (0.000000, 2.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ________________    ___    _____    _______
    (Intercept)    0.802484257967156    0       Inf       0
    x1             -1.33542614071276    0      -Inf       0
    x2             0.872994241130858    0       Inf       0
    x3             -0.321277987127409    0      -Inf       0
    x4              1.84176992996503    0       Inf       0
    x5             -2.52782154276741    0      -Inf       0
    x6              3.99281082785539    0       Inf       0
    x7             -4.10216184719049    0      -Inf       0
    x8              2.27955110617168    0       Inf       0
    x9             -0.502922845295886    0      -Inf       0
Number of observations: 201, Error degrees of freedom: 191
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ________________    ___    ______
    Total       15.9274939123248    200    0.0796374695616242
    Model       15.9274939123248      9      1.76972154581387    Inf       0
    Residual                   0    191                     0
A = 0.124145, B = 0.126788, C = 0.314221
order Q(x) = 3.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 2.000000
```



```
Ymin = 1.000000 and Ymax = 7.389056
List of powers for Q(x): [0.250933, 0.691941, 0.948884]
List of powers for D(x): [0.250933, 0.691941, 0.948884, 1.175543, 1.386526, 1.587492]
Fitting exp(x) in range (0.000000, 2.000000)
MSS of errors squared = 2.050448e-11
Corrected MSS of errors squared = 3.551479e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting exp(x) in range (0.000000, 2.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ____________________    __    _____    ______
    (Intercept)     0.85751899695683    0      Inf        0
    x1             -7.39045144686717    0     -Inf        0
    x2              11.2888846447925    0      Inf        0
    x3                             0    0      NaN      NaN
    x4             -10.2052976251332    0     -Inf        0
    x5              6.74405001293931    0      Inf        0
    x6             -1.25247778054921    0     -Inf        0
    x7              9.31677980827483    0      Inf        0
    x8             -17.4881997145372    0     -Inf        0
    x9              10.3200755390663    0      Inf        0
    x10                            0    0      NaN      NaN
    x11            -1.19088243494421    0     -Inf        0
Number of observations: 201, Error degrees of freedom: 191
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                   _____________    ___    ____________________    ___    ______
    Total       15.9274939123251    200    0.0796374695616253
    Model       15.9274939123251      9       1.7697215458139    Inf       0
    Residual                   0    191                     0
A = 0.217772, B = 0.366843, C = 0.034845
order Q(x) = 6.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 2.000000
Ymin = 1.000000 and Ymax = 7.389056
List of powers for Q(x): [0.584615, 0.771411, 0.922853, 1.055993, 1.177438, 1.290575]
List of powers for D(x): [0.584615, 0.771411, 0.922853, 1.055993, 1.177438]
Fitting exp(x) in range (0.000000, 2.000000)
MSS of errors squared = 1.916852e-12
Corrected MSS of errors squared = 3.320085e-12
R-Squared =   1.00000000
```



```
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting exp(x) in range (0.000000, 2.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ________________    __    _____    ______
    (Intercept)     0.864486529549491    0      Inf       0
    x1              -1.52727431920847    0     -Inf       0
    x2               1.90818235366201    0      Inf       0
    x3              -1.34013254327935    0     -Inf       0
    x4              0.341367897267805    0      Inf       0
    x5                1.8115333364399    0      Inf       0
    x6              -2.21725340201366    0     -Inf       0
    x7                1.5362273918055    0      Inf       0
    x8             -0.377137244196887    0     -Inf       0
Number of observations: 201, Error degrees of freedom: 192
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _______________    ___    ________________    ___    ______
    Total       15.9274939123247    200    0.0796374695616236
    Model       15.9274939123247      8      1.99093673904059    Inf       0
    Residual                   0    192                     0
A = 0.344317, B = 0.610584, C = 0.425166
order Q(x) = 4.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 2.000000
Ymin = 1.000000 and Ymax = 7.389056
List of powers for Q(x): [0.954901, 1.632979, 2.003155, 2.301894]
List of powers for D(x): [0.954901, 1.632979, 2.003155, 2.301894]
Fitting exp(x) in range (0.000000, 2.000000)
MSS of errors squared = 5.840689e-12
Corrected MSS of errors squared = 1.011637e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting exp(x) in range (0.000000, 2.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7
Estimated Coefficients:
```


	Estimate	SE	tStat	pValue
(Intercept)	0.868260038704826	0	Inf	0
x1	-74.1381465574504	0	-Inf	0
x2	85.6251554046478	0	Inf	0
x3	-12.560273457972	0	-Inf	0
x4	0.634939586622447	0	Inf	0
x5	75.3168060727542	0	Inf	0
x6	-85.1576069831208	0	-Inf	0
x7	10.4108659014881	0	Inf	0

```
Number of observations: 201, Error degrees of freedom: 193
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```

	SumSq	DF	MeanSq	F	pValue
Total	15.927493912321	200	0.0796374695616051		
Model	15.927493912321	7	2.27535627318872	Inf	0
Residual	0	193	0		

```
A = 1.580159, B = 3.050399
order Q(x) = 4.000000, order D(x) = 3.000000
Xmin = 0.000000 and Xmax = 2.000000
Ymin = 1.000000 and Ymax = 7.389056
List of powers for Q(x): [1.580159, 1.605209, 1.738237, 1.980948]
List of powers for D(x): [1.580159, 1.605209, 1.738237]
Fitting exp(x) in range (0.000000, 2.000000)
MSS of errors squared = 5.111299e-10
Corrected MSS of errors squared = 7.228469e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the exponential function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i+C*(i-1)	0.23	0.193	0.136	Inf	-Inf	1.13495600E-13	6, 6
A+B*sqrt(i)	0.878	0.631		Inf	-Inf	4.01033700E-13	5, 4
A+B*i	0.371	0.1		Inf	-Inf	9.29545600E-13	7, 5
A+B*sqrt(i)+C*(i-1)	0.217	0.366	0.034	Inf	-Inf	1.91685200E-12	6, 5
A+B*sqrt(i)+C*sqrt(i-1)	0.344	0.61	0.425	Inf	-Inf	5.84068900E-12	4, 4
A+B*i+C*sqrt(i-1)	0.124	0.126	0.314	Inf	-Inf	2.05044800E-11	3, 6
A+B*sqrt(i+C)	0.458	0.223	1.902	Inf	-Inf	3.84014300E-11	6, 4
A+B/i	1.789	1.336		Inf	-Inf	3.94904100E-11	3, 6
A+B*log10(i)^4	1.58	3.05		Inf	-Inf	5.11129900E-10	4, 3
A+B*i-C*(i-1)	0.719	0.197	0.194	Inf	-Inf	8.52759200E-09	4, 6

RESULTS FOR THE FRESNEL COSINE

Using Power A+B* i

```
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                    ___________________    ___    _____    ______
    (Intercept)    -9.91497710882089      0      -Inf       0
    x1               106.21155853048      0       Inf       0
    x2             -298.633793179952      0      -Inf       0
    x3               399.18957068255      0       Inf       0
    x4             -290.248552347805      0      -Inf       0
    x5               110.920277753763     0       Inf       0
    x6             -17.5240764331234      0      -Inf       0
    x7               8.05280991038779     0       Inf       0
    x8             -17.8598177114605      0      -Inf       0
    x9               17.6602096790966     0       Inf       0
    x10            -8.44587353747979      0      -Inf       0
    x11              1.59266493804265     0       Inf       0
Number of observations: 451, Error degrees of freedom: 439
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq          DF            MeanSq            F      pValue
                    ___________________    ___    ___________________    ___    ______
    Total          32.734528559633      450    0.0727433967991844
    Model          32.734528559633       11      2.97586623269391     Inf       0
    Residual                     0      439                     0
A = 0.104950, B = 0.207106
order Q(x) = 6.000000, order D(x) = 5.000000
Xmin = 0.500000 and Xmax = 5.000000
Ymin = 0.402386 and Ymax = 0.977438
List of powers for Q(x): [0.312055, 0.519161, 0.726266, 0.933372, 1.140477, 1.347583]
```



```
List of powers for D(x): [0.312055, 0.519161, 0.726266, 0.933372, 1.140477]
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
MSS of errors squared = 5.176640e-02
Corrected MSS of errors squared = 7.320875e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      -0.0267967059191378    0     -Inf       0
    x1              -0.000372160213393195   0     -Inf       0
    x2                 0.0791501590696987   0      Inf       0
    x3                  -1.26675133092731   0     -Inf       0
    x4                   5.33113374246644   0      Inf       0
    x5                  -7.96757028945082   0     -Inf       0
    x6                   3.85120662080338   0      Inf       0
    x7                 0.00328990715993449  0      Inf       0
    x8                  -1.70823875173547   0     -Inf       0
    x9                   51.6992361599366   0      Inf       0
    x10                 -388.682290374592   0     -Inf       0
    x11                  1092.57673112882   0      Inf       0
    x12                 -1270.19661872421   0     -Inf       0
    x13                  517.307890623894   0      Inf       0
Number of observations: 451, Error degrees of freedom: 437
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ___________________    ___    ____________________    ___    ______
    Total       32.7345285596452     450    0.0727433967992115
    Model       32.7345285596452      13      2.51804065843424     Inf       0
    Residual                   0     437                     0
A = 0.135342, B = 1.898560
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 0.500000 and Xmax = 5.000000
Ymin = 0.402386 and Ymax = 0.977438
List of powers for Q(x): [2.033902, 1.084622, 0.768195, 0.609982, 0.515054, 0.451768]
List of powers for D(x): [2.033902, 1.084622, 0.768195, 0.609982, 0.515054, 0.451768,
0.406565]
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
MSS of errors squared = 3.535563e-02
Corrected MSS of errors squared = 5.000042e-02
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      1.56149823367543    0      Inf        0
    x1              -1752.37662378436    0     -Inf        0
    x2               12120.6499418271    0      Inf        0
    x3              -31065.1935696201    0     -Inf        0
    x4               37999.2005766047    0      Inf        0
    x5              -22527.3195799967    0     -Inf        0
    x6               5223.47775717345    0      Inf        0
    x7                929.39658285935    0      Inf        0
    x8              -5714.27354576691    0     -Inf        0
    x9               12110.1891316804    0      Inf        0
    x10             -9819.87033179844    0     -Inf        0
    x11                            0    0      NaN      NaN
    x12              4017.26527487415    0      Inf        0
    x13             -1521.70711214935    0     -Inf        0
Number of observations: 451, Error degrees of freedom: 438
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______
    Total        32.7345285621684     450    0.0727433968048186
    Model        32.7345285621684      12       2.7278773801807     Inf       0
    Residual                    0     438                     0
A = 0.603852, B = 0.319003
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 0.500000 and Xmax = 5.000000
Ymin = 0.402386 and Ymax = 0.977438
List of powers for Q(x): [0.922855, 1.054990, 1.156382, 1.241858, 1.317165, 1.385247]
List of powers for D(x): [0.922855, 1.054990, 1.156382, 1.241858, 1.317165, 1.385247,
1.447855]
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
MSS of errors squared = 1.093302e-01
Corrected MSS of errors squared = 1.546162e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
```



```
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    ________________    __    _____    ______
    (Intercept)      1.11787013591088    0      Inf        0
    x1              -1376.44965438218    0     -Inf        0
    x2               5006.99732935691    0      Inf        0
    x3              -5429.41412108355    0     -Inf        0
    x4                             0    0      NaN      NaN
    x5               3135.36193239336    0      Inf        0
    x6              -1337.61335629779    0     -Inf        0
    x7               1227.50694890899    0      Inf        0
    x8              -4463.97086161884    0     -Inf        0
    x9               4839.40109001105    0      Inf        0
    x10                            0    0      NaN      NaN
    x11             -2793.46772055924    0     -Inf        0
    x12              1191.53054315735    0      Inf        0
Number of observations: 451, Error degrees of freedom: 440
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _________________    ___    ____________________    ___    ______
    Total        32.7345285600157    450    0.0727433968000348
    Model        32.7345285600157     10      3.27345285600157    Inf       0
    Residual                    0    440                     0
A = 0.127111, B = 0.100000, C = 2.000000
order Q(x) = 6.000000, order D(x) = 6.000000
Xmin = 0.500000 and Xmax = 5.000000
Ymin = 0.402386 and Ymax = 0.977438
List of powers for Q(x): [0.300316, 0.327111, 0.350717, 0.372060, 0.391686, 0.409953]
List of powers for D(x): [0.300316, 0.327111, 0.350717, 0.372060, 0.391686, 0.409953]
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
MSS of errors squared = 1.410089e-01
Corrected MSS of errors squared = 2.442345e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
```



```
Estimated Coefficients:
                        Estimate          SE     tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.00208271653970459     0     -Inf       0
    x1              0.00561070044351698     0      Inf       0
    x2             -0.00820631927627485     0     -Inf       0
    x3              0.00755113061566058     0      Inf       0
    x4             -0.00352426717241425     0     -Inf       0
    x5             0.000652251351217215     0      Inf       0
    x6                 2.70762620171167     0      Inf       0
    x7                -4.55713592304627     0     -Inf       0
    x8                 5.71104796196009     0      Inf       0
    x9                 -4.5340485285303     0     -Inf       0
    x10                2.20092861983062     0      Inf       0
    x11               -0.598414721396671    0     -Inf       0
    x12                0.0699957237720008    0      Inf       0
Number of observations: 451, Error degrees of freedom: 438
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______
    Total       32.7345285596265     450    0.0727433967991701
    Model       32.7345285596265      12      2.72787737996888    Inf       0
    Residual                   0     438                         0
A = 0.001708, B = 0.217252, C = 0.215824
order Q(x) = 5.000000, order D(x) = 7.000000
Xmin = 0.500000 and Xmax = 5.000000
Ymin = 0.402386 and Ymax = 0.977438
List of powers for Q(x): [0.218960, 0.652037, 1.085114, 1.518191, 1.951268]
List of powers for D(x): [0.218960, 0.652037, 1.085114, 1.518191, 1.951268, 2.384344,
2.817421]
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
MSS of errors squared = 1.059961e-01
Corrected MSS of errors squared = 1.835906e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE     tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -341.503399574888        0     -Inf       0
    x1              2508.58127505097        0      Inf       0
```



```
    x2              -4249.43677252272    0      -Inf        0
    x3                              0    0       NaN       NaN
    x4               7131.08366661897    0       Inf        0
    x5               -8354.8508397749    0      -Inf        0
    x6               4051.69978910278    0       Inf        0
    x7              -745.573711214686    0      -Inf        0
    x8               5.81198117776259    0       Inf        0
    x9              -9.83687873433242    0      -Inf        0
    x10              6.67356567728235    0       Inf        0
    x11             -1.64867474298715    0      -Inf        0
Number of observations: 451, Error degrees of freedom: 440
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F     pValue
                 _________________    ___    ___________________    ___    _______
    Total        32.7345285595298     450    0.072743396798955
    Model        32.7345285595298      10     3.27345285595298     Inf        0
    Residual                    0     440                    0
A = 0.000000, B = 0.155761, C = 0.047322
order Q(x) = 7.000000, order D(x) = 4.000000
Xmin = 0.500000 and Xmax = 5.000000
Ymin = 0.402386 and Ymax = 0.977438
List of powers for Q(x): [0.155761, 0.264201, 0.372640, 0.481080, 0.589520, 0.697959,
0.806399]
List of powers for D(x): [0.155761, 0.264201, 0.372640, 0.481080]
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
MSS of errors squared = 5.427324e-02
Corrected MSS of errors squared = 9.400402e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                     Estimate          SE    tStat    pValue
                 _________________    ___    _____    _______
    (Intercept)  0.306962816891222    0       Inf        0
    x1           -4.36790657559264    0      -Inf        0
    x2            27.7861584387455    0       Inf        0
    x3           -68.6858584537058    0      -Inf        0
    x4            86.1076246107663    0       Inf        0
    x5           -59.7846378904753    0      -Inf        0
    x6            22.0028561194393    0       Inf        0
    x7           -3.36519748048105    0      -Inf        0
```



```
    x8                12.97121586659    0      Inf       0
    x9               -77.0311541666266  0     -Inf       0
    x10               182.145561195394  0      Inf       0
    x11              -219.529619057064  0     -Inf       0
    x12               146.990333283722  0      Inf       0
    x13              -52.2916772516199  0     -Inf       0
    x14                7.7453387712995  0      Inf       0
Number of observations: 451, Error degrees of freedom: 436
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _________________   ___   ____________________   ___   ______
    Total        32.7345285596264    450   0.0727433967991698
    Model        32.7345285596264     14     2.33818061140189    Inf       0
    Residual                    0    436                     0
A = 0.340000, B = 0.188208, C = 0.229598
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 0.500000 and Xmax = 5.000000
Ymin = 0.402386 and Ymax = 0.977438
List of powers for Q(x): [0.528208, 0.946013, 1.229323, 1.490506, 1.740234, 1.982642,
2.219851]
List of powers for D(x): [0.528208, 0.946013, 1.229323, 1.490506, 1.740234, 1.982642,
2.219851]
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
MSS of errors squared = 3.155190e-02
Corrected MSS of errors squared = 5.464949e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________   ___   _____   _______
    (Intercept)    0.000304239491561854    0      Inf       0
    x1             -0.00079191961944295    0     -Inf       0
    x2              0.000689783116474203   0      Inf       0
    x3             -0.000201089560521787   0     -Inf       0
    x4                  6.50652227828995   0      Inf       0
    x5                -18.1065287985738    0     -Inf       0
    x6                 27.6369972536048    0      Inf       0
    x7                -24.9974843416379    0     -Inf       0
    x8                 13.4495697571746    0      Inf       0
    x9                -3.99535857001071    0     -Inf       0
```



```
    x10                0.506281540867039    0      Inf       0
Number of observations: 451, Error degrees of freedom: 440
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F      pValue
                 _________________    ___    _________________    ___    ______
    Total        32.7345285596273     450    0.0727433967991719
    Model        32.7345285596273      10      3.27345285596273    Inf       0
    Residual                    0     440                      0
A = 0.271738, B = 0.126056, C = 0.333655
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 0.500000 and Xmax = 5.000000
Ymin = 0.402386 and Ymax = 0.977438
List of powers for Q(x): [0.397794, 0.783663, 1.157384]
List of powers for D(x): [0.397794, 0.783663, 1.157384, 1.524816, 1.888229, 2.248787,
2.607182]
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
MSS of errors squared = 1.785756e-01
Corrected MSS of errors squared = 3.093021e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                         Estimate            SE    tStat    pValue
                     ____________________    __    _____    ______
    (Intercept)      3.29688235578368e-05    0      Inf       0
    x1               -0.00042896221848461    0     -Inf       0
    x2               0.000743239175796884    0      Inf       0
    x3               -0.000347105387429809   0     -Inf       0
    x4                    389.291889324968   0      Inf       0
    x5                   -3262.09332891314   0     -Inf       0
    x6                    10618.4481892604   0      Inf       0
    x7                   -17606.7806509429   0     -Inf       0
    x8                    15908.8568316793   0      Inf       0
    x9                   -7489.96887012445   0     -Inf       0
    x10                   1443.24593959247   0      Inf       0
Number of observations: 451, Error degrees of freedom: 440
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F      pValue
                 _________________    ___    _________________    ___    ______
    Total        32.7345285595145     450    0.0727433967989211
```



```
    Model       32.7345285595145     10      3.27345285595145    Inf       0
    Residual                   0    440                     0
A = 0.281671, B = 0.390641, C = 0.000000
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 0.500000 and Xmax = 5.000000
Ymin = 0.402386 and Ymax = 0.977438
List of powers for Q(x): [0.672313, 0.834122, 0.958282]
List of powers for D(x): [0.672313, 0.834122, 0.958282, 1.062954, 1.155172, 1.238543,
1.315211]
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
MSS of errors squared = 1.850127e+00
Corrected MSS of errors squared = 3.204515e+00
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.0277326937055072     0     -Inf       0
    x1                1.10697153994224     0      Inf       0
    x2               -1.2634657484104      0     -Inf       0
    x3               0.200870804020216     0      Inf       0
    x4              -0.0166437353677403     0     -Inf       0
    x5                51.8909714044676     0      Inf       0
    x6               -61.2177697414471     0     -Inf       0
    x7                11.8833554465435     0      Inf       0
    x8               -1.68512941971135     0     -Inf       0
    x9               0.128572167563915     0      Inf       0
Number of observations: 451, Error degrees of freedom: 441
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq           DF           MeanSq            F     pValue
                    ________________    ___    ____________________    ___    ______
    Total       32.7345285596276     450    0.0727433967991725
    Model       32.7345285596276       9      3.63716983995863    Inf       0
    Residual                   0     441                     0
A = 0.162402, B = 2.092596
order Q(x) = 4.000000, order D(x) = 5.000000
Xmin = 0.500000 and Xmax = 5.000000
Ymin = 0.402386 and Ymax = 0.977438
List of powers for Q(x): [0.162402, 0.179586, 0.270844, 0.437346]
List of powers for D(x): [0.162402, 0.179586, 0.270844, 0.437346, 0.661883]
```



```
Fitting FresnelCosine(x) in range (0.500000, 5.000000)
MSS of errors squared = 2.031549e+00
Corrected MSS of errors squared = 2.873044e+00
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the Fresnel cosine. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i+C*sqrt(i-1)	0.34	0.188	0.229	Inf	-Inf	3.15519000E-02	7, 7
A+B/i	0.135	1.898		Inf	-Inf	3.53556300E-02	6, 7
A+B*i	0.104	0.207		Inf	-Inf	5.17664000E-02	6, 5
A+B*i-C*(i-1)	0	0.155	0.047	Inf	-Inf	5.42732400E-02	7, 4
A+B*i+C*(i-1)	0.001	0.217	0.215	Inf	-Inf	1.05996100E-01	5, 7
A+B*sqrt(i)	0.603	0.319		Inf	-Inf	1.09330200E-01	6, 7
A+B*sqrt(i+C)	0.127	0.1	2	Inf	-Inf	1.41008900E-01	6, 6
A+B*sqrt(i)+C*(i-1)	0.271	0.126	0.333	Inf	-Inf	1.78575600E-01	3, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.281	0.39	0	Inf	-Inf	1.85012700E+00	3, 7
A+B*log10(i)^4	0.162	2.092		Inf	-Inf	2.03154900E+00	4, 5

RESULTS FOR THE FRESNEL SINE

Using Power A+B* i

```
Fitting FresnelSine(x) in range (1.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.0199633081919815     0     -Inf       0
    x1               0.192597539814225     0      Inf       0
    x2              -0.377027783135445     0     -Inf       0
    x3               0.276334099986173     0      Inf       0
    x4              -0.071940001120693     0     -Inf       0
    x5               14.0406876325129      0      Inf       0
    x6              -36.1027885301234      0     -Inf       0
    x7               39.124422861945       0      Inf       0
    x8              -20.0918109519404      0     -Inf       0
    x9               4.0294884374717       0      Inf       0
Number of observations: 401, Error degrees of freedom: 391
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ____________________    ___    ______
    Total       21.5526862273345    400    0.0538817155683362
    Model       21.5526862273345      9      2.39474291414827    Inf       0
    Residual                   0    391                     0
A = 0.101148, B = 0.100000
order Q(x) = 4.000000, order D(x) = 5.000000
Xmin = 1.000000 and Xmax = 5.000000
Ymin = 0.310268 and Ymax = 0.894821
List of powers for Q(x): [0.201148, 0.301148, 0.401148, 0.501148]
List of powers for D(x): [0.201148, 0.301148, 0.401148, 0.501148, 0.601148]
Fitting FresnelSine(x) in range (1.000000, 5.000000)
```



```
MSS of errors squared = 4.884967e-01
Corrected MSS of errors squared = 6.908386e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting FresnelSine(x) in range (1.000000, 5.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      -0.0163823178952607    0     -Inf        0
    x1               -0.00344212008978442   0     -Inf        0
    x2                  0.585454002455366   0      Inf        0
    x3                  -7.69173212348999   0     -Inf        0
    x4                   25.9554290416439   0      Inf        0
    x5                  -26.7693614851044   0     -Inf        0
    x6                                  0   0      NaN      NaN
    x7                   7.94003687517091   0      Inf        0
    x8                 0.0873918754668335   0      Inf        0
    x9                  -41.1366656799607   0     -Inf        0
    x10                  1125.89219200053   0      Inf        0
    x11                 -7767.05107511077   0     -Inf        0
    x12                  20302.7019621326   0      Inf        0
    x13                 -22187.6824875151   0     -Inf        0
    x14                  8568.18868042315   0      Inf        0
Number of observations: 401, Error degrees of freedom: 387
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq           DF            MeanSq            F      pValue
                  _________________    ___    ____________________    ___    ______
    Total         21.5526862273447     400    0.0538817155683617
    Model         21.5526862273447      13      1.65789894056498     Inf       0
    Residual                     0     387                     0
A = 0.652121, B = 2.000000
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 1.000000 and Xmax = 5.000000
Ymin = 0.310268 and Ymax = 0.894821
List of powers for Q(x): [2.652121, 1.652121, 1.318788, 1.152121, 1.052121, 0.985455,
0.937836]
List of powers for D(x): [2.652121, 1.652121, 1.318788, 1.152121, 1.052121, 0.985455,
0.937836]
Fitting FresnelSine(x) in range (1.000000, 5.000000)
MSS of errors squared = 4.136913e-01
Corrected MSS of errors squared = 5.850479e-01
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting FresnelSine(x) in range (1.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.0135023530292524     0     -Inf        0
    x1               0.403428264260569     0      Inf        0
    x2              -1.30238616783678      0     -Inf        0
    x3                                     0      NaN       NaN
    x4               5.41514743975448      0      Inf        0
    x5              -8.9908628138072       0     -Inf        0
    x6               5.9221418217137       0      Inf        0
    x7              -1.43396611273514      0     -Inf        0
    x8              66.0497286627924       0      Inf        0
    x9             -478.731163178422       0     -Inf        0
    x10             1446.07377318901       0      Inf        0
    x11            -2282.6103243941        0     -Inf        0
    x12             1988.93547977565       0      Inf        0
    x13            -909.977558482624       0     -Inf        0
    x14             171.260064349308       0      Inf        0
Number of observations: 401, Error degrees of freedom: 387
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F      pValue
                  _______________    ___    ____________________    ___    ______
    Total        21.5526862273292    400    0.0538817155683229
    Model        21.5526862273292     13      1.65789894056378    Inf       0
    Residual                    0    387                     0
A = 0.000000, B = 0.438991
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 1.000000 and Xmax = 5.000000
Ymin = 0.310268 and Ymax = 0.894821
List of powers for Q(x): [0.438991, 0.620827, 0.760354, 0.877982, 0.981613, 1.075304,
1.161461]
List of powers for D(x): [0.438991, 0.620827, 0.760354, 0.877982, 0.981613, 1.075304,
1.161461]
Fitting FresnelSine(x) in range (1.000000, 5.000000)
MSS of errors squared = 2.100689e-01
Corrected MSS of errors squared = 2.970822e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
```



```
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting FresnelSine(x) in range (1.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      9.79494331214366e-06    0      Inf       0
    x1              -7.72714209952317e-05    0     -Inf       0
    x2                0.00012199808072074    0      Inf       0
    x3              -5.45139172516801e-05    0     -Inf       0
    x4                    93.4667750919517    0      Inf       0
    x5                   -709.754783476286    0     -Inf       0
    x6                    2201.40217809475    0      Inf       0
    x7                   -3541.80636422198    0     -Inf       0
    x8                    3133.46109213803    0      Inf       0
    x9                   -1452.1504297356     0     -Inf       0
    x10                   276.381532101008    0      Inf       0
Number of observations: 401, Error degrees of freedom: 390
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______
    Total        21.5526862273335    400    0.0538817155683336
    Model        21.5526862273335     10      2.15526862273335    Inf       0
    Residual                    0    390                     0
A = 0.041569, B = 0.380845, C = 0.000000
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 1.000000 and Xmax = 5.000000
Ymin = 0.310268 and Ymax = 0.894821
List of powers for Q(x): [0.422414, 0.580165, 0.701212]
List of powers for D(x): [0.422414, 0.580165, 0.701212, 0.803259, 0.893164, 0.974445,
1.049190]
Fitting FresnelSine(x) in range (1.000000, 5.000000)
MSS of errors squared = 9.625642e-02
Corrected MSS of errors squared = 1.667210e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting FresnelSine(x) in range (1.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
```



```
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      0.00155659723220927    0      Inf       0
    x1              -0.00803193296387081    0     -Inf       0
    x2                0.0124270512280013    0      Inf       0
    x3              -0.00767295230564861    0     -Inf       0
    x4                0.0017221030420765    0      Inf       0
    x5                  11.5187404356773    0      Inf       0
    x6                  -36.081846512429    0     -Inf       0
    x7                  56.3839722876391    0      Inf       0
    x8                 -51.2469515782966    0     -Inf       0
    x9                  27.5718513101018    0      Inf       0
    x10                -8.18158053596006    0     -Inf       0
    x11                 1.03581372681344    0      Inf       0
Number of observations: 401, Error degrees of freedom: 389
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F      pValue
                 _____________    ___    ____________________    ___    ______
    Total        21.5526862273348    400    0.053881715568337
    Model        21.5526862273348     11     1.95933511157589    Inf       0
    Residual                    0    389                    0
A = 0.161086, B = 0.270000, C = 0.025180
order Q(x) = 4.000000, order D(x) = 7.000000
Xmin = 1.000000 and Xmax = 5.000000
Ymin = 0.310268 and Ymax = 0.894821
List of powers for Q(x): [0.431086, 0.726266, 1.021447, 1.316627]
List of powers for D(x): [0.431086, 0.726266, 1.021447, 1.316627, 1.611807, 1.906988,
2.202168]
Fitting FresnelSine(x) in range (1.000000, 5.000000)
MSS of errors squared = 4.068895e-02
Corrected MSS of errors squared = 7.047533e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting FresnelSine(x) in range (1.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      0.0959327348091791     0      Inf       0
    x1              0.00644054678057224     0      Inf       0
```



```
    x2              -0.0803039584220405     0      -Inf        0
    x3                                0     0       NaN      NaN
    x4               -0.056742516081309     0      -Inf        0
    x5               0.0550079331354889     0       Inf        0
    x6              -0.0248122254706347     0      -Inf        0
    x7              0.00447748528075562     0       Inf        0
    x8              -0.0285858534563755     0      -Inf        0
    x9                0.598234350929931     0       Inf        0
    x10               0.616556134580735     0       Inf        0
    x11              -0.25725025998348      0      -Inf        0
    x12              0.0837247234968887     0       Inf        0
    x13             -0.0126790955996165     0      -Inf        0
Number of observations: 401, Error degrees of freedom: 388
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ____________________    ___    ______
    Total       21.5526862273343    400    0.0538817155683357
    Model       21.5526862273343     12      1.79605718561119    Inf       0
    Residual                   0    388                     0
A = 0.027232, B = 0.119477, C = 0.229694
order Q(x) = 7.000000, order D(x) = 6.000000
Xmin = 1.000000 and Xmax = 5.000000
Ymin = 0.310268 and Ymax = 0.894821
List of powers for Q(x): [0.146709, 0.036493, -0.073724, -0.183940, -0.294157, -
0.404373, -0.514590]
List of powers for D(x): [0.146709, 0.036493, -0.073724, -0.183940, -0.294157, -
0.404373]
Fitting FresnelSine(x) in range (1.000000, 5.000000)
MSS of errors squared = 7.522062e-02
Corrected MSS of errors squared = 1.302859e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting FresnelSine(x) in range (1.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                   ________________    ___    _____    _______
    (Intercept)    -143.013876734168    0      -Inf        0
    x1              3266.44426439132    0       Inf        0
    x2             -21084.2940340669    0      -Inf        0
    x3              51468.3110296687    0       Inf        0
    x4             -63779.5884681584    0      -Inf        0
```



```
    x5              43843.9250037003    0      Inf       0
    x6             -16002.9081166412    0     -Inf       0
    x7              2431.12420512969    0      Inf       0
    x8              11.4363826399603    0      Inf       0
    x9             -40.7705780719786    0     -Inf       0
    x10             56.8101274915188    0      Inf       0
    x11            -34.1244192371873    0     -Inf       0
    x12             7.64847984028767    0      Inf       0
Number of observations: 401, Error degrees of freedom: 388
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ____________________    ___    ______
    Total       21.5526862258593    400    0.0538817155646481
    Model       21.5526862258593     12      1.79605718548827    Inf       0
    Residual                   0    388                     0
A = 0.251318, B = 0.110780, C = 0.113418
order Q(x) = 7.000000, order D(x) = 5.000000
Xmin = 1.000000 and Xmax = 5.000000
Ymin = 0.310268 and Ymax = 0.894821
List of powers for Q(x): [0.362098, 0.586296, 0.744056, 0.890884, 1.032055, 1.169609,
1.304596]
List of powers for D(x): [0.362098, 0.586296, 0.744056, 0.890884, 1.032055]
Fitting FresnelSine(x) in range (1.000000, 5.000000)
MSS of errors squared = 2.476061e-02
Corrected MSS of errors squared = 4.288663e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting FresnelSine(x) in range (1.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)     0.00590514171852965     0      Inf       0
    x1               -0.0179772644278586    0     -Inf       0
    x2                 0.024444648572387    0      Inf       0
    x3               -0.0181860046229547    0     -Inf       0
    x4               0.00684600116611732    0      Inf       0
    x5              -0.00103042794451558    0     -Inf       0
    x6                  4.32918510571595    0      Inf       0
    x7                  -9.8811093386017    0     -Inf       0
    x8                  13.6705272385631    0      Inf       0
    x9                 -11.4952099235621    0     -Inf       0
```



```
    x10                5.81271116402637    0      Inf       0
    x11               -1.63233519294939    0     -Inf       0
    x12                0.196228854052047   0      Inf       0
Number of observations: 401, Error degrees of freedom: 388
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ________________    ___    ____________________    ___    ______
    Total       21.5526862273341    400    0.0538817155683352
    Model       21.5526862273341     12      1.79605718561117    Inf       0
    Residual                   0    388                     0
A = 0.247566, B = 0.100000, C = 0.400000
order Q(x) = 5.000000, order D(x) = 7.000000
Xmin = 1.000000 and Xmax = 5.000000
Ymin = 0.310268 and Ymax = 0.894821
List of powers for Q(x): [0.347566, 0.788987, 1.220771, 1.647566, 2.071172]
List of powers for D(x): [0.347566, 0.788987, 1.220771, 1.647566, 2.071172, 2.492515,
2.912141]
Fitting FresnelSine(x) in range (1.000000, 5.000000)
MSS of errors squared = 8.138955e-02
Corrected MSS of errors squared = 1.409708e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting FresnelSine(x) in range (1.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.0243739962630539     0     -Inf       0
    x1                 39.118101756514     0      Inf       0
    x2               -208.030566729575     0     -Inf       0
    x3                 388.834100249419    0      Inf       0
    x4               -310.191167563703     0     -Inf       0
    x5                 90.2939067545389    0      Inf       0
    x6                 5816.87313080849    0      Inf       0
    x7               -42976.5980415035     0     -Inf       0
    x8                 115237.418402365    0      Inf       0
    x9               -146015.935955769     0     -Inf       0
    x10                89136.840600851     0      Inf       0
    x11              -21197.5981372247     0     -Inf       0
Number of observations: 401, Error degrees of freedom: 389
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```


	SumSq	DF	MeanSq	F	pValue
Total	21.5526862273381	400	0.0538817155683454		
Model	21.5526862273381	11	1.9593351115762	Inf	0
Residual	0	389	0		

A = 0.435139, B = 0.132367, C = 0.000000
order Q(x) = 5.000000, order D(x) = 6.000000
Xmin = 1.000000 and Xmax = 5.000000
Ymin = 0.310268 and Ymax = 0.894821
List of powers for Q(x): [0.567506, 0.622334, 0.664405, 0.699873, 0.731121]
List of powers for D(x): [0.567506, 0.622334, 0.664405, 0.699873, 0.731121, 0.759371]
Fitting FresnelSine(x) in range (1.000000, 5.000000)
MSS of errors squared = 4.057416e-02
Corrected MSS of errors squared = 7.027650e-02
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*log10(i)^4

Fitting FresnelSine(x) in range (1.000000, 5.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	0.00110534534718141	0	Inf	0
x1	-0.151617353583909	0	-Inf	0
x2	0.172619614277478	0	Inf	0
x3	-0.0220985443404026	0	-Inf	0
x4	1594.77001429882	0	Inf	0
x5	-2055.57573831108	0	-Inf	0
x6	577.857855575348	0	Inf	0
x7	-135.888495235071	0	-Inf	0
x8	21.4842789324274	0	Inf	0
x9	-1.64792437404363	0	-Inf	0

Number of observations: 401, Error degrees of freedom: 391
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	21.5526862273251	400	0.0538817155683128		
Model	21.5526862273251	9	2.39474291414723	Inf	0
Residual	0	391	0		

A = 0.775872, B = 2.931965
order Q(x) = 3.000000, order D(x) = 6.000000
Xmin = 1.000000 and Xmax = 5.000000
Ymin = 0.310268 and Ymax = 0.894821


```
List of powers for Q(x): [0.775872, 0.799949, 0.927813]
List of powers for D(x): [0.775872, 0.799949, 0.927813, 1.161100, 1.475703, 1.850888]
Fitting FresnelSine(x) in range (1.000000, 5.000000)
MSS of errors squared = 5.388407e-01
Corrected MSS of errors squared = 7.620358e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the Fresnel sine function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i+C*sqrt(i-1)	0.251	0.11	0.113	Inf	-Inf	2.47606100E-02	7, 5
A+B*sqrt(i)+C*sqrt(i-1)	0.435	0.132	0	Inf	-Inf	4.05741600E-02	5, 6
A+B*i+C*(i-1)	0.161	0.27	0.025	Inf	-Inf	4.06889500E-02	4, 7
A+B*i-C*(i-1)	0.027	0.119	0.229	Inf	-Inf	7.52206200E-02	7, 6
A+B*sqrt(i)+C*(i-1)	0.247	0.1	0.4	Inf	-Inf	8.13895500E-02	5, 7
A+B*sqrt(i+C)	0.041	0.38	0	Inf	-Inf	9.62564200E-02	3, 7
A+B*sqrt(i)	0	0.438		Inf	-Inf	2.10068900E-01	7, 7
A+B/i	0.652	2		Inf	-Inf	4.13691300E-01	7, 7
A+B*i	0.101	0.1		Inf	-Inf	4.88496700E-01	4, 5
A+B*log10(i)^4	0.775	2.931		Inf	-Inf	5.38840700E-01	3, 6

RESULTS FOR THE BESSEL J0(X)

Using Power A+B* i

```
Fitting J0(x) in range (2.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______
    (Intercept)    0.0108557981528733     0      Inf       0
    x1             -0.509941759867958     0     -Inf       0
    x2               1.50893936799137     0      Inf       0
    x3              -1.74979639310348     0     -Inf       0
    x4              0.928555771787081     0      Inf       0
    x5             -0.188612808296536     0     -Inf       0
    x6               90.1028290437102     0      Inf       0
    x7              -342.756386228218     0     -Inf       0
    x8               545.842766503611     0      Inf       0
    x9              -448.223368434618     0     -Inf       0
    x10              188.166425928376     0      Inf       0
    x11             -32.132266802992     0     -Inf       0
Number of observations: 281, Error degrees of freedom: 269
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                        SumSq          DF            MeanSq            F      pValue
                    ____________________    ___    ____________________    ___    ______
    Total          16.0887001722623     280    0.0574596434723654
    Model          16.0887001722623      11       1.4626091065693     Inf       0
    Residual                      0     269                     0
A = 0.273659, B = 0.100000
order Q(x) = 5.000000, order D(x) = 6.000000
Xmin = 2.000000 and Xmax = 30.000000
Ymin = -0.402556 and Ymax = 0.300079
List of powers for Q(x): [0.373659, 0.473659, 0.573659, 0.673659, 0.773659]
```



```
List of powers for D(x): [0.373659, 0.473659, 0.573659, 0.673659, 0.773659, 0.873659]
Fitting J0(x) in range (2.000000, 30.000000)
MSS of errors squared = 1.836766e-01
Corrected MSS of errors squared = 2.597579e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting J0(x) in range (2.000000, 30.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)        1.4387684459049      0      Inf        0
    x1              -0.0269606686123054     0     -Inf        0
    x2                 1.45572233207552     0      Inf        0
    x3                -7.03197122487377     0     -Inf        0
    x4                                0     0      NaN      NaN
    x5                 18.3429704233145     0      Inf        0
    x6                                0     0      NaN      NaN
    x7                -14.1785293077139     0     -Inf        0
    x8                0.018739802539286     0      Inf        0
    x9                -1.01181337426702     0     -Inf        0
    x10                4.88758961430678     0      Inf        0
    x11                               0     0      NaN      NaN
    x12               -12.7492286744598     0     -Inf        0
    x13                               0     0      NaN      NaN
    x14                9.85471263182176     0      Inf        0
Number of observations: 281, Error degrees of freedom: 270
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F     pValue
                  _______________    ___    ___________________    ___    ______
    Total       16.0887001722663     280    0.0574596434723796
    Model       16.0887001722663      10      1.60887001722663    Inf       0
    Residual                   0     270                     0
A = 0.000000, B = 0.322650
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 2.000000 and Xmax = 30.000000
Ymin = -0.402556 and Ymax = 0.300079
List of powers for Q(x): [0.322650, 0.161325, 0.107550, 0.080662, 0.064530, 0.053775,
0.046093]
List of powers for D(x): [0.322650, 0.161325, 0.107550, 0.080662, 0.064530, 0.053775,
0.046093]
Fitting J0(x) in range (2.000000, 30.000000)
```



```
MSS of errors squared = 4.953191e-01
Corrected MSS of errors squared = 7.004870e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting J0(x) in range (2.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -1.98020996898144e-05    0     -Inf       0
    x1              0.000391140171176068    0      Inf       0
    x2             -0.000713930535465916    0     -Inf       0
    x3              0.000342266714661437    0      Inf       0
    x4                  1794.53151426449    0      Inf       0
    x5                 -16078.5098191118    0     -Inf       0
    x6                  54355.4908420868    0      Inf       0
    x7                 -92478.8689027809    0     -Inf       0
    x8                  85177.7241881285    0      Inf       0
    x9                 -40710.9884601647    0     -Inf       0
    x10                   7941.620637732    0      Inf       0
Number of observations: 281, Error degrees of freedom: 270
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ________________    ___    ____________________    ___    ______
    Total       16.0887001716783    280    0.0574596434702798
    Model       16.0887001716783     10      1.60887001716783    Inf       0
    Residual                   0    270                     0
A = 0.653106, B = 0.409439
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 2.000000 and Xmax = 30.000000
Ymin = -0.402556 and Ymax = 0.300079
List of powers for Q(x): [1.062544, 1.232139, 1.362274]
List of powers for D(x): [1.062544, 1.232139, 1.362274, 1.471983, 1.568638, 1.656021,
1.736378]
Fitting J0(x) in range (2.000000, 30.000000)
MSS of errors squared = 6.335017e-01
Corrected MSS of errors squared = 8.959067e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```


Using Power A+B*sqrt(i+C)

```
Fitting J0(x) in range (2.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ________________    __    _____    ______
    (Intercept)      1.65611902817141    0      Inf        0
    x1              -2726.84473497512    0     -Inf        0
    x2               9991.56337690374    0      Inf        0
    x3              -10894.6557313063    0     -Inf        0
    x4                             0    0      NaN      NaN
    x5               6342.68749969711    0      Inf        0
    x6              -2714.40652929272    0     -Inf        0
    x7               954.872853503251    0      Inf        0
    x8              -2343.92670892559    0     -Inf        0
    x9                             0    0      NaN      NaN
    x10              3040.70652747675    0      Inf        0
    x11                            0    0      NaN      NaN
    x12             -3125.13750859662    0     -Inf        0
    x13              1474.4848365083     0      Inf        0
Number of observations: 281, Error degrees of freedom: 270
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq          DF           MeanSq           F     pValue
                    _______________    ___    ___________________    ___    ______
    Total          16.0887001724565    280    0.0574596434730589
    Model          16.0887001724565     10      1.60887001724565    Inf       0
    Residual                      0    270                     0
A = 0.155487, B = 0.100000, C = 2.000000
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 2.000000 and Xmax = 30.000000
Ymin = -0.402556 and Ymax = 0.300079
List of powers for Q(x): [0.328693, 0.355487, 0.379094, 0.400436, 0.420063, 0.438330]
List of powers for D(x): [0.328693, 0.355487, 0.379094, 0.400436, 0.420063, 0.438330,
0.455487]
Fitting J0(x) in range (2.000000, 30.000000)
MSS of errors squared = 3.968634e-02
Corrected MSS of errors squared = 6.873875e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting J0(x) in range (2.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
```



```
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ________________    __    _____    ______
    (Intercept)      1.46465992652267    0      Inf        0
    x1               -8.3980059720893    0     -Inf        0
    x2               12.6362742131101    0      Inf        0
    x3                              0    0      NaN      NaN
    x4              -17.4493141299223    0     -Inf        0
    x5               18.4242218630401    0      Inf        0
    x6              -7.96403387215698    0     -Inf        0
    x7               1.28619798484998    0      Inf        0
    x8               6.79315735396122    0      Inf        0
    x9              -14.7894389257613    0     -Inf        0
    x10              14.9325920659514    0      Inf        0
    x11             -7.38439440346922    0     -Inf        0
    x12              1.44808390113138    0      Inf        0
Number of observations: 281, Error degrees of freedom: 269
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ________________    ___    ______
    Total        16.0887001722657    280    0.0574596434723776
    Model        16.0887001722657     11      1.46260910656961    Inf       0
    Residual                    0    269                     0
A = 0.028511, B = 0.100000, C = 0.000000
order Q(x) = 7.000000, order D(x) = 5.000000
Xmin = 2.000000 and Xmax = 30.000000
Ymin = -0.402556 and Ymax = 0.300079
List of powers for Q(x): [0.128511, 0.228511, 0.328511, 0.428511, 0.528511, 0.628511,
0.728511]
List of powers for D(x): [0.128511, 0.228511, 0.328511, 0.428511, 0.528511]
Fitting J0(x) in range (2.000000, 30.000000)
MSS of errors squared = 7.406704e-02
Corrected MSS of errors squared = 1.282879e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting J0(x) in range (2.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                          Estimate          SE    tStat    pValue
                       ____________________    __    _____    ______
    (Intercept)    -0.00736039619646654    0     -Inf        0
```

```
    x1                0.650371066555557    0      Inf        0
    x2                -2.11903780601303    0     -Inf        0
    x3                 2.68164204073458    0      Inf        0
    x4                -1.54608507544208    0     -Inf        0
    x5                0.340470168990357    0      Inf        0
    x6                 82.6031281586599    0      Inf        0
    x7                -198.133603624848    0     -Inf        0
    x8                                0    0      NaN      NaN
    x9                 464.587804904394    0      Inf        0
    x10               -602.940103803301    0     -Inf        0
    x11                317.531065478777    0      Inf        0
    x12               -62.6482911130678    0     -Inf        0
Number of observations: 281, Error degrees of freedom: 269
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ____________________    ___    ______
    Total       16.0887001722828    280    0.0574596434724387
    Model       16.0887001722828     11      1.46260910657117    Inf       0
    Residual                   0    269                     0
A = 0.042719, B = 0.177161, C = 0.126788
order Q(x) = 5.000000, order D(x) = 7.000000
Xmin = 2.000000 and Xmax = 30.000000
Ymin = -0.402556 and Ymax = 0.300079
List of powers for Q(x): [0.219879, 0.270252, 0.320625, 0.370998, 0.421371]
List of powers for D(x): [0.219879, 0.270252, 0.320625, 0.370998, 0.421371, 0.471744,
0.522117]
Fitting J0(x) in range (2.000000, 30.000000)
MSS of errors squared = 2.707068e-01
Corrected MSS of errors squared = 4.688779e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting J0(x) in range (2.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -2.07619171403035e-05    0     -Inf       0
    x1              3.33880216470946e-05    0      Inf       0
    x2             -1.96296160658379e-05    0     -Inf       0
    x3              6.95551140560964e-06    0      Inf       0
    x4                     2.901193528382    0      Inf       0
    x5                   -10.4847827376202    0     -Inf       0
```



```
    x6                  23.3736308650673    0      Inf       0
    x7                 -27.5031092406052    0     -Inf       0
    x8                  18.190789780002     0      Inf       0
    x9                 -6.4247686981436     0     -Inf       0
    x10                0.947046525662645    0      Inf       0
Number of observations: 281, Error degrees of freedom: 270
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                _________________    ___    ___________________    ___    ______
    Total       16.0887001722667     280    0.0574596434723812
    Model       16.0887001722667      10      1.60887001722667    Inf       0
    Residual                   0     270                     0
A = 0.029265, B = 0.194738, C = 0.326291
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 2.000000 and Xmax = 30.000000
Ymin = -0.402556 and Ymax = 0.300079
List of powers for Q(x): [0.224002, 0.745032, 1.074924]
List of powers for D(x): [0.224002, 0.745032, 1.074924, 1.373369, 1.655536, 1.927301,
2.191676]
Fitting J0(x) in range (2.000000, 30.000000)
MSS of errors squared = 5.894056e-01
Corrected MSS of errors squared = 1.020880e+00
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting J0(x) in range (2.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ___________________    __    _____    ______
    (Intercept)    -0.274204652309771     0     -Inf       0
    x1               2.62179857840422     0      Inf       0
    x2              -7.97195395778391     0     -Inf       0
    x3               11.6826006320189     0      Inf       0
    x4              -9.15807779444823     0     -Inf       0
    x5               3.71541958353227     0      Inf       0
    x6              -0.615582840437399    0     -Inf       0
    x7               12.3592505864213     0      Inf       0
    x8              -47.2252626550115     0     -Inf       0
    x9               89.477112110509      0      Inf       0
    x10             -96.1642250825254     0     -Inf       0
    x11              59.9640067606875     0      Inf       0
    x12             -20.3130297614218     0     -Inf       0
```



```
    x13              2.90214826976337    0      Inf       0
Number of observations: 281, Error degrees of freedom: 267
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ________________    ___    ___________________    ___    ______
    Total       16.088700172266     280    0.0574596434723785
    Model       16.088700172266      13      1.23759232094354    Inf       0
    Residual                  0     267                        0
A = 0.151049, B = 0.386692, C = 0.225867
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 2.000000 and Xmax = 30.000000
Ymin = -0.402556 and Ymax = 0.300079
List of powers for Q(x): [0.537740, 0.923781, 1.272553, 1.602034, 1.919187, 2.227583]
List of powers for D(x): [0.537740, 0.923781, 1.272553, 1.602034, 1.919187, 2.227583,
2.529343]
Fitting J0(x) in range (2.000000, 30.000000)
MSS of errors squared = 1.282079e-01
Corrected MSS of errors squared = 2.220627e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting J0(x) in range (2.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate            SE     tStat     pValue
                    ____________________    ___    _____    _______
    (Intercept)    -6.04589968231978e-06    0      -Inf       0
    x1              1.03997640192368e-05    0       Inf       0
    x2             -8.20826327732458e-06    0      -Inf       0
    x3              3.83925745054357e-06    0       Inf       0
    x4                  3.99031120276622    0       Inf       0
    x5                 -43.4577619180545    0      -Inf       0
    x6                  171.823906052685    0       Inf       0
    x7                 -315.652254766557    0      -Inf       0
    x8                  304.296042310954    0       Inf       0
    x9                 -149.834967932201    0      -Inf       0
    x10                 29.8347250575246    0       Inf       0
Number of observations: 281, Error degrees of freedom: 270
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ________________    ___    ___________________    ___    ______
    Total       16.0887001722744    280    0.0574596434724087
```



```
    Model       16.0887001722744     10      1.60887001722744    Inf       0
    Residual                   0    270                     0
A = 0.138864, B = 0.100000, C = 0.460700
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 2.000000 and Xmax = 30.000000
Ymin = -0.402556 and Ymax = 0.300079
List of powers for Q(x): [0.238864, 0.740985, 0.963597]
List of powers for D(x): [0.238864, 0.740985, 0.963597, 1.136820, 1.283871, 1.413970,
1.531920]
Fitting J0(x) in range (2.000000, 30.000000)
MSS of errors squared = 3.885313e-01
Corrected MSS of errors squared = 6.729559e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting J0(x) in range (2.000000, 30.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)                        0    0      NaN      NaN
    x1              2.11600363760224e-05    0      Inf        0
    x2              -2.8676260105832e-05    0     -Inf        0
    x3              1.04231363197834e-05    0      Inf        0
    x4             -3.83451521381125e-06    0     -Inf        0
    x5              1.12361251890172e-06    0      Inf        0
    x6             -2.15680816053948e-07    0     -Inf        0
    x7              1.96709239546938e-08    0      Inf        0
    x8                 0.999985116674498    0      Inf        0
    x9              2.01696661275119e-05    0      Inf        0
    x10            -7.33054458816081e-06    0     -Inf        0
    x11             2.6963824352848e-06     0      Inf        0
    x12            -7.89957314862309e-07    0     -Inf        0
    x13             1.51602585460459e-07    0      Inf        0
    x14            -1.38237453509546e-08    0     -Inf        0
Number of observations: 281, Error degrees of freedom: 267
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                 _______________    ___    ___________________    ___    ______
    Total       16.0887001722662    280    0.0574596434723794
    Model       16.0887001722662     13     1.23759232094356    Inf       0
    Residual                   0    267                     0
A = 0.000000, B = 2.450515
```



```
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 2.000000 and Xmax = 30.000000
Ymin = -0.402556 and Ymax = 0.300079
List of powers for Q(x): [0.000000, 0.020123, 0.126991, 0.321971, 0.584913, 0.898490,
1.249930]
List of powers for D(x): [0.000000, 0.020123, 0.126991, 0.321971, 0.584913, 0.898490,
1.249930]
Fitting J0(x) in range (2.000000, 30.000000)
MSS of errors squared = 4.142190e-01
Corrected MSS of errors squared = 5.857941e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the Bessel $J_0(x)$ function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i+C)	0.155	0.1	2	Inf	-Inf	3.96863400E-02	6, 7
A+B*i+C*(i-1)	0.028	0.1	0	Inf	-Inf	7.40670400E-02	7, 5
A+B*sqrt(i)+C*(i-1)	0.151	0.386	0.225	Inf	-Inf	1.28207900E-01	6, 7
A+B*i	0.273	0.1		Inf	-Inf	1.83676600E-01	5, 6
A+B*i-C*(i-1)	0.042	0.177	0.126	Inf	-Inf	2.70706800E-01	5, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.138	0.1	0.46	Inf	-Inf	3.88531300E-01	3, 7
A+B*log10(i)^4	0	2.45		Inf	-Inf	4.14219000E-01	7, 7
A+B/i	0	0.322		Inf	-Inf	4.95319100E-01	7, 7
A+B*i+C*sqrt(i-1)	0.029	0.194	0.326	Inf	-Inf	5.89405600E-01	3, 7
A+B*sqrt(i)	0.653	0.409		Inf	-Inf	6.33501700E-01	3, 7

RESULTS FOR THE BESSEL J1(X)

Using Power A+B* i

```
Fitting J1(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate           SE     tStat     pValue
                    ____________________    ___    _____    _______
    (Intercept)      0.000100786177943514    0      Inf        0
    x1              -0.000556970852228897    0     -Inf        0
    x2               0.000696179606291272    0      Inf        0
    x3              -0.000239594044725246    0     -Inf        0
    x4                    31.0978553992936    0      Inf        0
    x5                   -117.195862385256    0     -Inf        0
    x6                    202.693689230154    0      Inf        0
    x7                   -197.603280870701    0     -Inf        0
    x8                    112.306471426531    0      Inf        0
    x9                   -34.9018843599902    0     -Inf        0
    x10                   4.60301127119509    0      Inf        0
Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F      pValue
                  _____________    ___    ____________________    ___    _______
    Total        14.5140717819757    300    0.0483802392732525
    Model        14.5140717819757     10      1.45140717819757    Inf        0
    Residual                    0    290                     0
A = 0.359556, B = 0.280068
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.345961 and Ymax = 0.581517
List of powers for Q(x): [0.639625, 0.919693, 1.199761]
```



```
List of powers for D(x): [0.639625, 0.919693, 1.199761, 1.479830, 1.759898, 2.039967,
2.320035]
Fitting J1(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.802941e-01
Corrected MSS of errors squared = 2.549743e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting J1(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      1.39047188315943e-08    0      Inf        0
    x1              -6.13849808655601e-09    0     -Inf        0
    x2               5.22768049932315e-08    0      Inf        0
    x3              -6.00426068884906e-08    0     -Inf        0
    x4               -0.00637987270736235    0     -Inf        0
    x5                   1.02247229748611    0      Inf        0
    x6                  -14.7653155644573    0     -Inf        0
    x7                   55.6990438869315    0      Inf        0
    x8                  -64.0127425081642    0     -Inf        0
    x9                                  0    0      NaN      NaN
    x10                  23.0629217609109    0      Inf        0
Number of observations: 301, Error degrees of freedom: 291
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______
    Total       14.5140717819763     300    0.0483802392732545
    Model       14.5140717819763       9      1.61267464244182    Inf       0
    Residual                   0     291                     0
A = 0.000000, B = 0.239687
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.345961 and Ymax = 0.581517
List of powers for Q(x): [0.239687, 0.119843, 0.079896]
List of powers for D(x): [0.239687, 0.119843, 0.079896, 0.059922, 0.047937, 0.039948,
0.034241]
Fitting J1(x) in range (0.000000, 30.000000)
MSS of errors squared = Inf
Corrected MSS of errors squared = Inf
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
```



```
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting J1(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      1.23718598875581    0      Inf        0
    x1              -715.304402998271    0     -Inf        0
    x2               4341.82048886536    0      Inf        0
    x3              -9142.98068737753    0     -Inf        0
    x4               7386.63193745349    0      Inf        0
    x5                             0    0      NaN      NaN
    x6              -3011.21079644771    0     -Inf        0
    x7                 1139.806274642    0      Inf        0
    x8                694.262903274916    0      Inf        0
    x9              -4743.88199429382    0     -Inf        0
    x10              12087.3295680596    0      Inf        0
    x11             -14737.424292104    0     -Inf        0
    x12              8720.89191411847    0      Inf        0
    x13             -2020.17809913993    0     -Inf        0
Number of observations: 301, Error degrees of freedom: 288
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF            MeanSq            F      pValue
                    _____________    ___    ____________________    ___    ______
    Total        14.514071782016    300    0.0483802392733868
    Model        14.514071782016     12      1.20950598183467    Inf        0
    Residual                   0    288                     0
A = 0.386682, B = 0.261732
order Q(x) = 7.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.345961 and Ymax = 0.581517
List of powers for Q(x): [0.648414, 0.756827, 0.840015, 0.910146, 0.971933, 1.027792,
1.079160]
List of powers for D(x): [0.648414, 0.756827, 0.840015, 0.910146, 0.971933, 1.027792]
Fitting J1(x) in range (0.000000, 30.000000)
MSS of errors squared = 7.017163e-02
Corrected MSS of errors squared = 9.923767e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)


```
Fitting J1(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.0688214433177956     0     -Inf       0
    x1                27.9797955619632     0      Inf       0
    x2               -78.8452646462715     0     -Inf       0
    x3                               0     0      NaN      NaN
    x4                122.426850356975     0      Inf       0
    x5                               0     0      NaN      NaN
    x6               -140.882286374654     0     -Inf       0
    x7                69.3897265480911     0      Inf       0
    x8                2060.41070904657     0      Inf       0
    x9               -11820.9011724239     0     -Inf       0
    x10               19467.5765922117     0      Inf       0
    x11                              0     0      NaN      NaN
    x12              -29503.4105748356     0     -Inf       0
    x13               27879.2882174062     0      Inf       0
    x14              -8081.96377140737     0     -Inf       0
Number of observations: 301, Error degrees of freedom: 289
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F      pValue
                  _______________    ___    ________________    ___    ______
    Total        14.5140717820848    300    0.048380239273616
    Model        14.5140717820848     11     1.31946107109862    Inf       0
    Residual                    0    289                     0
A = 0.250362, B = 0.100000, C = 0.000000
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.345961 and Ymax = 0.581517
List of powers for Q(x): [0.350362, 0.391784, 0.423567, 0.450362, 0.473969, 0.495311,
0.514937]
List of powers for D(x): [0.350362, 0.391784, 0.423567, 0.450362, 0.473969, 0.495311,
0.514937]
Fitting J1(x) in range (0.000000, 30.000000)
MSS of errors squared = 5.439688e-01
Corrected MSS of errors squared = 9.421816e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting J1(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
```



```
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      0.0018277003221808     0      Inf       0
    x1              -0.0180496752962457     0     -Inf       0
    x2               0.0425334933372349     0      Inf       0
    x3              -0.0436410888598072     0     -Inf       0
    x4               0.0214833932982054     0      Inf       0
    x5              -0.00415334540050292    0     -Inf       0
    x6                  15.4639879691859    0      Inf       0
    x7                  -51.520218533232    0     -Inf       0
    x8                  82.8506277030181    0      Inf       0
    x9                 -76.5945905366219    0     -Inf       0
    x10                 41.6817047230348    0      Inf       0
    x11                -12.4698455407057    0     -Inf       0
    x12                 1.58833386285485    0      Inf       0
Number of observations: 301, Error degrees of freedom: 288
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______
    Total       14.5140717819765     300    0.0483802392732551
    Model       14.5140717819765      12      1.20950598183138     Inf       0
    Residual                   0     288                     0
A = 0.389857, B = 0.195090, C = 0.149979
order Q(x) = 5.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.345961 and Ymax = 0.581517
List of powers for Q(x): [0.584946, 0.930015, 1.275083, 1.620152, 1.965220]
List of powers for D(x): [0.584946, 0.930015, 1.275083, 1.620152, 1.965220, 2.310289,
2.655357]
Fitting J1(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.207201e-01
Corrected MSS of errors squared = 2.090933e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting J1(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
```



```
    (Intercept)    -0.0275458697405825    0      -Inf       0
    x1              -0.452706994455724    0      -Inf       0
    x2               1.62684157309071     0       Inf       0
    x3               -2.004766947947      0      -Inf       0
    x4              0.858178378903648     0       Inf       0
    x5               12.2930277359377     0       Inf       0
    x6              -58.8101706585017     0      -Inf       0
    x7               108.548421095128     0       Inf       0
    x8              -92.8033168497321     0      -Inf       0
    x9               31.7720385397394     0       Inf       0
Number of observations: 301, Error degrees of freedom: 291
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F      pValue
                  _______________    ___    ____________________    ___    _______
    Total        14.5140717819758    300    0.0483802392732528
    Model        14.5140717819758      9      1.61267464244176    Inf        0
    Residual                    0    291                     0
A = 0.479452, B = 0.125135, C = 0.211940
order Q(x) = 4.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.345961 and Ymax = 0.581517
List of powers for Q(x): [0.604587, 0.517782, 0.430977, 0.344172]
List of powers for D(x): [0.604587, 0.517782, 0.430977, 0.344172, 0.257367]
Fitting J1(x) in range (0.000000, 30.000000)
MSS of errors squared = 4.067267e-01
Corrected MSS of errors squared = 7.044714e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting J1(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    _________________    ___    _____    _______
    (Intercept)      0.0011708564951521    0      Inf        0
    x1              -0.00426602063299722   0     -Inf        0
    x2               0.00930823522465774   0      Inf        0
    x3              -0.0109183523690333    0     -Inf        0
    x4               0.00595703119508192   0      Inf        0
    x5              -0.00125076605789509   0     -Inf        0
    x6                  4.30649071036499   0      Inf        0
    x7                 -13.0042223416379   0     -Inf        0
    x8                  22.4319851292373   0      Inf        0
```



```
    x9                -21.5895422878035    0      -Inf        0
    x10                12.0410492656431    0       Inf        0
    x11               -3.65465042753927    0      -Inf        0
    x12               0.468889223462207    0       Inf        0
Number of observations: 301, Error degrees of freedom: 288
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ________________    ___    ____________________    ___    _______
    Total       14.5140717819763    300    0.0483802392732545
    Model       14.5140717819763     12      1.20950598183136    Inf        0
    Residual                   0    288                     0
A = 0.040135, B = 0.390000, C = 0.216760
order Q(x) = 5.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.345961 and Ymax = 0.581517
List of powers for Q(x): [0.430135, 1.036895, 1.516680, 1.975575, 2.423656]
List of powers for D(x): [0.430135, 1.036895, 1.516680, 1.975575, 2.423656, 2.864826,
3.301087]
Fitting J1(x) in range (0.000000, 30.000000)
MSS of errors squared = 2.644013e-01
Corrected MSS of errors squared = 4.579565e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting J1(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                      Estimate          SE     tStat     pValue
                  _________________    ___    _____    _______
    (Intercept)    1.70245454837204    0       Inf         0
    x1            -340.127251060325    0      -Inf         0
    x2             2172.4485456768     0       Inf         0
    x3            -5324.26375488589    0      -Inf         0
    x4             6318.81075610748    0       Inf         0
    x5            -3663.36813436315    0      -Inf         0
    x6             834.797384030083    0       Inf         0
    x7             150.592351471055    0       Inf         0
    x8            -753.154408766448    0      -Inf         0
    x9             1138.25789784809    0       Inf         0
    x10                           0    0       NaN       NaN
    x11           -1539.37079208528    0      -Inf         0
    x12            1394.43766441565    0       Inf         0
    x13           -389.76271291885     0      -Inf         0
```



```
Number of observations: 301, Error degrees of freedom: 288
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ________________    ___    ____________________    ___    ______
    Total       14.5140717820148    300    0.0483802392733828
    Model       14.5140717820148     12      1.20950598183457    Inf       0
    Residual                   0    288                     0
A = 0.203187, B = 0.290057, C = 0.000000
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.345961 and Ymax = 0.581517
List of powers for Q(x): [0.493244, 0.613390, 0.705581, 0.783301, 0.851774, 0.913679]
List of powers for D(x): [0.493244, 0.613390, 0.705581, 0.783301, 0.851774, 0.913679,
0.970606]
Fitting J1(x) in range (0.000000, 30.000000)
MSS of errors squared = 4.226818e-02
Corrected MSS of errors squared = 7.321064e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting J1(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                   ________________    ___    _____    ______
    (Intercept)     1.40657584828276    0       Inf       0
    x1             -8.09743583484124    0      -Inf       0
    x2              50.1041862293858    0       Inf       0
    x3             -126.372322985057    0      -Inf       0
    x4              148.594396492689    0       Inf       0
    x5             -84.3939134916652    0      -Inf       0
    x6              18.758515311682     0       Inf       0
    x7              5.76351612572013    0       Inf       0
    x8             -35.7137541542332    0      -Inf       0
    x9              90.1432809445096    0       Inf       0
    x10            -106.056654770801    0      -Inf       0
    x11             60.2648057961379    0       Inf       0
    x12            -13.4011949877696    0      -Inf       0
Number of observations: 301, Error degrees of freedom: 288
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ________________    ___    ____________________    ___    ______
```



```
    Total       14.5140717819758    300    0.0483802392732526
    Model       14.5140717819758     12      1.20950598183131    Inf       0
    Residual                   0    288                     0
A = 0.039587, B = 0.446523, C = 0.423918
order Q(x) = 6.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.345961 and Ymax = 0.581517
List of powers for Q(x): [0.486110, 1.094984, 1.412498, 1.666881, 1.885879, 2.081251]
List of powers for D(x): [0.486110, 1.094984, 1.412498, 1.666881, 1.885879, 2.081251]
Fitting J1(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.301360e-01
Corrected MSS of errors squared = 2.254021e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting J1(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)    -0.212105407930097    0     -Inf       0
    x1               106.143217713856    0      Inf       0
    x2              -139.382452972011    0     -Inf       0
    x3               44.473656750622     0      Inf       0
    x4              -14.0800412861148    0     -Inf       0
    x5               3.63227161108978    0      Inf       0
    x6              -0.626491253819869    0     -Inf       0
    x7              0.0519449369049637    0      Inf       0
    x8               181.572945043911    0      Inf       0
    x9              -226.276637795373    0     -Inf       0
    x10              55.4788353054938    0      Inf       0
    x11             -11.2592192065669    0     -Inf       0
    x12              1.59814891975729    0      Inf       0
    x13             -0.114072330744621    0     -Inf       0
Number of observations: 301, Error degrees of freedom: 287
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                        SumSq          DF           MeanSq            F     pValue
                    ____________________    ___    ____________________    ___    _______
    Total       14.5140717819797    300    0.0483802392732657
    Model       14.5140717819797     13      1.11646706015229    Inf       0
    Residual                   0    287                     0
A = 0.429845, B = 3.500000
order Q(x) = 7.000000, order D(x) = 6.000000
```



```
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.345961 and Ymax = 0.581517
List of powers for Q(x): [0.429845, 0.458586, 0.611222, 0.889707, 1.265260, 1.713133,
2.215084]
List of powers for D(x): [0.429845, 0.458586, 0.611222, 0.889707, 1.265260, 1.713133]
Fitting J1(x) in range (0.000000, 30.000000)
MSS of errors squared = 2.371787e+00
Corrected MSS of errors squared = 3.354213e+00
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the Bessel $J_1(x)$ function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i)+C*(i-1)	0.203	0.29	0	Inf	-Inf	4.22681800E-02	6, 7
A+B*sqrt(i)	0.386	0.261		Inf	-Inf	7.01716300E-02	7, 6
A+B*i+C*(i-1)	0.389	0.195	0.149	Inf	-Inf	1.20720100E-01	5, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.039	0.446	0.423	Inf	-Inf	1.30136000E-01	6, 6
A+B*i	0.359	0.28		Inf	-Inf	1.80294100E-01	3, 7
A+B*i+C*sqrt(i-1)	0.04	0.39	0.216	Inf	-Inf	2.64401300E-01	5, 7
A+B*i-C*(i-1)	0.479	0.125	0.211	Inf	-Inf	4.06726700E-01	4, 5
A+B*sqrt(i+C)	0.25	0.1	0	Inf	-Inf	5.43968800E-01	7, 7
A+B*log10(i)^4	0.429	3.5		Inf	-Inf	2.37178700E+00	7, 6
A+B/i	0	0.239		Inf	-Inf	Inf	3, 7

RESULTS FOR THE BESSEL J2(X)

Using Power A+B* i

```
Fitting J2(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate           SE     tStat     pValue
                    ____________________    ___    _____    _______
    (Intercept)    -1.65516091478157e-05    0      -Inf       0
    x1              0.000432092092690906    0       Inf       0
    x2               -0.0010991373279836    0      -Inf       0
    x3              0.000985943034534821    0       Inf       0
    x4             -0.000302333610518223    0      -Inf       0
    x5                  19.5898051462971    0       Inf       0
    x6                 -62.9918930973898    0      -Inf       0
    x7                  91.1172962454144    0       Inf       0
    x8                 -70.2794582115411    0      -Inf       0
    x9                  28.2285318608306    0       Inf       0
    x10                -4.66428195323422    0      -Inf       0
Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F      pValue
                  _____________    ___    ____________________    ___    _______
    Total        16.2950552976581    300    0.0543168509921936
    Model        16.2950552976581     10      1.62950552976581    Inf        0
    Residual                    0    290                     0
A = 0.100296, B = 0.100000
order Q(x) = 4.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.313525 and Ymax = 0.486207
List of powers for Q(x): [0.200296, 0.300296, 0.400296, 0.500296]
List of powers for D(x): [0.200296, 0.300296, 0.400296, 0.500296, 0.600296, 0.700296]
```



```
Fitting J2(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.777762e+00
Corrected MSS of errors squared = 2.514135e+00
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting J2(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate          SE     tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    3.59234350717946e-08     0      Inf       0
    x1             -1.0012913863622e-08     0     -Inf       0
    x2             1.07663153826218e-07     0      Inf       0
    x3              -1.334587298759e-07     0     -Inf       0
    x4             0.000840033860462322     0      Inf       0
    x5               -0.415564954163396     0     -Inf       0
    x6                 12.8656266683595     0      Inf       0
    x7                -100.481301526967     0     -Inf       0
    x8                  294.73677610689     0      Inf       0
    x9                -357.924185984402     0     -Inf       0
    x10                152.217809656329     0      Inf       0
Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______
    Total        16.2950552976542     300    0.0543168509921805
    Model        16.2950552976542      10      1.62950552976542     Inf       0
    Residual                    0     290                     0
A = 0.000000, B = 1.343168
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.313525 and Ymax = 0.486207
List of powers for Q(x): [1.343168, 0.671584, 0.447723]
List of powers for D(x): [1.343168, 0.671584, 0.447723, 0.335792, 0.268634, 0.223861,
0.191881]
Fitting J2(x) in range (0.000000, 30.000000)
MSS of errors squared = 2.053236e-01
Corrected MSS of errors squared = 2.903714e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
```



```
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting J2(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    ________________    ___    _____    ______
    (Intercept)      1.02217716052734    0       Inf        0
    x1               -37.128233893559    0      -Inf        0
    x2               163.974900623632    0       Inf        0
    x3              -233.516772736195    0      -Inf        0
    x4                              0    0       NaN      NaN
    x5               296.858230384698    0       Inf        0
    x6              -263.909757572407    0      -Inf        0
    x7               72.6994560386683    0       Inf        0
    x8               45.1724215094255    0       Inf        0
    x9              -254.816200658371    0      -Inf        0
    x10              588.520126098463    0       Inf        0
    x11             -673.424378181639    0      -Inf        0
    x12              380.691108969417    0       Inf        0
    x13             -85.1430777411059    0      -Inf        0
Number of observations: 301, Error degrees of freedom: 288
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F      pValue
                 _______________    ___    ____________________    ___    ______
    Total        16.2950552976602    300    0.0543168509922007
    Model        16.2950552976602     12      1.35792127480502    Inf       0
    Residual                    0    288                     0
A = 0.000000, B = 0.256519
order Q(x) = 7.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.313525 and Ymax = 0.486207
List of powers for Q(x): [0.256519, 0.362772, 0.444304, 0.513038, 0.573594, 0.628340,
0.678685]
List of powers for D(x): [0.256519, 0.362772, 0.444304, 0.513038, 0.573594, 0.628340]
Fitting J2(x) in range (0.000000, 30.000000)
MSS of errors squared = 9.226799e-02
Corrected MSS of errors squared = 1.304866e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting J2(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
```



```
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)     0.268960671238098     0      Inf        0
    x1              -465.050832042134     0     -Inf        0
    x2               2508.03870853507     0      Inf        0
    x3              -4866.35785826453     0     -Inf        0
    x4               3697.39641530907     0      Inf        0
    x5                             0     0      NaN      NaN
    x6              -1378.07147885973     0     -Inf        0
    x7               503.776088211963     0      Inf        0
    x8               5601.87668522929     0      Inf        0
    x9                -42947.99938582     0     -Inf        0
    x10              131177.100972342     0      Inf        0
    x11             -207221.031578115     0     -Inf        0
    x12              180061.329500127     0      Inf        0
    x13             -82033.950862128     0     -Inf        0
    x14              15363.6746657613     0      Inf        0
Number of observations: 301, Error degrees of freedom: 287
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF          MeanSq            F      pValue
                    ____________________    ___    ____________________    ___    ______
    Total       16.2950552970708     300    0.054316850990236
    Model       16.2950552970708      13     1.25346579208237    Inf        0
    Residual                   0     287                    0
A = 0.670345, B = 0.418069, C = 0.845070
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.313525 and Ymax = 0.486207
List of powers for Q(x): [1.238222, 1.375516, 1.490131, 1.590579, 1.681093, 1.764143,
1.841317]
List of powers for D(x): [1.238222, 1.375516, 1.490131, 1.590579, 1.681093, 1.764143,
1.841317]
Fitting J2(x) in range (0.000000, 30.000000)
MSS of errors squared = 6.916737e-02
Corrected MSS of errors squared = 1.198014e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting J2(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
```



```
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)      1.37662957025731     0      Inf        0
    x1               -6.0560902325465     0     -Inf        0
    x2               12.3825378743775     0      Inf        0
    x3               -14.211253123109     0     -Inf        0
    x4               9.27781325437347     0      Inf        0
    x5              -3.24236059798818     0     -Inf        0
    x6              0.472723857213108     0      Inf        0
    x7               4.39859293218999     0      Inf        0
    x8              -8.99189883905635     0     -Inf        0
    x9               10.3177621938854     0      Inf        0
    x10             -6.73443864330712     0     -Inf        0
    x11              2.35294158140147     0      Inf        0
    x12             -0.342959649049818    0     -Inf        0
Number of observations: 301, Error degrees of freedom: 288
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F      pValue
                  _____________    ____    ____________________    ____    _______
    Total        16.2950552976593    300    0.0543168509921977
    Model        16.2950552976593     12      1.35792127480494    Inf        0
    Residual                    0    288                     0
A = 0.200872, B = 0.106024, C = 0.249691
order Q(x) = 6.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.313525 and Ymax = 0.486207
List of powers for Q(x): [0.306896, 0.662611, 1.018327, 1.374042, 1.729757, 2.085473]
List of powers for D(x): [0.306896, 0.662611, 1.018327, 1.374042, 1.729757, 2.085473]
Fitting J2(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.798606e-01
Corrected MSS of errors squared = 3.115278e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting J2(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)      1.38880827560574     0      Inf        0
    x1               -920.56847708315     0     -Inf        0
    x2               4463.75048536575     0      Inf        0
```



```
    x3              -9202.36141155796    0      -Inf       0
    x4               10274.9498814781    0       Inf       0
    x5              -6531.89491546136    0      -Inf       0
    x6               2236.32825355616    0       Inf       0
    x7              -321.592624200411    0      -Inf       0
    x8               661.223686001165    0       Inf       0
    x9              -3204.91414020605    0      -Inf       0
    x10              6604.50490879095    0       Inf       0
    x11             -7371.32452893295    0      -Inf       0
    x12              4684.1439018964     0       Inf       0
    x13             -1603.06903944932    0      -Inf       0
    x14              230.435211632938    0       Inf       0
Number of observations: 301, Error degrees of freedom: 286
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ____________________    ___    ______
    Total       16.2950552977866    300    0.0543168509926218
    Model       16.2950552977866     14      1.16393252127047    Inf       0
    Residual                   0    286                     0
A = 0.750000, B = 0.250000, C = 0.084335
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.313525 and Ymax = 0.486207
List of powers for Q(x): [1.000000, 1.165665, 1.331330, 1.496995, 1.662660, 1.828325,
1.993990]
List of powers for D(x): [1.000000, 1.165665, 1.331330, 1.496995, 1.662660, 1.828325,
1.993990]
Fitting J2(x) in range (0.000000, 30.000000)
MSS of errors squared = 3.376959e-01
Corrected MSS of errors squared = 5.849065e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting J2(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ________________    ___    _____    _______
    (Intercept)     0.28017518733621    0       Inf       0
    x1              -1.12653192687701   0      -Inf       0
    x2               3.80007360907895   0       Inf       0
    x3              -6.67720653421845   0      -Inf       0
    x4               5.82378686634195   0       Inf       0
```



```
    x5              -2.54843573769087    0      -Inf        0
    x6              0.448138643999548    0       Inf        0
    x7               4.55399018040262    0       Inf        0
    x8              -19.6027138330421    0      -Inf        0
    x9               42.9660975436654    0       Inf        0
    x10             -49.4144195685543    0      -Inf        0
    x11              31.9307796787805    0       Inf        0
    x12             -11.0224624543745    0      -Inf        0
    x13              1.58872837422189    0       Inf        0
Number of observations: 301, Error degrees of freedom: 287
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq           F      pValue
                ________________    ___    ________________    ___    ______
    Total       16.2950552976597    300    0.054316850992199
    Model       16.2950552976597     13     1.25346579212767    Inf       0
    Residual                   0    287                    0
A = 0.090272, B = 0.162282, C = 0.219264
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.313525 and Ymax = 0.486207
List of powers for Q(x): [0.252555, 0.634101, 0.887205, 1.119177, 1.340211, 1.554254]
List of powers for D(x): [0.252555, 0.634101, 0.887205, 1.119177, 1.340211, 1.554254,
1.763332]
Fitting J2(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.057525e-01
Corrected MSS of errors squared = 1.831686e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting J2(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   _________________    __    _____    ______
    (Intercept)     1.39104913912215    0      Inf       0
    x1             -18.8103191426586    0     -Inf       0
    x2              69.5936256407853    0      Inf       0
    x3             -126.545256194869    0     -Inf       0
    x4              131.127939902429    0      Inf       0
    x5              -79.231302148721    0     -Inf       0
    x6              26.1179752341194    0      Inf       0
    x7              -3.6437123741483    0     -Inf       0
    x8              13.5082910897138    0      Inf       0
```



```
    x9              -49.9423251987891    0      -Inf        0
    x10              90.7530571081387    0       Inf        0
    x11             -93.9805795725801    0      -Inf        0
    x12              56.7511747342043    0       Inf        0
    x13             -18.6964255600467    0      -Inf        0
    x14              2.60680735921536    0       Inf        0
Number of observations: 301, Error degrees of freedom: 286
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F      pValue
                  _________________    ___    ___________________    ___    _______
    Total        16.2950552976607     300    0.0543168509922022
    Model        16.2950552976607      14      1.16393252126148    Inf        0
    Residual                    0     286                     0
A = 0.230930, B = 0.198344, C = 0.207558
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.313525 and Ymax = 0.486207
List of powers for Q(x): [0.429273, 0.718988, 0.989586, 1.250290, 1.504670, 1.754559,
2.001043]
List of powers for D(x): [0.429273, 0.718988, 0.989586, 1.250290, 1.504670, 1.754559,
2.001043]
Fitting J2(x) in range (0.000000, 30.000000)
MSS of errors squared = 7.120698e-02
Corrected MSS of errors squared = 1.233341e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting J2(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                   _________________    ___    _____    _______
    (Intercept)    0.282554581928293    0       Inf        0
    x1              -28.1903202835333   0      -Inf        0
    x2               165.278859627232   0       Inf        0
    x3              -382.703724292825   0      -Inf        0
    x4               434.518644896389   0       Inf        0
    x5              -242.735346420348   0      -Inf        0
    x6               53.549333089925    0       Inf        0
    x7               168.11874054619    0       Inf        0
    x8              -1275.96203415573   0      -Inf        0
    x9               3883.38737309298   0       Inf        0
    x10             -6104.73495987365   0      -Inf        0
```



```
    x11              5271.37420072541     0      Inf       0
    x12             -2384.17493873595     0     -Inf       0
    x13              442.991617524542     0      Inf       0
Number of observations: 301, Error degrees of freedom: 287
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    _______
    Total        16.2950552976846     300    0.0543168509922821
    Model        16.2950552976846      13      1.25346579212959    Inf        0
    Residual                    0     287                     0
A = 0.246241, B = 0.579188, C = 0.000000
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.313525 and Ymax = 0.486207
List of powers for Q(x): [0.825429, 1.065336, 1.249424, 1.404617, 1.541344, 1.664956]
List of powers for D(x): [0.825429, 1.065336, 1.249424, 1.404617, 1.541344, 1.664956,
1.778628]
Fitting J2(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.377869e-01
Corrected MSS of errors squared = 2.386540e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting J2(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7
Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    ___    _____    _____
    (Intercept)                        0     0      NaN      NaN
    x1              2.22867652815638e-11     0      Inf        0
    x2             -2.62232641786104e-11     0     -Inf        0
    x3               3.9422766939245e-12     0      Inf        0
    x4                 0.999999999950844     0      Inf        0
    x5              6.06380876838813e-11     0      Inf        0
    x6             -1.28047292674129e-11     0     -Inf        0
    x7              1.31922335301649e-12     0      Inf        0
Number of observations: 301, Error degrees of freedom: 294
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    _______
    Total        16.2950552976593     300    0.0543168509921977
    Model        16.2950552976593       6      2.71584254960989    Inf        0
```



```
    Residual                  0    294                    0
A = 0.000000, B = 3.285270
order Q(x) = 3.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.313525 and Ymax = 0.486207
List of powers for Q(x): [0.000000, 0.026978, 0.170250]
List of powers for D(x): [0.000000, 0.026978, 0.170250, 0.431648]
Fitting J2(x) in range (0.000000, 30.000000)
MSS of errors squared = 9.100284e-01
Corrected MSS of errors squared = 1.286975e+00
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the Bessel $J_2(x)$ function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i+C)	0.67	0.418	0.845	Inf	-Inf	6.91673700E-02	7, 7
A+B*sqrt(i)+C*(i-1)	0.23	0.198	0.207	Inf	-Inf	7.12069800E-02	7, 7
A+B*sqrt(i)	0	0.256		Inf	-Inf	9.22679900E-02	7, 6
A+B*i+C*sqrt(i-1)	0.09	0.162	0.219	Inf	-Inf	1.05752500E-01	6, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.246	0.579	0	Inf	-Inf	1.37786900E-01	6, 7
A+B*i+C*(i-1)	0.2	0.106	0.249	Inf	-Inf	1.79860600E-01	6, 6
A+B/i	0	1.343		Inf	-Inf	2.05323600E-01	3, 7
A+B*i-C*(i-1)	0.75	0.25	0.084	Inf	-Inf	3.37695900E-01	7, 7
A+B*log10(i)^4	0	3.285		Inf	-Inf	9.10028400E-01	3, 4
A+B*i	0.1	0.1		Inf	-Inf	1.77776200E+00	4, 6

RESULTS FOR THE BESSEL J3(X)

Using Power A+B* i

```
Fitting J3(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ___________________    __    _____    _______
    (Intercept)     0.486290997977803      0      Inf        0
    x1              -2.48398942813439      0     -Inf        0
    x2               5.28035224714232      0      Inf        0
    x3              -5.97928602724679      0     -Inf        0
    x4               3.80397068073583      0      Inf        0
    x5              -1.28916199440345      0     -Inf        0
    x6              0.181825916652809      0      Inf        0
    x7               5.71150731811626      0      Inf        0
    x8               -13.936300760682      0     -Inf        0
    x9               18.8285437707202      0      Inf        0
    x10             -15.2087100207181      0     -Inf        0
    x11              7.34294136232817      0      Inf        0
    x12             -1.96156896485993      0     -Inf        0
    x13              0.22358556008343      0      Inf        0
Number of observations: 301, Error degrees of freedom: 287
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F      pValue
                  _______________    ___    ___________________    ___    _______
    Total         17.4514907484617    300    0.0581716358282057
    Model         17.4514907484617     13     1.34242236526629    Inf        0
    Residual                     0    287                     0
A = 0.000000, B = 0.412786
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
```



```
Ymin = -0.291132 and Ymax = 0.434394
List of powers for Q(x): [0.412786, 0.825572, 1.238357, 1.651143, 2.063929, 2.476715]
List of powers for D(x): [0.412786, 0.825572, 1.238357, 1.651143, 2.063929, 2.476715,
2.889500]
Fitting J3(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.304671e+00
Corrected MSS of errors squared = 1.845084e+00
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting J3(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                         Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -1.82984312374861e-07    0     -Inf       0
    x1              4.18623352309741e-08    0      Inf       0
    x2             -4.96983861992673e-07    0     -Inf       0
    x3              6.36640948068698e-07    0      Inf       0
    x4              0.000697408134605143    0      Inf       0
    x5                -0.379814121476382    0     -Inf       0
    x6                  12.1249987468698    0      Inf       0
    x7                 -96.1356087093877    0     -Inf       0
    x8                  284.529186296292    0      Inf       0
    x9                 -347.58769286301     0     -Inf       0
    x10                148.448233243576     0      Inf       0
Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______
    Total        17.4514907484652    300    0.0581716358282174
    Model        17.4514907484652     10     1.74514907484652    Inf       0
    Residual                    0    290                     0
A = 0.000000, B = 2.000000
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.291132 and Ymax = 0.434394
List of powers for Q(x): [2.000000, 1.000000, 0.666667]
List of powers for D(x): [2.000000, 1.000000, 0.666667, 0.500000, 0.400000, 0.333333,
0.285714]
Fitting J3(x) in range (0.000000, 30.000000)
MSS of errors squared = 7.924257e-01
Corrected MSS of errors squared = 1.120659e+00
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting J3(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ________________    __    _____    ______
    (Intercept)      1.30554533003381    0      Inf       0
    x1              -53.7809071746452    0     -Inf       0
    x2               291.894978803846    0      Inf       0
    x3              -654.388719356201    0     -Inf       0
    x4               730.164408933908    0      Inf       0
    x5              -403.672080033599    0     -Inf       0
    x6               88.4767740030667    0      Inf       0
    x7               41.1884803446963    0      Inf       0
    x8              -223.533430322892    0     -Inf       0
    x9               501.100335872124    0      Inf       0
    x10             -559.094367628528    0     -Inf       0
    x11              309.079630529735    0      Inf       0
    x12             -67.740649159092    0     -Inf       0
Number of observations: 301, Error degrees of freedom: 288
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _____________    ___    ____________________    ___    ______
    Total        17.4514907484757    300    0.0581716358282524
    Model        17.4514907484757     12      1.45429089570631    Inf       0
    Residual                    0    288                     0
A = 0.000000, B = 0.482904
order Q(x) = 6.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.291132 and Ymax = 0.434394
List of powers for Q(x): [0.482904, 0.682930, 0.836415, 0.965809, 1.079807, 1.182869]
List of powers for D(x): [0.482904, 0.682930, 0.836415, 0.965809, 1.079807, 1.182869]
Fitting J3(x) in range (0.000000, 30.000000)
MSS of errors squared = 6.285993e-02
Corrected MSS of errors squared = 8.889737e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)


```
Fitting J3(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    _________________    __    _____    ______
    (Intercept)      1.43236366276858    0      Inf        0
    x1              -91.3865560591525    0     -Inf        0
    x2               209.200865063315    0      Inf        0
    x3                             0    0      NaN      NaN
    x4              -253.658435856085    0     -Inf        0
    x5                             0    0      NaN      NaN
    x6               251.755430690397    0      Inf        0
    x7              -117.343667491004    0     -Inf        0
    x8               63.7995267195133    0      Inf        0
    x9              -146.048129967166    0     -Inf        0
    x10                            0    0      NaN      NaN
    x11              177.083296566308    0      Inf        0
    x12                            0    0      NaN      NaN
    x13             -175.753355967635    0     -Inf        0
    x14              81.9186626418052    0      Inf        0
Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF            MeanSq            F      pValue
                  _______________    ___    ___________________    ___    ______
    Total        17.451490748467     300    0.0581716358282232
    Model        17.451490748467      10       1.7451490748467    Inf        0
    Residual                   0     290                     0
A = 0.000000, B = 0.100000, C = 1.052808
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.291132 and Ymax = 0.434394
List of powers for Q(x): [0.143276, 0.174723, 0.201316, 0.224785, 0.246025, 0.265571,
0.283775]
List of powers for D(x): [0.143276, 0.174723, 0.201316, 0.224785, 0.246025, 0.265571,
0.283775]
Fitting J3(x) in range (0.000000, 30.000000)
MSS of errors squared = 4.259029e-02
Corrected MSS of errors squared = 7.376854e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting J3(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
```



```
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    0.000754919016278669    0      Inf       0
    x1              -0.0161708082656217    0     -Inf       0
    x2               0.0354407317694723    0      Inf       0
    x3              -0.0272449719016915    0     -Inf       0
    x4              0.00722020851230748    0      Inf       0
    x5                 107.214082439545    0      Inf       0
    x6                -470.392755188737    0     -Inf       0
    x7                 904.989195322858    0      Inf       0
    x8                -960.449346073628    0     -Inf       0
    x9                 587.237946404249    0      Inf       0
    x10               -194.934938447777    0     -Inf       0
    x11                27.3358154827072    0      Inf       0
Number of observations: 301, Error degrees of freedom: 289
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______
    Total        17.4514907484617    300    0.0581716358282056
    Model        17.4514907484617     11      1.58649915895106    Inf       0
    Residual                    0    289                     0
A = 0.500000, B = 0.100000, C = 0.072889
order Q(x) = 4.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.291132 and Ymax = 0.434394
List of powers for Q(x): [0.600000, 0.772889, 0.945779, 1.118668]
List of powers for D(x): [0.600000, 0.772889, 0.945779, 1.118668, 1.291558, 1.464447,
1.637337]
Fitting J3(x) in range (0.000000, 30.000000)
MSS of errors squared = 4.514983e-01
Corrected MSS of errors squared = 7.820180e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting J3(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    0.0205295772141704     0      Inf       0
```

```
    x1              -0.932382804796706    0      -Inf        0
    x2                2.69243854364706    0       Inf        0
    x3               -3.0392539134115     0      -Inf        0
    x4                1.56844737989852    0       Inf        0
    x5              -0.309776527972138    0      -Inf        0
    x6                 141.10300785343    0       Inf        0
    x7               -620.600585961102    0      -Inf        0
    x8                1186.31475287435    0       Inf        0
    x9               -1244.89300073525    0      -Inf        0
    x10               750.427484223547    0       Inf        0
    x11              -245.136714982388    0      -Inf        0
    x12               33.7850550585602    0       Inf        0
Number of observations: 301, Error degrees of freedom: 288
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ____________________    ___    ______
    Total       17.4514907484616    300    0.0581716358282052
    Model       17.4514907484616     12      1.45429089570513    Inf       0
    Residual                   0    288                     0
A = 0.666587, B = 0.228789, C = 0.000000
order Q(x) = 5.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.291132 and Ymax = 0.434394
List of powers for Q(x): [0.895376, 1.124165, 1.352954, 1.581743, 1.810532]
List of powers for D(x): [0.895376, 1.124165, 1.352954, 1.581743, 1.810532, 2.039321,
2.268111]
Fitting J3(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.930765e-01
Corrected MSS of errors squared = 3.344184e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting J3(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ________________    ___    _____    _______
    (Intercept)    0.394524950505017    0       Inf        0
    x1             -2.88360927642044    0      -Inf        0
    x2              8.16588336320457    0       Inf        0
    x3             -11.2336523424546    0      -Inf        0
    x4              8.14717661019774    0       Inf        0
    x5             -3.06235834474244    0      -Inf        0
```



```
    x6              0.472036207795871    0      Inf       0
    x7               8.58475389409843    0      Inf       0
    x8              -28.8985076377128    0     -Inf       0
    x9               48.3864055662304    0      Inf       0
    x10             -45.4143140807051    0     -Inf       0
    x11              24.8000803228631    0      Inf       0
    x12             -7.39146897560523    0     -Inf       0
    x13             0.933050063353867    0      Inf       0
Number of observations: 301, Error degrees of freedom: 287
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ________________    ___    ____________________    ___    ______
    Total       17.4514907484602    300    0.0581716358282007
    Model       17.4514907484602     13      1.34242236526617    Inf       0
    Residual                   0    287                     0
A = 0.144298, B = 0.290897, C = 0.074703
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.291132 and Ymax = 0.434394
List of powers for Q(x): [0.435194, 0.800794, 1.122634, 1.437274, 1.748187, 2.056719]
List of powers for D(x): [0.435194, 0.800794, 1.122634, 1.437274, 1.748187, 2.056719,
2.363559]
Fitting J3(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.442616e-01
Corrected MSS of errors squared = 2.498684e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting J3(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                    Estimate         SE    tStat    pValue
                ________________    ___    _____    ______
    (Intercept) 0.963636674451777    0      Inf       0
    x1          -6.86082979022807    0     -Inf       0
    x2           18.0684759194428    0      Inf       0
    x3          -21.8418881292229    0     -Inf       0
    x4           10.6091560801301    0      Inf       0
    x5           1.52101979468907    0      Inf       0
    x6          -3.36566249324843    0     -Inf       0
    x7          0.906092054504739    0      Inf       0
    x8           7.78719547020183    0      Inf       0
    x9            -23.34574385053    0     -Inf       0
```



```
    x10              35.1806873182927    0      Inf       0
    x11             -28.6315148573583    0     -Inf       0
    x12              12.1018183731562    0      Inf       0
    x13              -2.0924425332497    0     -Inf       0
Number of observations: 301, Error degrees of freedom: 287
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _________________    ___    ____________________    ___    _____
    Total        17.4514907484607     300    0.0581716358282022
    Model        17.4514907484607      13       1.3424223652662    Inf       0
    Residual                    0     287                     0
A = 0.028236, B = 0.220137, C = 0.119365
order Q(x) = 7.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.291132 and Ymax = 0.434394
List of powers for Q(x): [0.248373, 0.458921, 0.648254, 0.826605, 0.997937, 1.164285,
1.326854]
List of powers for D(x): [0.248373, 0.458921, 0.648254, 0.826605, 0.997937, 1.164285]
Fitting J3(x) in range (0.000000, 30.000000)
MSS of errors squared = 9.663683e-02
Corrected MSS of errors squared = 1.673799e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting J3(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    _____________________    __    _____    ______
    (Intercept)      0.0174805731136511    0      Inf       0
    x1              -0.0417313833688082    0     -Inf       0
    x2               0.132783384691782     0      Inf       0
    x3              -0.240423129733491     0     -Inf       0
    x4               0.181890962091673     0      Inf       0
    x5              -0.0499998322358638    0     -Inf       0
    x6                3.33681833434673     0      Inf       0
    x7              -32.4427539410899      0     -Inf       0
    x8               125.45697789833       0      Inf       0
    x9              -226.477893443164      0     -Inf       0
    x10              215.011882725041      0      Inf       0
    x11             -104.408039467586      0     -Inf       0
    x12              20.5230074689311      0      Inf       0
Number of observations: 301, Error degrees of freedom: 288
```



```
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```

	SumSq	DF	MeanSq	F	pValue
Total	17.451490748463	300	0.05817163582821		
Model	17.451490748463	12	1.45429089570525	Inf	0
Residual	0	288	0		

```
A = 0.197736, B = 0.100000, C = 0.676068
order Q(x) = 5.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.291132 and Ymax = 0.434394
List of powers for Q(x): [0.297736, 1.015225, 1.327046, 1.568720, 1.773478]
List of powers for D(x): [0.297736, 1.015225, 1.327046, 1.568720, 1.773478, 1.954419,
2.118332]
Fitting J3(x) in range (0.000000, 30.000000)
MSS of errors squared = 9.239660e-02
Corrected MSS of errors squared = 1.600356e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting J3(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
```

	Estimate	SE	tStat	pValue
(Intercept)	0	0	NaN	NaN
x1	-2.09796484820664e-05	0	-Inf	0
x2	0	0	NaN	NaN
x3	3.46587347665818e-05	0	Inf	0
x4	0	0	NaN	NaN
x5	-3.10046932447891e-05	0	-Inf	0
x6	2.25417908722731e-05	0	Inf	0
x7	-5.21618391488893e-06	0	-Inf	0
x8	1.0000001505763	0	Inf	0
x9	0	0	NaN	NaN
x10	-3.17119572599967e-07	0	-Inf	0
x11	2.1731847438633e-07	0	Inf	0
x12	-5.07752020441305e-08	0	-Inf	0

```
Number of observations: 301, Error degrees of freedom: 292
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
```

	SumSq	DF	MeanSq	F	pValue
Total	17.4514907484614	300	0.0581716358282046		


```
    Model       17.4514907484614      8      2.18143634355767    Inf       0
    Residual                   0    292                     0
A = 0.000000, B = 0.100000
order Q(x) = 7.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.291132 and Ymax = 0.434394
List of powers for Q(x): [0.000000, 0.000821, 0.005182, 0.013139, 0.023869, 0.036665,
0.051007]
List of powers for D(x): [0.000000, 0.000821, 0.005182, 0.013139, 0.023869]
Fitting J3(x) in range (0.000000, 30.000000)
MSS of errors squared = 6.620560e-01
Corrected MSS of errors squared = 9.362886e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the Bessel $J_3(x)$ function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i+C)	0.67	0.418	0.845	Inf	-Inf	6.91673700E-02	7, 7
A+B*sqrt(i)+C*(i-1)	0.23	0.198	0.207	Inf	-Inf	7.12069800E-02	7, 7
A+B*sqrt(i)	0	0.256		Inf	-Inf	9.22679900E-02	7, 6
A+B*i+C*sqrt(i-1)	0.09	0.162	0.219	Inf	-Inf	1.05752500E-01	6, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.246	0.579	0	Inf	-Inf	1.37786900E-01	6, 7
A+B*i+C*(i-1)	0.2	0.106	0.249	Inf	-Inf	1.79860600E-01	6, 6
A+B/i	0	1.343		Inf	-Inf	2.05323600E-01	3, 7
A+B*i-C*(i-1)	0.75	0.25	0.084	Inf	-Inf	3.37695900E-01	7, 7
A+B*log10(i)^4	0	3.285		Inf	-Inf	9.10028400E-01	3, 4
A+B*i	0.1	0.1		Inf	-Inf	1.77776200E+00	4, 6

RESULTS FOR THE BESSEL J4(X)

Using Power A+B* i

```
Fitting J4(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                    ________________    ___    _____    _______
    (Intercept)      1.37866566839934    0       Inf        0
    x1              -18.2985766746492    0      -Inf        0
    x2               56.9516712222215    0       Inf        0
    x3              -86.0506376302316    0      -Inf        0
    x4               74.8780126935164    0       Inf        0
    x5              -38.3754891268917    0      -Inf        0
    x6               10.8134845310348    0       Inf        0
    x7              -1.29712410110536    0      -Inf        0
    x8               13.2798170335375    0       Inf        0
    x9              -41.3456134214169    0      -Inf        0
    x10              62.4927898131179    0       Inf        0
    x11             -54.3985904239258    0      -Inf        0
    x12              27.8900558858867    0       Inf        0
    x13             -7.86190428158942    0      -Inf        0
    x14             0.943440703788741    0       Inf        0
Number of observations: 301, Error degrees of freedom: 286
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq          DF           MeanSq            F      pValue
                    ________________    ___    ________________    ___    _______
    Total           18.4699692869869    300    0.0615665642899562
    Model           18.4699692869869     14     1.31928352049906    Inf        0
    Residual                       0    286                     0
A = 0.302366, B = 0.469004
order Q(x) = 7.000000, order D(x) = 7.000000
```



```
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.274347 and Ymax = 0.399625
List of powers for Q(x): [0.771371, 1.240375, 1.709379, 2.178384, 2.647388, 3.116392,
3.585397]
List of powers for D(x): [0.771371, 1.240375, 1.709379, 2.178384, 2.647388, 3.116392,
3.585397]
Fitting J4(x) in range (0.000000, 30.000000)
MSS of errors squared = 4.231743e-02
Corrected MSS of errors squared = 5.984588e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting J4(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)       1.36703049253829      0      Inf       0
    x1               0.0418525450619954     0      Inf       0
    x2                -7.18782025450405     0     -Inf       0
    x3                 100.254675081901     0      Inf       0
    x4                -357.883174629845     0     -Inf       0
    x5                 387.358508691882     0      Inf       0
    x6                                0     0      NaN      NaN
    x7                -123.951071893818     0     -Inf       0
    x8               -0.0306133392668239    0     -Inf       0
    x9                 5.25778868339231     0      Inf       0
    x10               -73.3357152737472     0     -Inf       0
    x11                261.790870233694     0      Inf       0
    x12               -283.352889672212     0     -Inf       0
    x13                               0     0      NaN      NaN
    x14                90.6705593444264     0      Inf       0
Number of observations: 301, Error degrees of freedom: 288
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F     pValue
                  ___________________    ___    ____________________    ___    ______

    Total       18.4699692869907     300     0.0615665642899689
    Model       18.4699692869907      12       1.53916410724922     Inf       0
    Residual                   0     288                        0
A = 0.127162, B = 1.087961
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.274347 and Ymax = 0.399625
```



```
List of powers for Q(x): [1.215123, 0.671143, 0.489816, 0.399152, 0.344754, 0.308489,
0.282585]
List of powers for D(x): [1.215123, 0.671143, 0.489816, 0.399152, 0.344754, 0.308489,
0.282585]
Fitting J4(x) in range (0.000000, 30.000000)
MSS of errors squared = 2.986994e-01
Corrected MSS of errors squared = 4.224248e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting J4(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                         Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -2.59420358169088e-06    0     -Inf       0
    x1              2.00810890310628e-05    0      Inf       0
    x2             -3.21317881311176e-05    0     -Inf       0
    x3              1.46448215351717e-05    0      Inf       0
    x4                   47.4329775264507   0      Inf       0
    x5                  -273.070601460127   0     -Inf       0
    x6                   640.642385699807   0      Inf       0
    x7                  -742.842148181191   0     -Inf       0
    x8                   424.874658652476   0      Inf       0
    x9                  -96.0372722373699   0     -Inf       0
Number of observations: 301, Error degrees of freedom: 291
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______
    Total        18.4699692869714     300    0.0615665642899047
    Model        18.4699692869714       9      2.05221880966349     Inf       0
    Residual                    0     291                     0
A = 0.000000, B = 0.117348
order Q(x) = 3.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.274347 and Ymax = 0.399625
List of powers for Q(x): [0.117348, 0.165956, 0.203253]
List of powers for D(x): [0.117348, 0.165956, 0.203253, 0.234697, 0.262399, 0.287443]
Fitting J4(x) in range (0.000000, 30.000000)
MSS of errors squared = Inf
Corrected MSS of errors squared = Inf
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
```



```
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting J4(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)    0.0363229755706317    0      Inf       0
    x1              -2.29190835002059    0     -Inf       0
    x2               6.58033746525143    0      Inf       0
    x3              -6.4797881404681     0     -Inf       0
    x4               2.15503644742098    0      Inf       0
    x5               166.059348253577    0      Inf       0
    x6              -685.460696538738    0     -Inf       0
    x7               1076.08706928397    0      Inf       0
    x8              -753.696433796433    0     -Inf       0
    x9               198.010712500638    0      Inf       0
Number of observations: 301, Error degrees of freedom: 291
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF            MeanSq            F      pValue
                  _______________    ___    ____________________    ___    _______
    Total        18.4699692869785    300    0.0615665642899283
    Model        18.4699692869785      9      2.05221880966428    Inf        0
    Residual                    0    291                     0
A = 0.004836, B = 0.147565, C = 1.332661
order Q(x) = 4.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.274347 and Ymax = 0.399625
List of powers for Q(x): [0.230213, 0.274225, 0.311994, 0.345602]
List of powers for D(x): [0.230213, 0.274225, 0.311994, 0.345602, 0.376181]
Fitting J4(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.839665e-01
Corrected MSS of errors squared = 3.186392e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting J4(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
```


	Estimate	SE	tStat	pValue
(Intercept)	1.37523456344719	0	Inf	0
x1	-3.96044004085145	0	-Inf	0
x2	6.99894484043098	0	Inf	0
x3	-8.90151049468632	0	-Inf	0
x4	7.13606948317507	0	Inf	0
x5	-3.49221989633939	0	-Inf	0
x6	0.956633031127348	0	Inf	0
x7	-0.112711146880817	0	-Inf	0
x8	2.88044227052637	0	Inf	0
x9	-5.09239977764493	0	-Inf	0
x10	6.47934877509962	0	Inf	0
x11	-5.19646375370823	0	-Inf	0
x12	2.54410435715316	0	Inf	0
x13	-0.697214460880021	0	-Inf	0
x14	0.0821823476212291	0	Inf	0

Number of observations: 301, Error degrees of freedom: 286

R-squared: 1, Adjusted R-Squared: 1

F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	18.4699692869884	300	0.0615665642899612		
Model	18.4699692869884	14	1.31928352049917	Inf	0
Residual	0	286	0		

A = 0.003309, B = 0.241992, C = 0.212549

order Q(x) = 7.000000, order D(x) = 7.000000

Xmin = 0.000000 and Xmax = 30.000000

Ymin = -0.274347 and Ymax = 0.399625

List of powers for Q(x): [0.245301, 0.699843, 1.154384, 1.608926, 2.063467, 2.518009, 2.972550]

List of powers for D(x): [0.245301, 0.699843, 1.154384, 1.608926, 2.063467, 2.518009, 2.972550]

Fitting J4(x) in range (0.000000, 30.000000)

MSS of errors squared = 1.123535e+00

Corrected MSS of errors squared = 1.946020e+00

R-Squared = 1.00000000

R-Squared Adjusted = 1.00000000

Particle swarm AICc = -1.000000e+99

AIC = -Inf

AICc = -Inf

Using Power A+B*i–C*(i–1)

Fitting J4(x) in range (0.000000, 30.000000)

Pade-Shammas polynomial power is A+B*i-C*(i-1)

Linear regression model:

y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13

Estimated Coefficients:

	Estimate	SE	tStat	pValue


```
    (Intercept)    0.189703484953485    0      Inf        0
    x1             -12.9548123255588    0     -Inf        0
    x2               26.483710794807    0      Inf        0
    x3                             0    0      NaN      NaN
    x4             -38.1261035034192    0     -Inf        0
    x5              33.1694665686758    0      Inf        0
    x6             -8.76196491633621    0     -Inf        0
    x7               120.29023167642    0      Inf        0
    x8              -292.03446193109    0     -Inf        0
    x9                             0    0      NaN      NaN
    x10             686.020056965115    0      Inf        0
    x11            -885.681954730141    0     -Inf        0
    x12             462.866182490007    0      Inf        0
    x13            -90.4600545448963    0     -Inf        0
Number of observations: 301, Error degrees of freedom: 289
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ____________________    ___    ______
    Total       18.4699692869955    300    0.0615665642899849
    Model       18.4699692869955     11      1.67908811699959    Inf       0
    Residual                   0    289                     0
A = 0.187051, B = 0.106209, C = 0.049061
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.274347 and Ymax = 0.399625
List of powers for Q(x): [0.293260, 0.350408, 0.407556, 0.464705, 0.521853, 0.579001]
List of powers for D(x): [0.293260, 0.350408, 0.407556, 0.464705, 0.521853, 0.579001,
0.636150]
Fitting J4(x) in range (0.000000, 30.000000)
MSS of errors squared = 2.099337e-01
Corrected MSS of errors squared = 3.636159e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting J4(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)    0.00201982512161329     0      Inf        0
    x1             -0.00484571106353054    0     -Inf        0
    x2              0.00618962007825158    0      Inf        0
    x3             -0.00450303731625484    0     -Inf        0
```



```
    x4              0.00113924603125989    0      Inf       0
    x5                 3.79115864030086    0      Inf       0
    x6                -13.0246343467671    0     -Inf       0
    x7                 25.6260853288552    0      Inf       0
    x8                -27.2820930366882    0     -Inf       0
    x9                  16.566004184842    0      Inf       0
    x10               -5.42186906349429    0     -Inf       0
    x11               0.745348308894222    0      Inf       0
Number of observations: 301, Error degrees of freedom: 289
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ____________________    ___    ______
    Total       18.4699692869885    300    0.0615665642899615
    Model       18.4699692869885     11      1.67908811699895    Inf       0
    Residual                   0    289                     0
A = 0.052523, B = 0.288381, C = 0.285110
order Q(x) = 4.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.274347 and Ymax = 0.399625
List of powers for Q(x): [0.340903, 0.914394, 1.320871, 1.699871]
List of powers for D(x): [0.340903, 0.914394, 1.320871, 1.699871, 2.064646, 2.420332,
2.769562]
Fitting J4(x) in range (0.000000, 30.000000)
MSS of errors squared = 2.257034e-01
Corrected MSS of errors squared = 3.909298e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting J4(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)       1.37390141287536     0      Inf       0
    x1               -2.39224266725546     0     -Inf       0
    x2                2.60959837278287     0      Inf       0
    x3               -3.21344432000001     0     -Inf       0
    x4                2.59988980754074     0      Inf       0
    x5               -1.29751993857697     0     -Inf       0
    x6               0.363727692851466     0      Inf       0
    x7              -0.0439102867008939    0     -Inf       0
    x8                1.74138873390901     0      Inf       0
    x9               -1.90039489866274     0     -Inf       0
```



```
    x10              2.34109685284147    0      Inf       0
    x11             -1.89488342168001    0     -Inf       0
    x12              0.946066542113997   0      Inf       0
    x13             -0.265317235427967   0     -Inf       0
    x14             0.0320433745292815   0      Inf       0
Number of observations: 301, Error degrees of freedom: 286
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                _________________    ___    ___________________    ___    ______
    Total       18.4699692869884    300    0.0615665642899612
    Model       18.4699692869884     14      1.31928352049917    Inf       0
    Residual                   0    286                     0
A = 0.007398, B = 0.107864, C = 0.400000
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.274347 and Ymax = 0.399625
List of powers for Q(x): [0.115262, 0.559941, 0.994224, 1.423127, 1.848590, 2.271610,
2.692780]
List of powers for D(x): [0.115262, 0.559941, 0.994224, 1.423127, 1.848590, 2.271610,
2.692780]
Fitting J4(x) in range (0.000000, 30.000000)
MSS of errors squared = 4.866769e-02
Corrected MSS of errors squared = 8.429491e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting J4(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -3.79210338469013e-05    0     -Inf       0
    x1              4.96671789218521e-05    0      Inf       0
    x2             -2.09793955811755e-05    0     -Inf       0
    x3              9.16869453768938e-06    0      Inf       0
    x4                  1.96836851909806    0      Inf       0
    x5                 -11.2042053143033    0     -Inf       0
    x6                  41.0124429186387    0      Inf       0
    x7                 -71.9906434752628    0     -Inf       0
    x8                  67.1423799048936    0      Inf       0
    x9                 -32.1980521376599    0     -Inf       0
    x10                 6.26970962882523    0      Inf       0
Number of observations: 301, Error degrees of freedom: 290
```



```
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______
    Total       18.4699692869884    300    0.0615665642899613
    Model       18.4699692869884     10      1.84699692869884    Inf       0
    Residual                   0    290                     0
A = 0.000000, B = 0.145944, C = 0.602368
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.274347 and Ymax = 0.399625
List of powers for Q(x): [0.145944, 0.808765, 1.104661]
List of powers for D(x): [0.145944, 0.808765, 1.104661, 1.335222, 1.531079, 1.704426,
1.861628]
Fitting J4(x) in range (0.000000, 30.000000)
MSS of errors squared = 5.442639e-01
Corrected MSS of errors squared = 9.426928e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting J4(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)                        0    0      NaN      NaN
    x1              -7.80411174481891e-07    0     -Inf        0
    x2                                 0    0      NaN      NaN
    x3               2.0985682521574e-06    0      Inf        0
    x4              -2.27492842215156e-06    0     -Inf        0
    x5               1.32069773983143e-06    0      Inf        0
    x6              -4.22120642246712e-07    0     -Inf        0
    x7               5.81942658050108e-08    0      Inf        0
    x8                  0.999999611864284    0      Inf        0
    x9               5.06423249598921e-07    0      Inf        0
    x10             -1.48353598317133e-07    0     -Inf        0
    x11              3.39882331931902e-08    0      Inf        0
    x12              -3.9221834827902e-09    0     -Inf        0
Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______
    Total       18.4699692869884    300    0.0615665642899614
```



```
    Model       18.4699692869884     10      1.84699692869884    Inf       0
    Residual                   0    290                     0
A = 0.000000, B = 1.012828
order Q(x) = 7.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.274347 and Ymax = 0.399625
List of powers for Q(x): [0.000000, 0.008317, 0.052487, 0.133075, 0.241752, 0.371357,
0.516612]
List of powers for D(x): [0.000000, 0.008317, 0.052487, 0.133075, 0.241752]
Fitting J4(x) in range (0.000000, 30.000000)
MSS of errors squared = 5.471997e-01
Corrected MSS of errors squared = 7.738572e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the Bessel $J_4(x)$ function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i	0.302	0.469		Inf	-Inf	4.23174300E-02	7, 7
A+B*sqrt(i)+C*(i-1)	0.007	0.107	0.4	Inf	-Inf	4.86676900E-02	7, 7
A+B*sqrt(i+C)	0.004	0.147	1.332	Inf	-Inf	1.83966500E-01	4, 5
A+B*i-C*(i-1)	0.187	0.106	0.049	Inf	-Inf	2.09933700E-01	6, 7
A+B*i+C*sqrt(i-1)	0.052	0.288	0.285	Inf	-Inf	2.25703400E-01	4, 7
A+B/i	0.127	1.087		Inf	-Inf	2.98699400E-01	7, 7
A+B*sqrt(i)+C*sqrt(i-1)	0	0.145	0.602	Inf	-Inf	5.44263900E-01	3, 7
A+B*log10(i)^4	0	1.012		Inf	-Inf	5.47199700E-01	7, 5
A+B*i+C*(i-1)	0.003	0.241	0.212	Inf	-Inf	1.12353500E+00	7, 7
A+B*sqrt(i)	0	0.117		Inf	-Inf	Inf	3, 6

RESULTS FOR THE BESSEL J5(X)

Using Power A+B* i

```
Fitting J5(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate           SE     tStat     pValue
                    ____________________    ___    _____    _______
    (Intercept)    -6.51852162274733e-06    0      -Inf       0
    x1              7.66677331916458e-05    0       Inf       0
    x2             -0.000113314296449493    0      -Inf       0
    x3              4.30840129643547e-05    0       Inf       0
    x4                   291.469521228281   0       Inf       0
    x5                  -1372.7107025803    0      -Inf       0
    x6                   2779.14096218913   0       Inf       0
    x7                  -3068.90521800865   0      -Inf       0
    x8                    1938.7129044742   0       Inf       0
    x9                  -661.848983364191   0      -Inf       0
    x10                  95.1415161157621   0       Inf       0
Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F      pValue
                  _____________    ___    ____________________    ___    _______
    Total        18.6374025324351    300    0.0621246751081172
    Model        18.6374025324351     10      1.86374025324351    Inf        0
    Residual                    0    290                     0
A = 0.500000, B = 0.135552
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.261053 and Ymax = 0.374075
List of powers for Q(x): [0.635552, 0.771103, 0.906655]
```



```
List of powers for D(x): [0.635552, 0.771103, 0.906655, 1.042206, 1.177758, 1.313309,
1.448861]
Fitting J5(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.056317e-01
Corrected MSS of errors squared = 1.493858e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting J5(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)      -30.094856912677     0     -Inf         0
    x1              -4.44565761757783     0     -Inf         0
    x2               87.7176484116332     0      Inf         0
    x3              -215.834316609068     0     -Inf         0
    x4                             0     0      NaN       NaN
    x5               162.657184714686     0      Inf         0
    x6                             0     0      NaN       NaN
    x7                             0     0      NaN       NaN
    x8               0.47990155739329     0      Inf         0
    x9              -3.91885184381571     0     -Inf         0
    x10              4.43894886993918     0      Inf         0
Number of observations: 301, Error degrees of freedom: 293
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF            MeanSq            F      pValue
                  _____________    ____    ____________________    ____    _______
    Total        18.63740253242    300     0.0621246751080668
    Model        18.63740253242      7          2.66248607606    Inf        0
    Residual                  0    293                      0
A = 0.000000, B = 0.100000
order Q(x) = 7.000000, order D(x) = 3.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.261053 and Ymax = 0.374075
List of powers for Q(x): [0.100000, 0.050000, 0.033333, 0.025000, 0.020000, 0.016667,
0.014286]
List of powers for D(x): [0.100000, 0.050000, 0.033333]
Fitting J5(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.786131e+00
Corrected MSS of errors squared = 2.525970e+00
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
```



```
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting J5(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                    Estimate          SE    tStat    pValue
                  ____________________    __    _____    ______

    (Intercept)     1.40124683405661    0      Inf       0
    x1             -1331.07229197226    0     -Inf       0
    x2              11210.7830265475    0      Inf       0
    x3             -36223.0068221978    0     -Inf       0
    x4              59383.7666157882    0      Inf       0
    x5             -52967.6173049993    0     -Inf       0
    x6              24601.2545327191    0      Inf       0
    x7             -4675.50898352438    0     -Inf       0
    x8              950.503408193962    0      Inf       0
    x9             -8006.42296866888    0     -Inf       0
    x10             25871.7961234983    0      Inf       0
    x11            -42417.3557396867    0     -Inf       0
    x12             37836.9728835176    0      Inf       0
    x13            -17574.8117428181    0     -Inf       0
    x14             3340.31802234306    0      Inf       0
Number of observations: 301, Error degrees of freedom: 286
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                  _____________    ___    ____________________    ___    ______

    Total       18.6374025319843    300    0.0621246751066144
    Model       18.6374025319843     14     1.33124303799888    Inf       0
    Residual                   0    286                     0
A = 0.894476, B = 0.656747
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.261053 and Ymax = 0.374075
List of powers for Q(x): [1.551223, 1.823256, 2.031995, 2.207970, 2.363006, 2.503170,
2.632065]
List of powers for D(x): [1.551223, 1.823256, 2.031995, 2.207970, 2.363006, 2.503170,
2.632065]
Fitting J5(x) in range (0.000000, 30.000000)
MSS of errors squared = 4.729087e-02
Corrected MSS of errors squared = 6.687939e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
```



```
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting J5(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    0.00154425590986507     0      Inf        0
    x1             -0.0236118142128988     0     -Inf        0
    x2              0.0641589891801824     0      Inf        0
    x3             -0.0636689004237698     0     -Inf        0
    x4              0.0215774704289579     0      Inf        0
    x5                 37.642823213303     0      Inf        0
    x6                -168.754972661942    0     -Inf        0
    x7                 243.118175453753    0      Inf        0
    x8                                0    0      NaN      NaN
    x9                -314.822209835218    0     -Inf        0
    x10                282.089653216646    0      Inf        0
    x11               -78.2734693872129    0     -Inf        0
Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF            MeanSq            F     pValue
                 _______________    ___    ____________________    ___    ______
    Total        18.6374025324131    300    0.0621246751080436
    Model        18.6374025324131     10      1.86374025324131    Inf       0
    Residual                    0    290                     0
A = 0.000000, B = 0.153465, C = 0.000000
order Q(x) = 4.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.261053 and Ymax = 0.374075
List of powers for Q(x): [0.153465, 0.217033, 0.265810, 0.306930]
List of powers for D(x): [0.153465, 0.217033, 0.265810, 0.306930, 0.343159, 0.375912,
0.406031]
Fitting J5(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.378509e-01
Corrected MSS of errors squared = 2.387648e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting J5(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
```



```
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.00019793956452299     0     -Inf       0
    x1              0.00294692187764056     0      Inf       0
    x2             -0.00616140605914979     0     -Inf       0
    x3              0.00462091338637798     0      Inf       0
    x4             -0.00120850327834522     0     -Inf       0
    x5                 58.1224365202746     0      Inf       0
    x6                -244.720612964649     0     -Inf       0
    x7                 461.946413646449     0      Inf       0
    x8                -486.467539562654     0     -Inf       0
    x9                 297.075340544448     0      Inf       0
    x10               -98.9061396199396     0     -Inf       0
    x11                13.9501014455667     0      Inf       0
Number of observations: 301, Error degrees of freedom: 289
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                 _______________    ___    ____________________    ___    ______
    Total        18.6374025324188    300    0.0621246751080628
    Model        18.6374025324188     11      1.69430932112899    Inf       0
    Residual                    0    289                     0
A = 0.232482, B = 0.120473, C = 0.007449
order Q(x) = 4.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.261053 and Ymax = 0.374075
List of powers for Q(x): [0.352955, 0.480878, 0.608800, 0.736723]
List of powers for D(x): [0.352955, 0.480878, 0.608800, 0.736723, 0.864645, 0.992568,
1.120491]
Fitting J5(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.321961e-01
Corrected MSS of errors squared = 2.289704e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting J5(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)     1.42364164222983     0      Inf       0
    x1             -1178.40856503723     0     -Inf       0
    x2              3089.54886845438     0      Inf       0
```


x3	0	0	NaN	NaN
x4	-8064.2910637987	0	-Inf	0
x5	10810.5275578113	0	Inf	0
x6	-5827.74655383447	0	-Inf	0
x7	1168.94612038226	0	Inf	0
x8	826.665291143069	0	Inf	0
x9	-2167.05627648729	0	-Inf	0
x10	0	0	NaN	NaN
x11	5654.8879251508	0	Inf	0
x12	-7579.59658225669	0	-Inf	0
x13	4085.46317162004	0	Inf	0
x14	-819.363533160305	0	-Inf	0

Number of observations: 301, Error degrees of freedom: 288
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	18.6374025325971	300	0.0621246751086571		
Model	18.6374025325971	12	1.55311687771643	Inf	0
Residual	0	288	0		

A = 0.613703, B = 0.248362, C = 0.158983
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.261053 and Ymax = 0.374075
List of powers for Q(x): [0.862065, 0.951444, 1.040823, 1.130203, 1.219582, 1.308961, 1.398340]
List of powers for D(x): [0.862065, 0.951444, 1.040823, 1.130203, 1.219582, 1.308961, 1.398340]
Fitting J5(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.083844e-01
Corrected MSS of errors squared = 1.877272e-01
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*i+C*sqrt(i–1)

Fitting J5(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	0.0279967918115528	0	Inf	0
x1	-0.571954942479092	0	-Inf	0
x2	1.90423706528413	0	Inf	0
x3	-2.80367259073352	0	-Inf	0
x4	2.168239744962	0	Inf	0


```
    x5              -0.866659393450059     0      -Inf        0
    x6               0.141813191522357     0       Inf        0
    x7               26.2259876753545      0       Inf        0
    x8              -103.163415811193      0      -Inf        0
    x9               186.412572500023      0       Inf        0
    x10             -188.868338620128      0      -Inf        0
    x11              111.372003510569      0       Inf        0
    x12             -35.8830642370404      0      -Inf        0
    x13              4.90425507553085      0       Inf        0
Number of observations: 301, Error degrees of freedom: 287
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F      pValue
                ________________    ___    ____________________    ___    ______
    Total       18.6374025324214    300    0.0621246751080714
    Model       18.6374025324214     13       1.4336463486478    Inf       0
    Residual                   0    287                     0
A = 0.224459, B = 0.195007, C = 0.011335
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.261053 and Ymax = 0.374075
List of powers for Q(x): [0.419466, 0.625808, 0.825510, 1.024120, 1.222164, 1.419847]
List of powers for D(x): [0.419466, 0.625808, 0.825510, 1.024120, 1.222164, 1.419847,
1.617274]
Fitting J5(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.410791e-01
Corrected MSS of errors squared = 2.443562e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting J5(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.000221689278205659    0     -Inf       0
    x1                0.0277944871111791    0      Inf       0
    x2               -0.0591352145308205    0     -Inf       0
    x3                0.0315617510794341    0      Inf       0
    x4                  22456.2177649596    0      Inf       0
    x5                 -134886.039756222    0     -Inf       0
    x6                  228893.290494422    0      Inf       0
    x7                                 0    0      NaN      NaN
    x8                 -361094.634620073    0     -Inf       0
```



```
    x9                  346285.750947901    0      Inf        0
    x10                -101653.584830528    0     -Inf        0
Number of observations: 301, Error degrees of freedom: 291
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                 _______________    ___    ___________________    ___    ______
    Total        18.6374025408031    300    0.0621246751360104
    Model        18.6374025408031      9      2.07082250453368    Inf       0
    Residual                    0    291                     0
A = 0.550000, B = 0.100000, C = 0.000000
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.261053 and Ymax = 0.374075
List of powers for Q(x): [0.650000, 0.691421, 0.723205]
List of powers for D(x): [0.650000, 0.691421, 0.723205, 0.750000, 0.773607, 0.794949,
0.814575]
Fitting J5(x) in range (0.000000, 30.000000)
MSS of errors squared = 1.075523e+00
Corrected MSS of errors squared = 1.862860e+00
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting J5(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                   _________________    ___    _____    _______
    (Intercept)     1.20884795699238    0       Inf         0
    x1             -419.613342937823    0      -Inf         0
    x2              1739.60022380502    0       Inf         0
    x3             -2075.26778242595    0      -Inf         0
    x4                             0    0       NaN       NaN
    x5              1362.7265482715     0       Inf         0
    x6             -608.654492285513    0      -Inf         0
    x7              452.052740382063    0       Inf         0
    x8             -2285.36403669572    0      -Inf         0
    x9              4130.90231299804    0       Inf         0
    x10            -3212.65658724589    0      -Inf         0
    x11             916.065568864087    0       Inf         0
Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
```


Total	18.6374025324004	300	0.0621246751080012		
Model	18.6374025324004	10	1.86374025324004	Inf	0
Residual	0	290	0		

A = 0.291046, B = 0.150345, C = 0.000000
order Q(x) = 6.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.261053 and Ymax = 0.374075
List of powers for Q(x): [0.441391, 0.503666, 0.551451, 0.591735, 0.627227, 0.659314]
List of powers for D(x): [0.441391, 0.503666, 0.551451, 0.591735, 0.627227]
Fitting J5(x) in range (0.000000, 30.000000)
MSS of errors squared = 5.557458e-02
Corrected MSS of errors squared = 9.625800e-02
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*log10(i)^4

Fitting J5(x) in range (0.000000, 30.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	-0.000542648881871099	0	-Inf	0
x1	0.0715954043939782	0	Inf	0
x2	-0.0815706464839074	0	-Inf	0
x3	0.0105168592737584	0	Inf	0
x4	1262.68711247	0	Inf	0
x5	-1627.98843026872	0	-Inf	0
x6	459.830456727527	0	Inf	0
x7	-110.092198751518	0	-Inf	0
x8	18.0133232780653	0	Inf	0
x9	-1.45026274567098	0	-Inf	0

Number of observations: 301, Error degrees of freedom: 291
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	18.6374025324071	300	0.0621246751080238		
Model	18.6374025324071	9	2.07082250360079	Inf	0
Residual	0	291	0		

A = 0.543018, B = 2.296990
order Q(x) = 3.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 30.000000
Ymin = -0.261053 and Ymax = 0.374075
List of powers for Q(x): [0.543018, 0.561881, 0.662053]


```
List of powers for D(x): [0.543018, 0.561881, 0.662053, 0.844817, 1.091286, 1.385218]
Fitting J5(x) in range (0.000000, 30.000000)
MSS of errors squared = 9.352232e-01
Corrected MSS of errors squared = 1.322605e+00
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the Bessel $J_5(x)$ function. The results are sorted in ascending order using the MSSE values.

Power	*A*	*B*	*C*	*F*	*AICc*	*MSSE*	*Q/D Orders*
A+B*sqrt(i)	0.894	0.656		Inf	-Inf	4.72908700E-02	7, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.291	0.15	0	Inf	-Inf	5.55745800E-02	6, 5
A+B*i	0.5	0.135		Inf	-Inf	1.05631700E-01	3, 7
A+B*i-C*(i-1)	0.613	0.248	0.158	Inf	-Inf	1.08384400E-01	7, 7
A+B*i+C*(i-1)	0.232	0.12	0.007	Inf	-Inf	1.32196100E-01	4, 7
A+B*sqrt(i+C)	0	0.153	0	Inf	-Inf	1.37850900E-01	4, 7
A+B*i+C*sqrt(i-1)	0.224	0.195	0.011	Inf	-Inf	1.41079100E-01	6, 7
A+B*log10(i)^4	0.543	2.296		Inf	-Inf	9.35223200E-01	3, 6
A+B*sqrt(i)+C*(i-1)	0.55	0.1	0	Inf	-Inf	1.07552300E+00	3, 7
A+B/i	0	0.1		Inf	-Inf	1.78613100E+00	7, 3

RESULTS FOR THE NATURAL LOGRITHM

Using Power A+B* i

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                    ________________    ___    _____    _______
    (Intercept)      2.67467183384446    0       Inf        0
    x1              -44.0749644143026    0      -Inf        0
    x2               75.3170369364355    0       Inf        0
    x3                             0    0       NaN       NaN
    x4              -72.278573027121     0      -Inf        0
    x5               43.3997704939648    0       Inf        0
    x6              -5.03469920225113    0      -Inf        0
    x7               27.8122449290219    0       Inf        0
    x8              -75.2535695029877    0      -Inf        0
    x9               74.3147611365159    0       Inf        0
    x10             -27.4549408922359    0      -Inf        0
    x11              1.57826170921259    0       Inf        0
Number of observations: 901, Error degrees of freedom: 890
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq          DF           MeanSq            F      pValue
                    ________________    ___    ____________________    ___    _______
    Total           58.9211587782479    900    0.0654679541980533
    Model           58.9211587782479     10      5.89211587782479    Inf        0
    Residual                       0    890                     0
A = 0.380124, B = 0.253972
order Q(x) = 6.000000, order D(x) = 5.000000
Xmin = 1.000000 and Xmax = 10.000000
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.634096, 0.888069, 1.142041, 1.396014, 1.649986, 1.903959]
```



```
List of powers for D(x): [0.634096, 0.888069, 1.142041, 1.396014, 1.649986]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 6.567432e-11
Corrected MSS of errors squared = 9.287752e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)      1.80028737252176     0      Inf         0
    x1              -333.063315599088     0     -Inf         0
    x2               2982.20796607073     0      Inf         0
    x3              -5909.64363639739     0     -Inf         0
    x4               3258.63649268694     0      Inf         0
    x5               47.7230895892327     0      Inf         0
    x6              -60.0772154549506     0     -Inf         0
    x7              -161.994859726033     0     -Inf         0
    x8                              0     0      NaN       NaN
    x9               175.411191394725     0      Inf         0
Number of observations: 901, Error degrees of freedom: 892
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F      pValue
                  ___________________    ___    ___________________    ___    ______
    Total        58.9211587780913     900    0.0654679541978792
    Model        58.9211587780913       8      7.36514484726141    Inf       0
    Residual                    0     892                     0
A = 1.300549, B = 0.502634
order Q(x) = 4.000000, order D(x) = 5.000000
Xmin = 1.000000 and Xmax = 10.000000
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [1.803183, 1.551866, 1.468094, 1.426208]
List of powers for D(x): [1.803183, 1.551866, 1.468094, 1.426208, 1.401076]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 2.464101e-09
Corrected MSS of errors squared = 3.484765e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```


Using Power A+B*sqrt(i)

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.0626247387460643     0     -Inf       0
    x1                929.492236251089     0      Inf       0
    x2               -1972.65982158393     0     -Inf       0
    x3                                0     0      NaN      NaN
    x4                                0     0      NaN      NaN
    x5                2210.72475142652     0      Inf       0
    x6                                0     0      NaN      NaN
    x7               -1167.65849811485     0     -Inf       0
    x8               -488.562424829699     0     -Inf       0
    x9                1810.04650885202     0      Inf       0
    x10              -1613.27148903306     0     -Inf       0
    x11                               0     0      NaN      NaN
    x12               -36.261539601484     0     -Inf       0
    x13                329.21290133685     0      Inf       0
Number of observations: 901, Error degrees of freedom: 891
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ____    ____________________    ___    ______
    Total        58.9211587782107    900    0.0654679541980119
    Model        58.9211587782107      9      6.54679541980119    Inf       0
    Residual                    0    891                     0
A = 1.083553, B = 0.449056
order Q(x) = 7.000000, order D(x) = 6.000000
Xmin = 1.000000 and Xmax = 10.000000
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [1.532609, 1.718614, 1.861341, 1.981665, 2.087673, 2.183512,
2.271644]
List of powers for D(x): [1.532609, 1.718614, 1.861341, 1.981665, 2.087673, 2.183512]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 4.883163e-10
Corrected MSS of errors squared = 6.905835e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
```



```
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ________________    __    _____    ______
    (Intercept)    0.500996189786673    0      Inf        0
    x1              516.264572300994    0      Inf        0
    x2             -1714.90827710893    0     -Inf        0
    x3                             0    0      NaN      NaN
    x4              3220.52864073482    0      Inf        0
    x5                             0    0      NaN      NaN
    x6             -4166.3291794692     0     -Inf        0
    x7              2143.85550109706    0      Inf        0
    x8             -143.163093066991    0     -Inf        0
    x9              658.028172267728    0      Inf        0
    x10                            0    0      NaN      NaN
    x11            -2678.69219240598    0     -Inf        0
    x12             3433.81720677349    0      Inf        0
    x13            -1268.90234732063    0     -Inf        0
Number of observations: 901, Error degrees of freedom: 890
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F      pValue
                   _______________    ___    ___________________    ___    ______
    Total       58.9211587799612     900    0.0654679541999569
    Model       58.9211587799612      10      5.89211587799612    Inf       0
    Residual                   0     890                     0
A = 0.998277, B = 0.788682, C = 0.000000
order Q(x) = 7.000000, order D(x) = 6.000000
Xmin = 1.000000 and Xmax = 10.000000
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [1.786959, 2.113642, 2.364314, 2.575641, 2.761823, 2.930145,
3.084933]
List of powers for D(x): [1.786959, 2.113642, 2.364314, 2.575641, 2.761823, 2.930145]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 2.048688e-10
Corrected MSS of errors squared = 3.548432e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ________________    __    _____    ______
```



```
    (Intercept)      1.79728690942035    0      Inf        0
    x1              -9.45245013900747    0     -Inf        0
    x2               11.7517201958889    0      Inf        0
    x3                             0    0      NaN      NaN
    x4              -5.58465197725168    0     -Inf        0
    x5                             0    0      NaN      NaN
    x6               1.48890443147367    0      Inf        0
    x7               7.36469346642871    0      Inf        0
    x8              -15.4743162737197    0     -Inf        0
    x9               14.1433941794616    0      Inf        0
    x10             -6.87752709886107    0     -Inf        0
    x11              2.85721887726109    0      Inf        0
    x12             -1.1062962820007    0     -Inf        0
    x13            0.0920237109115982    0      Inf        0
Number of observations: 901, Error degrees of freedom: 889
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ___________________    ___    ____________________    ___    ______
    Total       58.9211587781873     900    0.0654679541979859
    Model       58.9211587781873      11      5.35646897983521    Inf       0
    Residual                   0     889                     0
A = 0.326697, B = 0.100000, C = 0.220387
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 1.000000 and Xmax = 10.000000
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.426697, 0.747084, 1.067472, 1.387859, 1.708247, 2.028634]
List of powers for D(x): [0.426697, 0.747084, 1.067472, 1.387859, 1.708247, 2.028634,
2.349022]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 1.383734e-11
Corrected MSS of errors squared = 2.396697e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)    -0.573155792821806     0     -Inf       0
    x1               186.139278334171     0      Inf       0
    x2              -694.737546197472     0     -Inf       0
    x3               1015.21134844508     0      Inf       0
```



```
    x4              -724.652158563193    0     -Inf       0
    x5               257.539023268733    0      Inf       0
    x6               -38.929488013564    0     -Inf       0
    x7               21.5444805869092    0      Inf       0
    x8              -37.0218088676552    0     -Inf       0
    x9               16.4800267992347    0      Inf       0
Number of observations: 901, Error degrees of freedom: 891
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                 _______________    ___    ____________________    ___    ______
    Total        58.9211587780104    900    0.0654679541977893
    Model        58.9211587780104      9      6.54679541977893    Inf       0
    Residual                    0    891                     0
A = 0.441023, B = 0.164881, C = 0.023663
order Q(x) = 6.000000, order D(x) = 3.000000
Xmin = 1.000000 and Xmax = 10.000000
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.605903, 0.747121, 0.888338, 1.029556, 1.170773, 1.311990]
List of powers for D(x): [0.605903, 0.747121, 0.888338]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 4.974676e-10
Corrected MSS of errors squared = 8.616392e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    _____________________    __    _____    _______
    (Intercept)      2.52091840502023    0      Inf        0
    x1               -7.1611631800354    0     -Inf        0
    x2               8.67124990273302    0      Inf        0
    x3                              0    0      NaN      NaN
    x4               -7.90674793248278    0     -Inf        0
    x5               3.87586005347834    0      Inf        0
    x6               3.20633440389659    0      Inf        0
    x7               -4.88079541288549    0     -Inf        0
    x8                              0    0      NaN      NaN
    x9               8.02449498192711    0      Inf        0
    x10              -7.97019129505762    0     -Inf        0
    x11              3.12471241029978    0      Inf        0
    x12             -0.504672336890351    0     -Inf        0
```



```
Number of observations: 901, Error degrees of freedom: 890
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ___________________    ___    ______
    Total       58.9211587781886    900    0.0654679541979873
    Model       58.9211587781886     10      5.89211587781886    Inf       0
    Residual                   0    890                     0
A = 0.040067, B = 0.127866, C = 0.134648
order Q(x) = 5.000000, order D(x) = 7.000000
Xmin = 1.000000 and Xmax = 10.000000
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.167932, 0.430446, 0.614085, 0.784747, 0.948692]
List of powers for D(x): [0.167932, 0.430446, 0.614085, 0.784747, 0.948692, 1.108344,
1.264946]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 6.568430e-11
Corrected MSS of errors squared = 1.137685e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)       3.02556944560295     0      Inf       0
    x1               -11.1091946851446     0     -Inf       0
    x2                13.9806545134238     0      Inf       0
    x3               -5.90765714839657     0     -Inf       0
    x4                4.50831662208753     0      Inf       0
    x5               -7.17406078845333     0     -Inf       0
    x6                4.51693686831404     0      Inf       0
    x7              -0.964470122265525     0     -Inf       0
    x8               0.123905277274495     0      Inf       0
Number of observations: 901, Error degrees of freedom: 892
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ___________________    ___    ______
    Total       58.9211587781889    900    0.0654679541979877
    Model       58.9211587781889      8      7.36514484727362    Inf       0
    Residual                   0    892                     0
A = 0.122034, B = 0.312982, C = 0.325539
```



```
order Q(x) = 3.000000, order D(x) = 5.000000
Xmin = 1.000000 and Xmax = 10.000000
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.435016, 0.890196, 1.315212]
List of powers for D(x): [0.435016, 0.890196, 1.315212, 1.724614, 2.124038]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 4.196279e-09
Corrected MSS of errors squared = 7.268168e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    _________________    __    _____    ______
    (Intercept)      1.36300793890357    0      Inf        0
    x1               35.3281604493715    0      Inf        0
    x2               -250.71977093116    0     -Inf        0
    x3               381.851277739143    0      Inf        0
    x4                             0     0      NaN      NaN
    x5              -388.759575488175    0     -Inf        0
    x6               290.915290697027    0      Inf        0
    x7              -70.0048769656512    0     -Inf        0
    x8               27.0578895072909    0      Inf        0
    x9              -56.2507653837849    0     -Inf        0
    x10              30.2193624291958    0      Inf        0
Number of observations: 901, Error degrees of freedom: 891
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F      pValue
                  _______________    ___    ________________    ___    ______
    Total        58.9211587782024    900    0.0654679541980026
    Model        58.9211587782024      9      6.54679541980026    Inf        0
    Residual                    0    891                      0
A = 0.433912, B = 0.700000, C = 0.000000
order Q(x) = 7.000000, order D(x) = 3.000000
Xmin = 1.000000 and Xmax = 10.000000
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [1.133912, 1.423862, 1.646348, 1.833912, 1.999160, 2.148555,
2.285938]
List of powers for D(x): [1.133912, 1.423862, 1.646348]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 6.929069e-10
```



```
Corrected MSS of errors squared = 1.200150e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)       1.2825257935788    0      Inf        0
    x1               -10.0266230163606    0     -Inf        0
    x2                              0    0      NaN      NaN
    x3                11.5931584619824    0      Inf        0
    x4               -2.20529593021609    0     -Inf        0
    x5               -0.719365041242909    0     -Inf        0
    x6               0.0743582243100353    0      Inf        0
    x7                79.9200953097517    0      Inf        0
    x8               -90.7943734337573    0     -Inf        0
    x9                11.8755196305335    0      Inf        0
Number of observations: 901, Error degrees of freedom: 892
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______
    Total       58.9211587782052    900    0.0654679541980058
    Model       58.9211587782052      8      7.36514484727565    Inf       0
    Residual                   0    892                     0
A = 0.409269, B = 3.041922
order Q(x) = 6.000000, order D(x) = 3.000000
Xmin = 1.000000 and Xmax = 10.000000
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.409269, 0.434248, 0.566907, 0.808944, 1.135345, 1.524600]
List of powers for D(x): [0.409269, 0.434248, 0.566907]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 2.842604e-09
Corrected MSS of errors squared = 4.020049e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the natural logarithm function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i+C*(i-1)	0.208	0.161	0.146	Inf	-Inf	1.04284400E-11	7, 4
A+B*sqrt(i)+C*sqrt(i-1)	0.38	0.531	0.621	Inf	-Inf	1.32559400E-11	6, 5
A+B*i+C*sqrt(i-1)	0.074	0.336	0.111	Inf	-Inf	7.25503800E-11	6, 4
A+B*sqrt(i+C)	0.893	0.959	0.413	Inf	-Inf	1.41042700E-10	4, 6
A+B/i	2	1.942		Inf	-Inf	1.31647200E-09	5, 7
A+B*i	0.416	0.111		Inf	-Inf	2.64185100E-09	4, 5
A+B*log10(i)^4	0.587	1.37		Inf	-Inf	2.78316800E-09	3, 5
A+B*sqrt(i)+C*(i-1)	0.477	0.161	0.058	Inf	-Inf	2.94548900E-09	3, 5
A+B*sqrt(i)	0	0.137		Inf	-Inf	1.37512400E-08	3, 4
A+B*i-C*(i-1)	0.158	0.1	0.096	Inf	-Inf	1.38183200E-08	5, 5

RESULTS FOR THE COMMON LOGARITHM

Using Power A+B* i

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ________________    ___    ______    _______
    (Intercept)      2.67467183384446    0       Inf         0
    x1              -44.0749644143026    0      -Inf         0
    x2               75.3170369364355    0       Inf         0
    x3                             0    0       NaN       NaN
    x4              -72.278573027121     0      -Inf         0
    x5               43.3997704939648    0       Inf         0
    x6              -5.03469920225113    0      -Inf         0
    x7               27.8122449290219    0       Inf         0
    x8              -75.2535695029877    0      -Inf         0
    x9               74.3147611365159    0       Inf         0
    x10             -27.4549408922359    0      -Inf         0
    x11              1.57826170921259    0       Inf         0
Number of observations: 901, Error degrees of freedom: 890
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                        SumSq          DF            MeanSq            F      pValue
                    _______________    ___    ____________________    ___    _______
    Total           58.9211587782479    900    0.0654679541980533
    Model           58.9211587782479     10      5.89211587782479    Inf         0
    Residual                       0    890                     0
A = 0.380124, B = 0.253972
order Q(x) = 6.000000, order D(x) = 5.000000
Xmin = 1.000000 and Xmax = 10.000000
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.634096, 0.888069, 1.142041, 1.396014, 1.649986, 1.903959]
```



```
List of powers for D(x): [0.634096, 0.888069, 1.142041, 1.396014, 1.649986]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 6.567432e-11
Corrected MSS of errors squared = 9.287752e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)      1.80028737252176     0      Inf         0
    x1              -333.063315599088     0     -Inf         0
    x2               2982.20796607073     0      Inf         0
    x3              -5909.64363639739     0     -Inf         0
    x4               3258.63649268694     0      Inf         0
    x5               47.7230895892327     0      Inf         0
    x6              -60.0772154549506     0     -Inf         0
    x7              -161.994859726033     0     -Inf         0
    x8                              0     0      NaN       NaN
    x9               175.411191394725     0      Inf         0
Number of observations: 901, Error degrees of freedom: 892
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F      pValue
                   ____________________    ____    ____________________    ____    _______
    Total       58.9211587780913     900    0.0654679541978792
    Model       58.9211587780913       8      7.36514484726141    Inf         0
    Residual                   0     892                     0
A = 1.300549, B = 0.502634
order Q(x) = 4.000000, order D(x) = 5.000000
Xmin = 1.000000 and Xmax = 10.000000
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [1.803183, 1.551866, 1.468094, 1.426208]
List of powers for D(x): [1.803183, 1.551866, 1.468094, 1.426208, 1.401076]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 2.464101e-09
Corrected MSS of errors squared = 3.484765e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```


Using Power A+B*sqrt(i)

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.0626247387460643     0     -Inf       0
    x1                929.492236251089     0      Inf       0
    x2               -1972.65982158393     0     -Inf       0
    x3                                0     0      NaN      NaN
    x4                                0     0      NaN      NaN
    x5                2210.72475142652     0      Inf       0
    x6                                0     0      NaN      NaN
    x7               -1167.65849811485     0     -Inf       0
    x8               -488.562424829699     0     -Inf       0
    x9                1810.04650885202     0      Inf       0
    x10              -1613.27148903306     0     -Inf       0
    x11                               0     0      NaN      NaN
    x12               -36.261539601484     0     -Inf       0
    x13                329.21290133685     0      Inf       0
Number of observations: 901, Error degrees of freedom: 891
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ___________________    ___    ____________________    ___    ______
    Total       58.9211587782107     900    0.0654679541980119
    Model       58.9211587782107       9      6.54679541980119    Inf       0
    Residual                   0     891                     0
A = 1.083553, B = 0.449056
order Q(x) = 7.000000, order D(x) = 6.000000
Xmin = 1.000000 and Xmax = 10.000000
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [1.532609, 1.718614, 1.861341, 1.981665, 2.087673, 2.183512,
2.271644]
List of powers for D(x): [1.532609, 1.718614, 1.861341, 1.981665, 2.087673, 2.183512]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 4.883163e-10
Corrected MSS of errors squared = 6.905835e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
```



```
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ________________    __    _____    ______
    (Intercept)    0.500996189786673    0      Inf        0
    x1              516.264572300994    0      Inf        0
    x2             -1714.90827710893    0     -Inf        0
    x3                             0    0      NaN      NaN
    x4              3220.52864073482    0      Inf        0
    x5                             0    0      NaN      NaN
    x6             -4166.3291794692     0     -Inf        0
    x7              2143.85550109706    0      Inf        0
    x8             -143.163093066991    0     -Inf        0
    x9              658.028172267728    0      Inf        0
    x10                            0    0      NaN      NaN
    x11            -2678.69219240598    0     -Inf        0
    x12             3433.81720677349    0      Inf        0
    x13            -1268.90234732063    0     -Inf        0
Number of observations: 901, Error degrees of freedom: 890
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F      pValue
                   _______________    ___    ___________________    ___    ______
    Total       58.9211587799612     900    0.0654679541999569
    Model       58.9211587799612      10      5.89211587799612    Inf       0
    Residual                   0     890                     0
A = 0.998277, B = 0.788682, C = 0.000000
order Q(x) = 7.000000, order D(x) = 6.000000
Xmin = 1.000000 and Xmax = 10.000000
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [1.786959, 2.113642, 2.364314, 2.575641, 2.761823, 2.930145,
3.084933]
List of powers for D(x): [1.786959, 2.113642, 2.364314, 2.575641, 2.761823, 2.930145]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 2.048688e-10
Corrected MSS of errors squared = 3.548432e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ________________    __    _____    ______
```



```
    (Intercept)      1.79728690942035    0      Inf        0
    x1              -9.45245013900747    0     -Inf        0
    x2               11.7517201958889    0      Inf        0
    x3                             0    0      NaN      NaN
    x4              -5.58465197725168    0     -Inf        0
    x5                             0    0      NaN      NaN
    x6               1.48890443147367    0      Inf        0
    x7               7.36469346642871    0      Inf        0
    x8              -15.4743162737197    0     -Inf        0
    x9               14.1433941794616    0      Inf        0
    x10             -6.87752709886107    0     -Inf        0
    x11              2.85721887726109    0      Inf        0
    x12             -1.1062962820007    0     -Inf        0
    x13            0.0920237109115982    0      Inf        0
Number of observations: 901, Error degrees of freedom: 889
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ___________________    ___    ____________________    ___    ______
    Total       58.9211587781873     900    0.0654679541979859
    Model       58.9211587781873      11      5.35646897983521    Inf       0
    Residual                   0     889                     0
A = 0.326697, B = 0.100000, C = 0.220387
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 1.000000 and Xmax = 10.000000
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.426697, 0.747084, 1.067472, 1.387859, 1.708247, 2.028634]
List of powers for D(x): [0.426697, 0.747084, 1.067472, 1.387859, 1.708247, 2.028634,
2.349022]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 1.383734e-11
Corrected MSS of errors squared = 2.396697e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)    -0.573155792821806     0     -Inf       0
    x1               186.139278334171     0      Inf       0
    x2              -694.737546197472     0     -Inf       0
    x3               1015.21134844508     0      Inf       0
```



```
    x4              -724.652158563193    0     -Inf       0
    x5               257.539023268733    0      Inf       0
    x6               -38.929488013564    0     -Inf       0
    x7               21.5444805869092    0      Inf       0
    x8              -37.0218088676552    0     -Inf       0
    x9               16.4800267992347    0      Inf       0
Number of observations: 901, Error degrees of freedom: 891
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                 _______________    ___    ____________________    ___    ______
    Total        58.9211587780104    900    0.0654679541977893
    Model        58.9211587780104      9      6.54679541977893    Inf       0
    Residual                    0    891                     0
A = 0.441023, B = 0.164881, C = 0.023663
order Q(x) = 6.000000, order D(x) = 3.000000
Xmin = 1.000000 and Xmax = 10.000000
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.605903, 0.747121, 0.888338, 1.029556, 1.170773, 1.311990]
List of powers for D(x): [0.605903, 0.747121, 0.888338]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 4.974676e-10
Corrected MSS of errors squared = 8.616392e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    _____________________    __    _____    _______
    (Intercept)      2.52091840502023    0      Inf        0
    x1               -7.1611631800354    0     -Inf        0
    x2               8.67124990273302    0      Inf        0
    x3                              0    0      NaN       NaN
    x4               -7.90674793248278    0     -Inf        0
    x5               3.87586005347834    0      Inf        0
    x6               3.20633440389659    0      Inf        0
    x7               -4.88079541288549    0     -Inf        0
    x8                              0    0      NaN       NaN
    x9               8.02449498192711    0      Inf        0
    x10              -7.97019129505762    0     -Inf        0
    x11              3.12471241029978    0      Inf        0
    x12             -0.504672336890351    0     -Inf        0
```



```
Number of observations: 901, Error degrees of freedom: 890
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ___________________    ___    ______
    Total       58.9211587781886    900    0.0654679541979873
    Model       58.9211587781886     10      5.89211587781886    Inf       0
    Residual                   0    890                     0
A = 0.040067, B = 0.127866, C = 0.134648
order Q(x) = 5.000000, order D(x) = 7.000000
Xmin = 1.000000 and Xmax = 10.000000
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.167932, 0.430446, 0.614085, 0.784747, 0.948692]
List of powers for D(x): [0.167932, 0.430446, 0.614085, 0.784747, 0.948692, 1.108344,
1.264946]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 6.568430e-11
Corrected MSS of errors squared = 1.137685e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)       3.02556944560295     0      Inf       0
    x1               -11.1091946851446     0     -Inf       0
    x2                13.9806545134238     0      Inf       0
    x3               -5.90765714839657     0     -Inf       0
    x4                4.50831662208753     0      Inf       0
    x5               -7.17406078845333     0     -Inf       0
    x6                4.51693686831404     0      Inf       0
    x7              -0.964470122265525     0     -Inf       0
    x8               0.123905277274495     0      Inf       0
Number of observations: 901, Error degrees of freedom: 892
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ___________________    ___    ______
    Total       58.9211587781889    900    0.0654679541979877
    Model       58.9211587781889      8      7.36514484727362    Inf       0
    Residual                   0    892                     0
A = 0.122034, B = 0.312982, C = 0.325539
```



```
order Q(x) = 3.000000, order D(x) = 5.000000
Xmin = 1.000000 and Xmax = 10.000000
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.435016, 0.890196, 1.315212]
List of powers for D(x): [0.435016, 0.890196, 1.315212, 1.724614, 2.124038]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 4.196279e-09
Corrected MSS of errors squared = 7.268168e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    _________________    __    _____    ______
    (Intercept)      1.36300793890357    0      Inf        0
    x1               35.3281604493715    0      Inf        0
    x2               -250.71977093116    0     -Inf        0
    x3               381.851277739143    0      Inf        0
    x4                             0     0      NaN      NaN
    x5              -388.759575488175    0     -Inf        0
    x6               290.915290697027    0      Inf        0
    x7              -70.0048769656512    0     -Inf        0
    x8               27.0578895072909    0      Inf        0
    x9              -56.2507653837849    0     -Inf        0
    x10              30.2193624291958    0      Inf        0
Number of observations: 901, Error degrees of freedom: 891
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F      pValue
                  _______________    ___    ________________    ___    ______
    Total        58.9211587782024    900    0.0654679541980026
    Model        58.9211587782024      9      6.54679541980026    Inf        0
    Residual                    0    891                      0
A = 0.433912, B = 0.700000, C = 0.000000
order Q(x) = 7.000000, order D(x) = 3.000000
Xmin = 1.000000 and Xmax = 10.000000
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [1.133912, 1.423862, 1.646348, 1.833912, 1.999160, 2.148555,
2.285938]
List of powers for D(x): [1.133912, 1.423862, 1.646348]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 6.929069e-10
```



```
Corrected MSS of errors squared = 1.200150e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting log(x) in range (1.000000, 10.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)       1.2825257935788    0      Inf        0
    x1               -10.0266230163606    0     -Inf        0
    x2                              0    0      NaN      NaN
    x3                11.5931584619824    0      Inf        0
    x4               -2.20529593021609    0     -Inf        0
    x5               -0.719365041242909    0     -Inf        0
    x6               0.0743582243100353    0      Inf        0
    x7                79.9200953097517    0      Inf        0
    x8               -90.7943734337573    0     -Inf        0
    x9                11.8755196305335    0      Inf        0
Number of observations: 901, Error degrees of freedom: 892
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______
    Total       58.9211587782052    900    0.0654679541980058
    Model       58.9211587782052      8      7.36514484727565    Inf       0
    Residual                   0    892                     0
A = 0.409269, B = 3.041922
order Q(x) = 6.000000, order D(x) = 3.000000
Xmin = 1.000000 and Xmax = 10.000000
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.409269, 0.434248, 0.566907, 0.808944, 1.135345, 1.524600]
List of powers for D(x): [0.409269, 0.434248, 0.566907]
Fitting log(x) in range (1.000000, 10.000000)
MSS of errors squared = 2.842604e-09
Corrected MSS of errors squared = 4.020049e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the common logarithm function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i+C*(i-1)	0.326	0.1	0.22	Inf	-Inf	1.38373400E-11	6, 7
A+B*i	0.38	0.253		Inf	-Inf	6.56743200E-11	6, 5
A+B*i+C*sqrt(i-1)	0.04	0.127	0.134	Inf	-Inf	6.56843000E-11	5, 7
A+B*sqrt(i+C)	0.998	0.788	0	Inf	-Inf	2.04868800E-10	7, 6
A+B*sqrt(i)	1.083	0.449		Inf	-Inf	4.88316300E-10	7, 6
A+B*i-C*(i-1)	0.441	0.164	0.023	Inf	-Inf	4.97467600E-10	6, 3
A+B*sqrt(i)+C*sqrt(i-1)	0.433	0.7	0	Inf	-Inf	6.92906900E-10	7, 3
A+B/i	1.3	0.502		Inf	-Inf	2.46410100E-09	4, 5
A+B*log10(i)^4	0.409	3.041		Inf	-Inf	2.84260400E-09	6, 3
A+B*sqrt(i)+C*(i-1)	0.122	0.312	0.325	Inf	-Inf	4.19627900E-09	3, 5

RESULTS FOR THE COMMON LOGARITHM OF THE GAMMA FUNCTION

Using Power A+B* i

```
Fitting log10Gamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)       1.32389144155301      0      Inf        0
    x1               -7.92102843249225      0     -Inf        0
    x2                              0      0      NaN       NaN
    x3                22.3332548443079      0      Inf        0
    x4                              0      0      NaN       NaN
    x5               -51.7692477797598      0     -Inf        0
    x6                51.2576460282081      0      Inf        0
    x7               -15.2245586330304      0     -Inf        0
    x8                7.24658949796895      0      Inf        0
    x9               -11.6356617909128      0     -Inf        0
    x10                6.3051453493908      0      Inf        0
    x11              -0.780262275315588     0     -Inf        0
    x12              -0.135768249794691     0     -Inf        0
Number of observations: 981, Error degrees of freedom: 970
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                        SumSq          DF           MeanSq           F      pValue
                    ____________________    ___    ____________________    ___    _____
    Total         88.4325686540131     980    0.0902373149530746
    Model         88.4325686540131      10      8.84325686540131    Inf       0
    Residual                     0     970                     0
A = 0.157261, B = 0.171325
order Q(x) = 7.000000, order D(x) = 5.000000
Xmin = 2.000000 and Xmax = 100.000000
```



```
Ymin = 0.000000 and Ymax = 155.970004
List of powers for Q(x): [0.328587, 0.499912, 0.671237, 0.842562, 1.013887, 1.185213,
1.356538]
List of powers for D(x): [0.328587, 0.499912, 0.671237, 0.842562, 1.013887]
Fitting log10Gamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 4.691761e-09
Corrected MSS of errors squared = 6.635151e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting log10Gamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)       3.92230707724888      0      Inf         0
    x1                147.563731550395      0      Inf         0
    x2               -1803.18872538321      0     -Inf         0
    x3                3226.25846971408      0      Inf         0
    x4                               0      0      NaN       NaN
    x5               -1574.55247700242      0     -Inf         0
    x6              -0.250508455978616      0     -Inf         0
    x7                43.7205989488894      0      Inf         0
    x8               -153.201964155386      0     -Inf         0
    x9                110.728567602589      0      Inf         0
Number of observations: 981, Error degrees of freedom: 972
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F      pValue
                  _______________    ___    ___________________    ___    _______
    Total       88.4325686545408     980    0.090237314953613
    Model       88.4325686545408       8     11.0540710818176    Inf        0
    Residual                   0     972                    0
A = 0.659637, B = 0.409970
order Q(x) = 5.000000, order D(x) = 4.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 155.970004
List of powers for Q(x): [1.069607, 0.864622, 0.796294, 0.762130, 0.741631]
List of powers for D(x): [1.069607, 0.864622, 0.796294, 0.762130]
Fitting log10Gamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 5.846056e-08
Corrected MSS of errors squared = 8.267571e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
```



```
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting log10Gamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      0.63770053729962     0      Inf        0
    x1              -91.6938168561544     0     -Inf        0
    x2               418.837542806067     0      Inf        0
    x3              -552.758554941568     0     -Inf        0
    x4                             0     0      NaN      NaN
    x5               440.424330976359     0      Inf        0
    x6              -215.450570102719     0     -Inf        0
    x7                 32.00717086908     0      Inf        0
    x8              -82.0071239125274     0     -Inf        0
    x9                71.631445498659     0      Inf        0
    x10              -23.184493219906     0     -Inf        0
    x11              2.55636835669193     0      Inf        0
Number of observations: 981, Error degrees of freedom: 970
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                        SumSq          DF           MeanSq            F      pValue
                    _________________    ___    ____________________    ___    ______
    Total        88.432568653841     980    0.090237314952899
    Model        88.432568653841      10      8.8432568653841    Inf        0
    Residual                   0     970                    0
A = 0.583178, B = 0.843044
order Q(x) = 6.000000, order D(x) = 5.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 155.970004
List of powers for Q(x): [1.426223, 1.775423, 2.043374, 2.269267, 2.468283, 2.648207]
List of powers for D(x): [1.426223, 1.775423, 2.043374, 2.269267, 2.468283]
Fitting log10Gamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 5.445125e-09
Corrected MSS of errors squared = 7.700570e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting log10Gamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
```



```
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    0.00512498147587384     0      Inf       0
    x1              -0.071098074538937     0     -Inf       0
    x2               0.125261324593977     0      Inf       0
    x3              -0.0592870276689582    0     -Inf       0
    x4                 113.592237835875    0      Inf       0
    x5                -637.733654302079    0     -Inf       0
    x6                 1435.56523543161    0      Inf       0
    x7                -1599.96505485029    0     -Inf       0
    x8                 882.482763016931    0      Inf       0
    x9                -192.941528335915    0     -Inf       0
Number of observations: 99, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF          MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______
    Total        9.07288178684332    98    0.0925804263963604
    Model        9.07288178684332     9      1.00809797631592    Inf       0
    Residual                    0    89                     0
A = 0.050806, B = 0.254133, C = 0.315127
order Q(x) = 3.000000, order D(x) = 6.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 155.970004
List of powers for Q(x): [0.342244, 0.437484, 0.513519]
List of powers for D(x): [0.342244, 0.437484, 0.513519, 0.578714, 0.636699, 0.689441]
Fitting log10Gamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 2.207106e-06
Corrected MSS of errors squared = 3.822820e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting log10Gamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      1.13583049147716      0      Inf       0
    x1              -12.1896095644351      0     -Inf       0
    x2               34.4152974908992      0      Inf       0
    x3              -44.646989004031       0     -Inf       0
    x4               28.7564891627147      0      Inf       0
```



```
    x5              -7.73260368957459    0      -Inf        0
    x6              0.415909748431839    0       Inf        0
    x7             -0.154987932007273    0      -Inf        0
    x8               5.06217324557584    0       Inf        0
    x9              -6.71176708348553    0      -Inf        0
    x10              2.6502571364407     0       Inf        0
Number of observations: 981, Error degrees of freedom: 970
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F      pValue
                 _______________    ___    ____________________    ___    _______
    Total        88.4325686540442    980    0.0902373149531063
    Model        88.4325686540442     10      8.84325686540442    Inf        0
    Residual                    0    970                     0
A = 0.412685, B = 0.226145, C = 0.210078
order Q(x) = 7.000000, order D(x) = 3.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 155.970004
List of powers for Q(x): [0.638830, 1.075052, 1.511274, 1.947497, 2.383719, 2.819941,
3.256163]
List of powers for D(x): [0.638830, 1.075052, 1.511274]
Fitting log10Gamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 4.914471e-09
Corrected MSS of errors squared = 8.512113e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting log10Gamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                     Estimate          SE    tStat    pValue
                 _______________    ___    _____    _______
    (Intercept)   4.94549402112946    0      Inf        0
    x1            4411.28820169287    0      Inf        0
    x2                           0    0      NaN      NaN
    x3           -15357.8269558344    0     -Inf        0
    x4                           0    0      NaN      NaN
    x5            24163.6890700485    0      Inf        0
    x6           -13222.0936828002    0     -Inf        0
    x7           -784.277073758815    0     -Inf        0
    x8            2670.33864682915    0      Inf        0
    x9           -3020.9890568382     0     -Inf        0
    x10           1135.9253565747     0      Inf        0
Number of observations: 981, Error degrees of freedom: 972
```



```
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______
    Total       88.4325686548984    980    0.090237314953978
    Model       88.4325686548984      8     11.0540710818623    Inf       0
    Residual                    0    972                    0
A = 0.691478, B = 0.109871, C = 0.133253
order Q(x) = 6.000000, order D(x) = 4.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 155.970004
List of powers for Q(x): [0.801350, 0.777967, 0.754585, 0.731202, 0.707820, 0.684438]
List of powers for D(x): [0.801350, 0.777967, 0.754585, 0.731202]
Fitting log10Gamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 5.704937e-08
Corrected MSS of errors squared = 9.881240e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting log10Gamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    ___    _____    ______
    (Intercept)       1.27537012172562    0       Inf       0
    x1                -4.8554057803359    0      -Inf       0
    x2                14.0040034323189    0       Inf       0
    x3               -21.5307947358521    0      -Inf       0
    x4                15.3024999022088    0       Inf       0
    x5               -4.19595415857397    0      -Inf       0
    x6                2.49852368211484    0       Inf       0
    x7               -2.83585219597909    0      -Inf       0
    x8                 1.4791332196062    0       Inf       0
    x9              -0.141523486203168    0      -Inf       0
Number of observations: 981, Error degrees of freedom: 971
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF          MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______
    Total       88.432568654003    980    0.0902373149530643
    Model       88.432568654003      9      9.82584096155589    Inf       0
    Residual                  0    971                    0
A = 0.041676, B = 0.320862, C = 0.258723
order Q(x) = 5.000000, order D(x) = 4.000000
```



```
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 155.970004
List of powers for Q(x): [0.362538, 0.942122, 1.370151, 1.773244, 2.163431]
List of powers for D(x): [0.362538, 0.942122, 1.370151, 1.773244]
Fitting log10Gamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 5.931556e-09
Corrected MSS of errors squared = 1.027376e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting log10Gamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                   _________________    __    _____    ______
    (Intercept)     4.69167463172154    0      Inf       0
    x1             -25.3244695680497    0     -Inf       0
    x2              46.6647580813279    0      Inf       0
    x3             -36.8098836842556    0     -Inf       0
    x4              10.7795461897988    0      Inf       0
    x5              2.45226904482806    0      Inf       0
    x6             -1.45922129620917    0     -Inf       0
    x7             -0.13382335146598    0     -Inf       0
    x8             0.139149901076173    0      Inf       0
Number of observations: 981, Error degrees of freedom: 972
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______
    Total        88.4325686540239    980    0.0902373149530856
    Model        88.4325686540239      8       11.054071081753    Inf       0
    Residual                    0    972                     0
A = 0.231715, B = 0.109736, C = 0.239446
order Q(x) = 4.000000, order D(x) = 4.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 155.970004
List of powers for Q(x): [0.341452, 0.626352, 0.900676, 1.169525]
List of powers for D(x): [0.341452, 0.626352, 0.900676, 1.169525]
Fitting log10Gamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 5.868448e-08
Corrected MSS of errors squared = 1.016445e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
```



```
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting log10Gamma(x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ________________    __    _____    ______
    (Intercept)      3.10001710017301    0      Inf        0
    x1              -28.3970240803704    0     -Inf        0
    x2               135.577072647558    0      Inf        0
    x3              -183.261738424293    0     -Inf        0
    x4                             0    0      NaN      NaN
    x5               105.741781436687    0      Inf        0
    x6                             0    0      NaN      NaN
    x7              -32.7632850273389    0     -Inf        0
    x8              -3.50954098152083    0     -Inf        0
    x9               61.7613472362187    0      Inf        0
    x10             -153.821480336199    0     -Inf        0
    x11              151.316813571541    0      Inf        0
    x12             -66.6950147360912    0     -Inf        0
    x13              11.9510515987451    0      Inf        0
Number of observations: 981, Error degrees of freedom: 969
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______
    Total        88.4325686540398     980    0.0902373149531018
    Model        88.4325686540398      11      8.03932442309453     Inf       0
    Residual                    0     969                     0
A = 0.156337, B = 0.692084, C = 0.438251
order Q(x) = 7.000000, order D(x) = 6.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 155.970004
List of powers for Q(x): [0.848421, 1.573343, 1.974843, 2.299578, 2.580386, 2.831549,
3.060911]
List of powers for D(x): [0.848421, 1.573343, 1.974843, 2.299578, 2.580386, 2.831549]
Fitting log10Gamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 2.520678e-09
Corrected MSS of errors squared = 4.365942e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting log10Gamma(x) in range (2.000000, 100.000000)
```



```
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)    -1.16143810648241     0     -Inf        0
    x1              300.691521932572     0      Inf        0
    x2                             0     0      NaN      NaN
    x3                             0     0      NaN      NaN
    x4                             0     0      NaN      NaN
    x5             -1472.08701191558     0     -Inf        0
    x6              1734.68999407638     0      Inf        0
    x7             -562.133483241747     0     -Inf        0
    x8              5257.94256567956     0      Inf        0
    x9                             0     0      NaN      NaN
    x10             -8316.0324706419     0     -Inf        0
    x11                            0     0      NaN      NaN
    x12             6383.68914719271     0      Inf        0
    x13            -4192.53646963423     0     -Inf        0
    x14             867.937644662627     0      Inf        0
Number of observations: 981, Error degrees of freedom: 971
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F      pValue
                  _________________    ___    ____________________    ___    _______
    Total        88.4325686523672     980    0.0902373149513951
    Model        88.4325686523672       9      9.82584096137413    Inf        0
    Residual                    0     971                      0
A = 0.759170, B = 0.431638
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 0.000000 and Ymax = 155.970004
List of powers for Q(x): [0.759170, 0.762715, 0.781539, 0.815883, 0.862198, 0.917432,
0.979335]
List of powers for D(x): [0.759170, 0.762715, 0.781539, 0.815883, 0.862198, 0.917432,
0.979335]
Fitting log10Gamma(x) in range (2.000000, 100.000000)
MSS of errors squared = 1.691520e-08
Corrected MSS of errors squared = 2.392170e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the

common logarithm of the gamma function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i)+C*sqrt(i-1)	0.156	0.692	0.438	Inf	-Inf	2.52067800E-09	7, 6
A+B*i	0.157	0.171		Inf	-Inf	4.69176100E-09	7, 5
A+B*i+C*(i-1)	0.412	0.226	0.21	Inf	-Inf	4.91447100E-09	7, 3
A+B*sqrt(i)	0.583	0.843		Inf	-Inf	5.44512500E-09	6, 5
A+B*i+C*sqrt(i-1)	0.041	0.32	0.258	Inf	-Inf	5.93155600E-09	5, 4
A+B*log10(i)^4	0.759	0.431		Inf	-Inf	1.69152000E-08	7, 7
A+B*i-C*(i-1)	0.691	0.109	0.133	Inf	-Inf	5.70493700E-08	6, 4
A+B/i	0.659	0.409		Inf	-Inf	5.84605600E-08	5, 4
A+B*sqrt(i)+C*(i-1)	0.231	0.109	0.239	Inf	-Inf	5.86844800E-08	4, 4
A+B*sqrt(i+C)	0.05	0.254	0.315	Inf	-Inf	2.20710600E-06	3, 6

RESULTS FOR THE COMMON EXPONENT (10^X)

Using Power A+B* i

```
Fitting 10^x in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ___________________    __    _____    _______
    (Intercept)    0.919766622276344      0      Inf         0
    x1             -6.14955739651597      0     -Inf         0
    x2              5.74661389488999      0      Inf         0
    x3                            0      0      NaN       NaN
    x4                            0      0      NaN       NaN
    x5              2.08550877484302      0      Inf         0
    x6             -4.38156019068492      0     -Inf         0
    x7              1.79302657969865      0      Inf         0
    x8              4.94667235643913      0      Inf         0
    x9                            0      0      NaN       NaN
    x10            -7.10874468513462      0     -Inf         0
    x11                           0      0      NaN       NaN
    x12             3.92922324912594      0      Inf         0
    x13                           0      0      NaN       NaN
    x14            -0.7809492049379       0     -Inf         0
Number of observations: 101, Error degrees of freedom: 91
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F      pValue
                   _____________    ____    ____________________    ___    _______
    Total          7.96302009058554    100    0.0796302009058554
    Model          7.96302009058554      9      0.88478001006506    Inf        0
    Residual                      0     91                     0
A = 0.252205, B = 0.100000
order Q(x) = 7.000000, order D(x) = 7.000000
```



```
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 1.000000 and Ymax = 10.000000
List of powers for Q(x): [0.352205, 0.452205, 0.552205, 0.652205, 0.752205, 0.852205,
0.952205]
List of powers for D(x): [0.352205, 0.452205, 0.552205, 0.652205, 0.752205, 0.852205,
0.952205]
Fitting 10^x in range (0.000000, 1.000000)
MSS of errors squared = 3.675915e-12
Corrected MSS of errors squared = 5.198529e-12
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting 10^x in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ________________    ___    _____    ______

    (Intercept)      0.894750367680852    0      Inf       0
    x1               0.037438580846343    0      Inf       0
    x2               -2.14578836946712    0     -Inf       0
    x3                 9.3916418258248    0      Inf       0
    x4               -8.12373150608686    0     -Inf       0
    x5              -0.0360129955820038    0     -Inf       0
    x6                2.27736118507373    0      Inf       0
    x7               -9.85482173733647    0     -Inf       0
    x8                8.00438423458999    0      Inf       0
    x9               0.554778414454632    0      Inf       0
Number of observations: 101, Error degrees of freedom: 91
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _______________    ___    ___________________    ___    ______

    Total        7.96302009058574    100    0.0796302009058574
    Model        7.96302009058574      9     0.884780010065082    Inf       0
    Residual                    0     91                     0
A = 0.223710, B = 1.591826
order Q(x) = 4.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 1.000000 and Ymax = 10.000000
List of powers for Q(x): [1.815536, 1.019623, 0.754319, 0.621667]
List of powers for D(x): [1.815536, 1.019623, 0.754319, 0.621667, 0.542075]
Fitting 10^x in range (0.000000, 1.000000)
MSS of errors squared = 4.920321e-12
Corrected MSS of errors squared = 6.958385e-12
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting 10^x in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                   ________________    ___    _____    ______
    (Intercept)    0.901258818003302    0      Inf       0
    x1             -6.46823407850491    0     -Inf       0
    x2              13.2915772641198    0      Inf       0
    x3             -10.5478903821634    0     -Inf       0
    x4              2.98041698727665    0      Inf       0
    x5              6.95113802558171    0      Inf       0
    x6             -12.7616002749915    0     -Inf       0
    x7              6.03777663657156    0      Inf       0
    x8              4.28960296851549    0      Inf       0
    x9             -4.94241992277822    0     -Inf       0
    x10             1.26837395836982    0      Inf       0
Number of observations: 101, Error degrees of freedom: 90
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______
    Total        7.96302009058566     100    0.0796302009058566
    Model        7.96302009058566      10     0.796302009058566     Inf       0
    Residual                    0      90                      0
A = 0.288385, B = 0.733693
order Q(x) = 4.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 1.000000 and Ymax = 10.000000
List of powers for Q(x): [1.022078, 1.325983, 1.559178, 1.755771]
List of powers for D(x): [1.022078, 1.325983, 1.559178, 1.755771, 1.928972, 2.085558]
Fitting 10^x in range (0.000000, 1.000000)
MSS of errors squared = 2.132034e-13
Corrected MSS of errors squared = 3.015151e-13
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting 10^x in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
```



```
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8
Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                    ________________    __    _____    ______
    (Intercept)    0.901074091581643    0      Inf       0
    x1             -11.2282887222123    0     -Inf       0
    x2              25.9140694901539    0      Inf       0
    x3             -21.4152583677532    0     -Inf       0
    x4              6.11899637118558    0      Inf       0
    x5              12.7358850828111    0      Inf       0
    x6              -29.325122056276    0     -Inf       0
    x7              24.1767185549874    0      Inf       0
    x8             -6.87807444426172    0     -Inf       0
Number of observations: 101, Error degrees of freedom: 92
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F      pValue
                   _________________    ___    ____________________    ___    ______
    Total       7.96302009058563     100    0.0796302009058563
    Model       7.96302009058563       8     0.995377511323204     Inf       0
    Residual                   0      92                     0
A = 0.747660, B = 0.690779, C = 0.414478
order Q(x) = 4.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 1.000000 and Ymax = 10.000000
List of powers for Q(x): [1.569217, 1.821034, 2.024104, 2.199034]
List of powers for D(x): [1.569217, 1.821034, 2.024104, 2.199034]
Fitting 10^x in range (0.000000, 1.000000)
MSS of errors squared = 6.311628e-11
Corrected MSS of errors squared = 1.093206e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting 10^x in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                     ________________    __    _____    ______
    (Intercept)     0.898544337774634    0      Inf       0
    x1               -1.40740515147669    0     -Inf       0
    x2               0.728260455776103    0      Inf       0
    x3              -0.208115712012735    0     -Inf       0
    x4              0.00641379157551743    0      Inf       0
```



```
    x5               0.0179508899012702    0      Inf       0
    x6              -0.00832241169969219   0     -Inf       0
    x7               0.0019098935671714    0      Inf       0
    x8                 1.56871899917246    0      Inf       0
    x9               -0.814610010060146    0     -Inf       0
    x10               0.246387839890478    0      Inf       0
    x11              -0.0297329224083777   0     -Inf       0
Number of observations: 101, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ________________    ___    ________________    ___    ______
    Total       7.9630200905856     100    0.079630200905856
    Model       7.9630200905856      11    0.723910917325964    Inf       0
    Residual                  0      89                    0
A = 0.029086, B = 0.212076, C = 0.233648
order Q(x) = 7.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 1.000000 and Ymax = 10.000000
List of powers for Q(x): [0.241161, 0.686885, 1.132608, 1.578332, 2.024055, 2.469778,
2.915502]
List of powers for D(x): [0.241161, 0.686885, 1.132608, 1.578332]
Fitting 10^x in range (0.000000, 1.000000)
MSS of errors squared = 1.218389e-14
Corrected MSS of errors squared = 2.110313e-14
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting 10^x in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                    Estimate          SE    tStat    pValue
                ________________    ___    _____    ______
    (Intercept)  0.896921325603404    0      Inf       0
    x1          -26.2056881862555     0     -Inf       0
    x2           55.7508182932546     0      Inf       0
    x3          -39.2180466349612     0     -Inf       0
    x4           8.11556416568969     0      Inf       0
    x5          0.706605535350084     0      Inf       0
    x6           30.0150493692972     0      Inf       0
    x7          -65.2065098379546     0     -Inf       0
    x8           48.2396449125217     0      Inf       0
    x9          -12.094358942547      0     -Inf       0
Number of observations: 101, Error degrees of freedom: 91
```



```
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______
    Total       7.96302009058544    100    0.0796302009058544
    Model       7.96302009058544      9     0.884780010065049    Inf       0
    Residual                   0     91                     0
A = 0.490450, B = 0.231869, C = 0.111970
order Q(x) = 5.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 1.000000 and Ymax = 10.000000
List of powers for Q(x): [0.722319, 0.842218, 0.962117, 1.082016, 1.201916]
List of powers for D(x): [0.722319, 0.842218, 0.962117, 1.082016]
Fitting 10^x in range (0.000000, 1.000000)
MSS of errors squared = 4.744567e-12
Corrected MSS of errors squared = 8.217832e-12
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting 10^x in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      0.902802004402803     0      Inf       0
    x1               -2.58399372662551     0     -Inf       0
    x2                3.71454572968625     0      Inf       0
    x3               -2.62262499623898     0     -Inf       0
    x4               0.634432549908156     0      Inf       0
    x5                2.83024605995413     0      Inf       0
    x6               -3.83615356049312     0     -Inf       0
    x7                2.24945645230425     0      Inf       0
    x8                               0     0      NaN      NaN
    x9              -0.356583903898095     0     -Inf       0
    x10             0.0600290144067353     0      Inf       0
    x11             0.00784437659343278     0      Inf       0
Number of observations: 101, Error degrees of freedom: 90
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______
    Total       7.96302009058555    100    0.0796302009058555
    Model       7.96302009058555     10     0.796302009058555    Inf       0
    Residual                   0     90                     0
```



```
A = 0.307779, B = 0.164933, C = 0.246574
order Q(x) = 4.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 1.000000 and Ymax = 10.000000
List of powers for Q(x): [0.472712, 0.884219, 1.151286, 1.394590]
List of powers for D(x): [0.472712, 0.884219, 1.151286, 1.394590, 1.625592, 1.848733,
2.066291]
Fitting 10^x in range (0.000000, 1.000000)
MSS of errors squared = 2.072190e-13
Corrected MSS of errors squared = 3.589139e-13
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting 10^x in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      0.897974203285303     0      Inf       0
    x1               -2.36615619330276     0     -Inf       0
    x2                2.49980740668566     0      Inf       0
    x3               -1.23475139571357     0     -Inf       0
    x4               0.234410608821668     0      Inf       0
    x5                 2.6427523259614     0      Inf       0
    x6               -2.80182202039215     0     -Inf       0
    x7                1.40525481262566     0      Inf       0
    x8              -0.286725833589026     0     -Inf       0
    x9              0.00925608561684534    0      Inf       0
Number of observations: 101, Error degrees of freedom: 91
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                 ________________    ___    ___________________    ___    ______
    Total       7.96302009058562     100    0.0796302009058562
    Model       7.96302009058562       9     0.884780010065069    Inf       0
    Residual                    0      91                     0
A = 0.026736, B = 0.361759, C = 0.206900
order Q(x) = 4.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 1.000000 and Ymax = 10.000000
List of powers for Q(x): [0.388495, 0.745240, 1.067121, 1.370953]
List of powers for D(x): [0.388495, 0.745240, 1.067121, 1.370953, 1.663253]
Fitting 10^x in range (0.000000, 1.000000)
MSS of errors squared = 4.089368e-12
```



```
Corrected MSS of errors squared = 7.082994e-12
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting 10^x in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                   _________________    __    _____    ______
    (Intercept)     0.88178051114636    0      Inf        0
    x1             -3.88541074582807    0     -Inf        0
    x2              7.06390427841895    0      Inf        0
    x3                             0    0      NaN      NaN
    x4             -10.0392540711331    0     -Inf        0
    x5              7.28927017018356    0      Inf        0
    x6             -1.30480125140333    0     -Inf        0
    x7              4.62396985225755    0      Inf        0
    x8              -10.914995970216    0     -Inf        0
    x9              8.85795386306517    0      Inf        0
    x10                            0    0      NaN      NaN
    x11            -1.5724166364914     0     -Inf        0
Number of observations: 101, Error degrees of freedom: 91
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq             F      pValue
                  _______________    ___    _________________    ___    _______
    Total        7.9630200905856    100    0.079630200905856
    Model        7.9630200905856      9    0.884780010065067    Inf        0
    Residual                   0     91                    0
A = 0.161870, B = 0.100000, C = 0.163235
order Q(x) = 6.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 1.000000 and Ymax = 10.000000
List of powers for Q(x): [0.261870, 0.466527, 0.565925, 0.644602, 0.711948, 0.771825]
List of powers for D(x): [0.261870, 0.466527, 0.565925, 0.644602, 0.711948]
Fitting 10^x in range (0.000000, 1.000000)
MSS of errors squared = 6.036171e-12
Corrected MSS of errors squared = 1.045495e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4


```
Fitting 10^x in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      0.898358181629598     0      Inf       0
    x1               -14.0824524062879     0     -Inf       0
    x2                             0       0      NaN      NaN
    x3                             0       0      NaN      NaN
    x4                31.8085548469396     0      Inf       0
    x5               -23.0629414445364     0     -Inf       0
    x6                4.69165333563953     0      Inf       0
    x7                29.7425591439327     0      Inf       0
    x8                             0       0      NaN      NaN
    x9                -35.477905876287     0     -Inf       0
    x10                            0       0      NaN      NaN
    x11               6.95047092684547     0      Inf       0
    x12              0.182172871624916     0      Inf       0
    x13             -0.650469579516287     0     -Inf       0
Number of observations: 101, Error degrees of freedom: 91
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _________________    ___    ____________________    ___    ______
    Total        7.96302009058642     100    0.0796302009058642
    Model        7.96302009058642       9     0.884780010065158     Inf       0
    Residual                    0      91                     0
A = 1.426501, B = 1.607372
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 1.000000 and Ymax = 10.000000
List of powers for Q(x): [1.426501, 1.439700, 1.509798, 1.637692, 1.810164, 2.015850]
List of powers for D(x): [1.426501, 1.439700, 1.509798, 1.637692, 1.810164, 2.015850,
2.246370]
Fitting 10^x in range (0.000000, 1.000000)
MSS of errors squared = 5.676590e-12
Corrected MSS of errors squared = 8.027911e-12
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the commn exponent function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i+C*(i-1)	0.029	0.212	0.233	Inf	-Inf	1.21838900E-14	7, 4
A+B*i+C*sqrt(i-1)	0.307	0.164	0.246	Inf	-Inf	2.07219000E-13	4, 7
A+B*sqrt(i)	0.288	0.733		Inf	-Inf	2.13203400E-13	4, 6
A+B*i	0.252	0.1		Inf	-Inf	3.67591500E-12	7, 7
A+B*sqrt(i)+C*(i-1)	0.026	0.361	0.206	Inf	-Inf	4.08936800E-12	4, 5
A+B*i-C*(i-1)	0.49	0.231	0.111	Inf	-Inf	4.74456700E-12	5, 4
A+B/i	0.223	1.591		Inf	-Inf	4.92032100E-12	4, 5
A+B*log10(i)^4	1.426	1.607		Inf	-Inf	5.67659000E-12	6, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.161	0.1	0.163	Inf	-Inf	6.03617100E-12	6, 5
A+B*sqrt(i+C)	0.747	0.69	0.414	Inf	-Inf	6.31162800E-11	4, 4

RESULTS FOR THE SINE INTEGRAL

Using Power A+B* i

```
Fitting Si(x) in range (1.300000, 20.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)    -0.263386292209533     0     -Inf       0
    x1               3.56559976573978     0      Inf       0
    x2              -9.95053222021364     0     -Inf       0
    x3               12.6370580413236     0      Inf       0
    x4              -8.58601728091688     0     -Inf       0
    x5               3.03977972773586     0      Inf       0
    x6             -0.442498526947656     0     -Inf       0
    x7               18.0874083319448     0      Inf       0
    x8              -61.3233156306723     0     -Inf       0
    x9               98.6290396995566     0      Inf       0
    x10             -90.5836964101917     0     -Inf       0
    x11              48.8004247313992     0      Inf       0
    x12             -14.4214889051348     0     -Inf       0
    x13              1.81162498855643     0      Inf       0
Number of observations: 188, Error degrees of freedom: 174
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F      pValue
                  ____________________    ___    ____________________    ___    _______
    Total        5.02479642278229     187    0.0268705691057877
    Model        5.02479642278229      13     0.386522801752484     Inf       0
    Residual                    0     174                     0
A = 0.347046, B = 0.407133
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 1.300000 and Xmax = 20.000000
```



```
Ymin = 1.183958 and Ymax = 1.851659
List of powers for Q(x): [0.754179, 1.161313, 1.568446, 1.975579, 2.382713, 2.789846]
List of powers for D(x): [0.754179, 1.161313, 1.568446, 1.975579, 2.382713, 2.789846,
3.196979]
Fitting Si(x) in range (1.300000, 20.000000)
MSS of errors squared = 6.448576e-04
Corrected MSS of errors squared = 9.119664e-04
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting Si(x) in range (1.300000, 20.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      5.11499345212828e-07    0      Inf       0
    x1              -2.94978324959247e-07    0     -Inf       0
    x2               2.42212736073163e-06    0      Inf       0
    x3              -2.63569659600471e-06    0     -Inf       0
    x4                -0.0281579660578962    0     -Inf       0
    x5                   4.85375489360277    0      Inf       0
    x6                  -71.9850032496523    0     -Inf       0
    x7                   290.178784563107    0      Inf       0
    x8                  -421.141964859111    0     -Inf       0
    x9                   199.122586615288    0      Inf       0
Number of observations: 188, Error degrees of freedom: 178
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _______________    ___    ___________________    ___    ______
    Total       5.02479642278776    187    0.0268705691058169
    Model       5.02479642278776      9     0.558310713643084    Inf       0
    Residual                   0    178                     0
A = 0.074391, B = 0.800935
order Q(x) = 3.000000, order D(x) = 6.000000
Xmin = 1.300000 and Xmax = 20.000000
Ymin = 1.183958 and Ymax = 1.851659
List of powers for Q(x): [0.875326, 0.474859, 0.341370]
List of powers for D(x): [0.875326, 0.474859, 0.341370, 0.274625, 0.234578, 0.207881]
Fitting Si(x) in range (1.300000, 20.000000)
MSS of errors squared = 6.444973e-01
Corrected MSS of errors squared = 9.114568e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
```



```
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting Si(x) in range (1.300000, 20.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)    -0.073270245734814    0     -Inf        0
    x1               46.0389956125483    0      Inf        0
    x2              -196.013415914015    0     -Inf        0
    x3               237.932578626365    0      Inf        0
    x4                              0    0      NaN       NaN
    x5              -159.97171533279     0     -Inf        0
    x6               72.0868273379914    0      Inf        0
    x7               2861.65007867525    0      Inf        0
    x8              -20808.2140573016    0     -Inf        0
    x9               55239.4022573566    0      Inf        0
    x10             -69487.6535955928    0     -Inf        0
    x11              42181.2481522608    0      Inf        0
    x12             -9985.43283548174    0     -Inf        0
Number of observations: 188, Error degrees of freedom: 176
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F     pValue
                  ___________________    ___    ____________________    ___    _______
    Total        5.02479642324733    187    0.0268705691082745
    Model        5.02479642324733     11     0.456799674840666    Inf        0
    Residual                    0    176                      0
A = 0.373021, B = 0.146777
order Q(x) = 6.000000, order D(x) = 6.000000
Xmin = 1.300000 and Xmax = 20.000000
Ymin = 1.183958 and Ymax = 1.851659
List of powers for Q(x): [0.519798, 0.580595, 0.627246, 0.666575, 0.701224, 0.732549]
List of powers for D(x): [0.519798, 0.580595, 0.627246, 0.666575, 0.701224, 0.732549]
Fitting Si(x) in range (1.300000, 20.000000)
MSS of errors squared = 1.751249e-02
Corrected MSS of errors squared = 2.476640e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting Si(x) in range (1.300000, 20.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
```



```
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.000134194570512062    0     -Inf       0
    x1                0.0258478153837241    0      Inf       0
    x2               -0.0780037749530631    0     -Inf       0
    x3                0.0792394266597353    0      Inf       0
    x4               -0.0269492717722698    0     -Inf       0
    x5                  1945.85228520867    0      Inf       0
    x6                 -10964.0075301555    0     -Inf       0
    x7                  24488.2696281825    0      Inf       0
    x8                 -27113.1514047379    0     -Inf       0
    x9                  14896.0277988848    0      Inf       0
    x10                 -3251.9907773833    0     -Inf       0
Number of observations: 188, Error degrees of freedom: 177
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______
    Total       5.02479642275814    187    0.0268705691056585
    Model       5.02479642275814     10     0.502479642275814    Inf       0
    Residual                   0    177                      0
A = 0.048460, B = 0.100000, C = 2.000000
order Q(x) = 4.000000, order D(x) = 6.000000
Xmin = 1.300000 and Xmax = 20.000000
Ymin = 1.183958 and Ymax = 1.851659
List of powers for Q(x): [0.221666, 0.248460, 0.272067, 0.293409]
List of powers for D(x): [0.221666, 0.248460, 0.272067, 0.293409, 0.313036, 0.331303]
Fitting Si(x) in range (1.300000, 20.000000)
MSS of errors squared = 3.636823e-01
Corrected MSS of errors squared = 6.299162e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting Si(x) in range (1.300000, 20.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)     -64.8881405996048     0     -Inf       0
    x1               177.606782229037     0      Inf       0
    x2              -319.743087177926     0     -Inf       0
```



```
    x3               438.376834027671    0      Inf       0
    x4              -382.312702483938    0     -Inf       0
    x5               204.297874403982    0      Inf       0
    x6              -61.2490111422642    0     -Inf       0
    x7               7.91145083368351    0      Inf       0
    x8                2.5420814454544    0      Inf       0
    x9              -3.84562260800332    0     -Inf       0
    x10              4.25477927170477    0      Inf       0
    x11             -2.80822251524827    0     -Inf       0
    x12              1.00982754980465    0      Inf       0
    x13            -0.152843234347091    0     -Inf       0
Number of observations: 188, Error degrees of freedom: 174
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ________________    ___    ___________________    ___    _______
    Total       5.02479642278401    187    0.0268705691057968
    Model       5.02479642278401     13     0.386522801752616    Inf        0
    Residual                   0    174                     0
A = 0.000000, B = 0.110135, C = 0.116647
order Q(x) = 7.000000, order D(x) = 6.000000
Xmin = 1.300000 and Xmax = 20.000000
Ymin = 1.183958 and Ymax = 1.851659
List of powers for Q(x): [0.110135, 0.336917, 0.563699, 0.790481, 1.017263, 1.244045,
1.470827]
List of powers for D(x): [0.110135, 0.336917, 0.563699, 0.790481, 1.017263, 1.244045]
Fitting Si(x) in range (1.300000, 20.000000)
MSS of errors squared = 9.781057e-05
Corrected MSS of errors squared = 1.694129e-04
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting Si(x) in range (1.300000, 20.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                    Estimate          SE     tStat     pValue
                ________________    ___    _____    _______
    (Intercept)   1.73510157905874    0      Inf        0
    x1           -470.164719218018    0     -Inf        0
    x2            1575.02284529896    0      Inf        0
    x3           -1682.80878061083    0     -Inf        0
    x4                           0    0      NaN      NaN
    x5            1284.05584092911    0      Inf        0
    x6           -911.869226550302    0     -Inf        0
```



```
    x7              204.028938578621    0      Inf        0
    x8              203.112360570558    0      Inf        0
    x9              -510.29350849937    0     -Inf        0
    x10                            0    0      NaN      NaN
    x11             1262.16066696723    0      Inf        0
    x12            -1663.93136873797    0     -Inf        0
    x13             886.200359343318    0      Inf        0
    x14            -176.248509649008    0     -Inf        0
Number of observations: 188, Error degrees of freedom: 175
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ________________    ___    ___________________    ___    _______
    Total       5.02479642275862    187    0.0268705691056611
    Model       5.02479642275862     12     0.418733035229885    Inf        0
    Residual                   0    175                     0
A = 0.212895, B = 0.243498, C = 0.167764
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 1.300000 and Xmax = 20.000000
Ymin = 1.183958 and Ymax = 1.851659
List of powers for Q(x): [0.456393, 0.532127, 0.607861, 0.683595, 0.759329, 0.835063,
0.910798]
List of powers for D(x): [0.456393, 0.532127, 0.607861, 0.683595, 0.759329, 0.835063,
0.910798]
Fitting Si(x) in range (1.300000, 20.000000)
MSS of errors squared = 1.251730e-01
Corrected MSS of errors squared = 2.168060e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting Si(x) in range (1.300000, 20.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    _________________    __    _____    ______
    (Intercept)      2.57920160080527    0      Inf        0
    x1              -7.38597859079561    0     -Inf        0
    x2               19.2730679755168    0      Inf        0
    x3              -30.5043596424311    0     -Inf        0
    x4               23.8256726676935    0      Inf        0
    x5              -9.16239259030943    0     -Inf        0
    x6               1.37478860663754    0      Inf        0
    x7               2.70753126807601    0      Inf        0
    x8              -5.98839413438377    0     -Inf        0
```



```
    x9              7.93327206860612    0      Inf       0
    x10            -4.71112003695335    0     -Inf       0
    x11             1.05871081500678    0      Inf       0
Number of observations: 188, Error degrees of freedom: 176
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ___________________    ___    ______
    Total       5.02479642278409    187    0.0268705691057973
    Model       5.02479642278409     11     0.456799674798554    Inf       0
    Residual                   0    176                      0
A = 0.025847, B = 0.100000, C = 0.152562
order Q(x) = 6.000000, order D(x) = 5.000000
Xmin = 1.300000 and Xmax = 20.000000
Ymin = 1.183958 and Ymax = 1.851659
List of powers for Q(x): [0.125847, 0.378408, 0.541602, 0.690091, 0.830970, 0.966985]
List of powers for D(x): [0.125847, 0.378408, 0.541602, 0.690091, 0.830970]
Fitting Si(x) in range (1.300000, 20.000000)
MSS of errors squared = 1.050198e-01
Corrected MSS of errors squared = 1.818997e-01
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting Si(x) in range (1.300000, 20.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      3.26655557588161e-06    0      Inf       0
    x1              -1.46242183655251e-05    0     -Inf       0
    x2               1.02386333013118e-05    0      Inf       0
    x3               1.28370773517113e-06    0      Inf       0
    x4                   1218.43316910134    0      Inf       0
    x5                  -10807.5561245532    0     -Inf       0
    x6                   36356.6429353261    0      Inf       0
    x7                  -61682.5745922882    0     -Inf       0
    x8                   56717.6263480091    0      Inf       0
    x9                  -27081.6878626359    0     -Inf       0
    x10                  5280.11612688259    0      Inf       0
Number of observations: 188, Error degrees of freedom: 177
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ___________________    ___    ______
```



```
    Total       5.02479642348581    187    0.0268705691095498
    Model       5.02479642348581     10     0.502479642348581    Inf       0
    Residual                   0    177                      0
A = 0.472415, B = 0.355159, C = 0.000000
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 1.300000 and Xmax = 20.000000
Ymin = 1.183958 and Ymax = 1.851659
List of powers for Q(x): [0.827575, 0.974686, 1.087569]
List of powers for D(x): [0.827575, 0.974686, 1.087569, 1.182734, 1.266576, 1.342374,
1.412079]
Fitting Si(x) in range (1.300000, 20.000000)
MSS of errors squared = 4.121103e-03
Corrected MSS of errors squared = 7.137960e-03
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting Si(x) in range (1.300000, 20.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.086695936585189    0     -Inf       0
    x1              0.310349402273788    0      Inf       0
    x2             -1.40217120157086     0     -Inf       0
    x3              2.74031958836985     0      Inf       0
    x4             -2.19550806338449     0     -Inf       0
    x5              0.633706351531674    0      Inf       0
    x6              4.46675672557867     0      Inf       0
    x7             -34.7133719031912     0     -Inf       0
    x8              101.024622534531     0      Inf       0
    x9             -131.256323522235     0     -Inf       0
    x10             80.5905727660334     0      Inf       0
    x11            -19.112256738452     0     -Inf       0
Number of observations: 188, Error degrees of freedom: 176
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                        SumSq          DF           MeanSq            F      pValue
                    ____________________    ___    ____________________    ___    ______
    Total       5.02479642278299    187    0.0268705691057914
    Model       5.02479642278299     11     0.456799674798453    Inf       0
    Residual                   0    176                      0
A = 0.124247, B = 0.100000, C = 0.334108
order Q(x) = 5.000000, order D(x) = 6.000000
Xmin = 1.300000 and Xmax = 20.000000
```



```
Ymin = 1.183958 and Ymax = 1.851659
List of powers for Q(x): [0.224247, 0.599777, 0.769953, 0.902940, 1.016071]
List of powers for D(x): [0.224247, 0.599777, 0.769953, 0.902940, 1.016071, 1.116285]
Fitting Si(x) in range (1.300000, 20.000000)
MSS of errors squared = 3.036457e-02
Corrected MSS of errors squared = 5.259298e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting Si(x) in range (1.300000, 20.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)    -0.121909799277819    0     -Inf        0
    x1               751.221866685218    0      Inf        0
    x2              -966.185671132857    0     -Inf        0
    x3               263.622147117999    0      Inf        0
    x4               -54.078167992136    0     -Inf        0
    x5               5.54173517696537    0      Inf        0
    x6               461.269055095479    0      Inf        0
    x7                              0    0      NaN      NaN
    x8              -1113.94149599053    0     -Inf        0
    x9               1044.61417908943    0      Inf        0
    x10             -513.512483409898    0     -Inf        0
    x11              138.940070270028    0      Inf        0
    x12             -16.3693251093014    0     -Inf        0
Number of observations: 188, Error degrees of freedom: 176
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F      pValue
                  _______________    ___    ___________________    ___    ______
    Total       5.0247964227864    187    0.0268705691058096
    Model       5.0247964227864     11     0.456799674798764    Inf        0
    Residual                  0    176                      0
A = 0.410113, B = 0.905686
order Q(x) = 5.000000, order D(x) = 7.000000
Xmin = 1.300000 and Xmax = 20.000000
Ymin = 1.183958 and Ymax = 1.851659
List of powers for Q(x): [0.410113, 0.417550, 0.457048, 0.529110, 0.626291]
List of powers for D(x): [0.410113, 0.417550, 0.457048, 0.529110, 0.626291, 0.742186,
0.872075]
Fitting Si(x) in range (1.300000, 20.000000)
MSS of errors squared = 2.198276e-02
```



```
Corrected MSS of errors squared = 3.108832e-02
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the sine Integral. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i+C*(i-1)	0	0.11	0.116	Inf	-Inf	9.78105700E-05	7, 6
A+B*i	0.347	0.407		Inf	-Inf	6.44857600E-04	6, 7
A+B*sqrt(i)+C*(i-1)	0.472	0.355	0	Inf	-Inf	4.12110300E-03	3, 7
A+B*sqrt(i)	0.373	0.146		Inf	-Inf	1.75124900E-02	6, 6
A+B*log10(i)^4	0.41	0.905		Inf	-Inf	2.19827600E-02	5, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.124	0.1	0.334	Inf	-Inf	3.03645700E-02	5, 6
A+B*i+C*sqrt(i-1)	0.025	0.1	0.152	Inf	-Inf	1.05019800E-01	6, 5
A+B*i-C*(i-1)	0.212	0.243	0.167	Inf	-Inf	1.25173000E-01	7, 7
A+B*sqrt(i+C)	0.048	0.1	2	Inf	-Inf	3.63682300E-01	4, 6
A+B/i	0.074	0.8		Inf	-Inf	6.44497300E-01	3, 6

RESULTS FOR THE SINE FUNCTION

Using Power A+B* i

```
Fitting sin(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.000126689287540351    0     -Inf       0
    x1              -0.00724162728284178    0     -Inf       0
    x2                0.0358266776595765    0      Inf       0
    x3                                 0    0      NaN      NaN
    x4                  1.34627383007926    0      Inf       0
    x5                 -1.22317422936916    0     -Inf       0
    x6                  0.19805297559106    0      Inf       0
    x7                 0.0375955270727612    0      Inf       0
    x8                  0.840127863277127    0      Inf       0
    x9                                 0    0      NaN      NaN
    x10                -0.407423947082917    0     -Inf       0
    x11                 0.222602488537148    0      Inf       0
    x12               -0.0425128691981529    0     -Inf       0
Number of observations: 501, Error degrees of freedom: 490
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ___________________    ___    ____________________    ___    ______
    Total       47.6265936125812       500    0.0952531872251624
    Model       47.6265936125812        10      4.76265936125812     Inf       0
    Residual                   0       490                     0
A = 0.067802, B = 0.500000
order Q(x) = 7.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 1.570796
Ymin = 0.000000 and Ymax = 1.000000
```



```
List of powers for Q(x): [0.567802, 1.067802, 1.567802, 2.067802, 2.567802, 3.067802,
3.567802]
List of powers for D(x): [0.567802, 1.067802, 1.567802, 2.067802, 2.567802]
Fitting sin(x) in range (0.000000, 1.570796)
MSS of errors squared = 1.333491e-13
Corrected MSS of errors squared = 1.885842e-13
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting sin(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ________________    __    _____    ______
    (Intercept)    0.105618880022669    0      Inf       0
    x1              23.9565583259145    0      Inf       0
    x2             -343.469350010307    0     -Inf       0
    x3              815.756912973027    0      Inf       0
    x4             -496.155619569195    0     -Inf       0
    x5             -27.6262003399197    0     -Inf       0
    x6               447.73353120726    0      Inf       0
    x7              -1120.9280466069    0     -Inf       0
    x8              701.626595143429    0      Inf       0
Number of observations: 501, Error degrees of freedom: 492
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ___________________    ___    ______
    Total       47.6265936124747    500    0.0952531872249495
    Model       47.6265936124747      8      5.95332420155934    Inf       0
    Residual                   0    492                     0
A = 1.093989, B = 0.355697
order Q(x) = 4.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 1.570796
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [1.449686, 1.271838, 1.212555, 1.182913]
List of powers for D(x): [1.449686, 1.271838, 1.212555, 1.182913]
Fitting sin(x) in range (0.000000, 1.570796)
MSS of errors squared = 2.524835e-10
Corrected MSS of errors squared = 3.570656e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
```



```
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting sin(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.100748596247958     0     -Inf        0
    x1                3.8787451772709     0      Inf        0
    x2               -16.052879050569     0     -Inf        0
    x3                19.5646787121628    0      Inf        0
    x4                               0    0      NaN      NaN
    x5               -9.52903248310897    0     -Inf        0
    x6                               0    0      NaN      NaN
    x7                2.42416244619567    0      Inf        0
    x8                4.72789137878303    0      Inf        0
    x9               -8.59337196252756    0     -Inf        0
    x10               6.43129016873346    0      Inf        0
    x11              -1.75073579075718    0     -Inf        0
Number of observations: 501, Error degrees of freedom: 491
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF            MeanSq            F      pValue
                  _________________    ___    ____________________    ___    ______
    Total        47.6265936125814     500    0.0952531872251628
    Model        47.6265936125814       9      5.29184373473126    Inf       0
    Residual                     0    491                      0
A = 0.000000, B = 0.759214
order Q(x) = 7.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 1.570796
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.759214, 1.073691, 1.314998, 1.518428, 1.697654, 1.859687,
2.008692]
List of powers for D(x): [0.759214, 1.073691, 1.314998, 1.518428]
Fitting sin(x) in range (0.000000, 1.570796)
MSS of errors squared = 4.886702e-12
Corrected MSS of errors squared = 6.910841e-12
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting sin(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
```



```
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    -0.0275879183616435     0     -Inf        0
    x1               1.45319512365956      0      Inf        0
    x2              -7.35748261744666      0     -Inf        0
    x3               11.1579858516947      0      Inf        0
    x4                             0      0      NaN      NaN
    x5              -9.25617276237659      0     -Inf        0
    x6               4.31917735416886      0      Inf        0
    x7                 3.218735905595      0      Inf        0
    x8              -3.95207710265015      0     -Inf        0
    x9                             0      0      NaN      NaN
    x10                2.304644611863      0      Inf        0
    x11                            0      0      NaN      NaN
    x12             -1.38075058045002      0     -Inf        0
    x13             0.520332134296255      0      Inf        0
Number of observations: 501, Error degrees of freedom: 490
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______

    Total        47.6265936125856     500    0.0952531872251712
    Model        47.6265936125856      10      4.76265936125856    Inf       0
    Residual                    0     490                     0
A = 0.000000, B = 0.926496, C = 0.000000
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 1.570796
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.926496, 1.310264, 1.604739, 1.852993, 2.071709, 2.269444]
List of powers for D(x): [0.926496, 1.310264, 1.604739, 1.852993, 2.071709, 2.269444,
2.451279]
Fitting sin(x) in range (0.000000, 1.570796)
MSS of errors squared = 3.727613e-13
Corrected MSS of errors squared = 6.456414e-13
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting sin(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)    0.0282733864075474      0      Inf        0
```

```
    x1              -0.094180557407189     0      -Inf         0
    x2                               0     0       NaN       NaN
    x3               0.157717207578665     0       Inf         0
    x4                               0     0       NaN       NaN
    x5               -0.16471685848232     0      -Inf         0
    x6              0.0749953638749124     0       Inf         0
    x7                 5.4743614085965     0       Inf         0
    x8               -10.0659885220152     0      -Inf         0
    x9                8.61096518463414     0       Inf         0
    x10              -3.63496953107559     0      -Inf         0
    x11              0.613542917890074     0       Inf         0
Number of observations: 501, Error degrees of freedom: 491
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F      pValue
                 _______________    ___    ___________________    ___    _______
    Total        47.6265936125817    500    0.0952531872251633
    Model        47.6265936125817      9      5.29184373473129    Inf         0
    Residual                    0    491                      0
A = 0.055048, B = 0.172234, C = 0.000000
order Q(x) = 6.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 1.570796
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.227282, 0.399516, 0.571750, 0.743984, 0.916218, 1.088452]
List of powers for D(x): [0.227282, 0.399516, 0.571750, 0.743984, 0.916218]
Fitting sin(x) in range (0.000000, 1.570796)
MSS of errors squared = 1.815680e-11
Corrected MSS of errors squared = 3.144851e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting sin(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ___________________    __    _____    _______
    (Intercept)    0.0605132097095761     0      Inf         0
    x1               -24.8078908695915    0     -Inf         0
    x2                               0    0      NaN       NaN
    x3                137.523056139117    0      Inf         0
    x4               -176.233575127568    0     -Inf         0
    x5                63.4957748008163    0      Inf         0
    x6                403.114602964499    0      Inf         0
    x7               -793.807403994502    0     -Inf         0
```



```
    x8                              0     0      NaN      NaN
    x9                685.879648920016    0      Inf        0
    x10                             0     0      NaN      NaN
    x11              -488.385749770035    0     -Inf        0
    x12               194.161023727593    0      Inf        0
Number of observations: 501, Error degrees of freedom: 491
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                 _________________    ___    ___________________    ___    ______
    Total        47.6265936125662    500    0.0952531872251323
    Model        47.6265936125662      9      5.29184373472957    Inf       0
    Residual                    0    491                      0
A = 0.649672, B = 0.246738, C = 0.185604
order Q(x) = 5.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 1.570796
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.896410, 0.957543, 1.018677, 1.079810, 1.140944]
List of powers for D(x): [0.896410, 0.957543, 1.018677, 1.079810, 1.140944, 1.202077,
1.263211]
Fitting sin(x) in range (0.000000, 1.570796)
MSS of errors squared = 3.038286e-11
Corrected MSS of errors squared = 5.262466e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting sin(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)      0.132603224386218      0      Inf        0
    x1              -0.482606882925445      0     -Inf        0
    x2               1.19163369182456       0      Inf        0
    x3              -1.24613145304791       0     -Inf        0
    x4               0.411448695343453      0      Inf        0
    x5               2.61275266254878       0      Inf        0
    x6              -4.30089988943049       0     -Inf        0
    x7               3.44705436119723       0      Inf        0
    x8                                      0      NaN      NaN
    x9              -1.14902424327613       0     -Inf        0
    x10              0.383169833393542      0      Inf        0
Number of observations: 501, Error degrees of freedom: 491
R-squared: 1,  Adjusted R-Squared: 1
```



```
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ________________    ___    ___________________    ___    ______
    Total       47.626593612581     500    0.0952531872251619
    Model       47.626593612581       9      5.29184373473122    Inf        0
    Residual                  0     491                     0
A = 0.134740, B = 0.119050, C = 0.263301
order Q(x) = 4.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 1.570796
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.253791, 0.636142, 0.864255, 1.066993]
List of powers for D(x): [0.253791, 0.636142, 0.864255, 1.066993, 1.256594, 1.437802]
Fitting sin(x) in range (0.000000, 1.570796)
MSS of errors squared = 5.652255e-11
Corrected MSS of errors squared = 9.789994e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting sin(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                   ________________    ___    _____    _______
    (Intercept)    0.0317485653197154    0      Inf        0
    x1             -0.270000775254134    0     -Inf        0
    x2               1.15740873213979    0      Inf        0
    x3              -2.80953879857603    0     -Inf        0
    x4               4.08045036131953    0      Inf        0
    x5              -2.76202115868436    0     -Inf        0
    x6              0.668333497128153    0      Inf        0
    x7                1.9920332844264    0      Inf        0
    x8              -1.69816217461709    0     -Inf        0
    x9              0.740874789628898    0      Inf        0
    x10            -0.131126322831451    0     -Inf        0
Number of observations: 501, Error degrees of freedom: 490
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ________________    ___    ___________________    ___    ______
    Total       47.6265936125814    500    0.0952531872251629
    Model       47.6265936125814     10     4.76265936125814    Inf        0
    Residual                  0    490                     0
A = 0.077539, B = 0.306030, C = 0.276378
order Q(x) = 6.000000, order D(x) = 4.000000
```



```
Xmin = 0.000000 and Xmax = 1.570796
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.383569, 0.786709, 1.160356, 1.518734, 1.867357, 2.209048]
List of powers for D(x): [0.383569, 0.786709, 1.160356, 1.518734]
Fitting sin(x) in range (0.000000, 1.570796)
MSS of errors squared = 1.216547e-13
Corrected MSS of errors squared = 2.107121e-13
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting sin(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    0.0544154931295988    0      Inf       0
    x1             -0.228426932176108    0     -Inf       0
    x2              0.520586813986515    0      Inf       0
    x3             -0.308429623888085    0     -Inf       0
    x4               2.38729566651854    0      Inf       0
    x5              -3.97780341239419    0     -Inf       0
    x6               3.76147675861556    0      Inf       0
    x7              -1.20911476730517    0     -Inf       0
Number of observations: 501, Error degrees of freedom: 493
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _______________    ___    ___________________    ___    ______
    Total        47.6265936125815    500    0.0952531872251631
    Model        47.6265936125815      7      6.80379908751165    Inf       0
    Residual                    0    493                     0
A = 0.040290, B = 0.287265, C = 0.325173
order Q(x) = 3.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 1.570796
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.327555, 0.771718, 0.997712]
List of powers for D(x): [0.327555, 0.771718, 0.997712, 1.178037]
Fitting sin(x) in range (0.000000, 1.570796)
MSS of errors squared = 1.827267e-09
Corrected MSS of errors squared = 3.164919e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
```



```
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting sin(x) in range (0.000000, 1.570796)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    ________________    ___    _____    ______
    (Intercept)    0.209211091047452    0       Inf        0
    x1             -39.4013015886544    0      -Inf        0
    x2                             0    0       NaN      NaN
    x3              75.0814926572471    0       Inf        0
    x4             -44.4002412899302    0      -Inf        0
    x5              8.54800564726482    0       Inf        0
    x6              79.2307786043584    0       Inf        0
    x7                             0    0       NaN      NaN
    x8             -141.976529277363    0      -Inf        0
    x9              77.3793800817668    0       Inf        0
    x10            -13.6707959251254    0      -Inf        0
Number of observations: 501, Error degrees of freedom: 492
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F      pValue
                  _______________    ___    ____________________    ___    ______
    Total        47.6265936125779    500    0.0952531872251559
    Model        47.6265936125779      8      5.95332420157224    Inf        0
    Residual                    0    492                     0
A = 0.622031, B = 1.216700
order Q(x) = 5.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 1.570796
Ymin = 0.000000 and Ymax = 1.000000
List of powers for Q(x): [0.622031, 0.632023, 0.685083, 0.781892, 0.912446]
List of powers for D(x): [0.622031, 0.632023, 0.685083, 0.781892, 0.912446]
Fitting sin(x) in range (0.000000, 1.570796)
MSS of errors squared = 2.886809e-10
Corrected MSS of errors squared = 4.082564e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the sine function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i)+C*(i-1)	0.077	0.306	0.276	Inf	-Inf	1.21654700E-13	6, 4
A+B*i	0.067	0.5		Inf	-Inf	1.33349100E-13	7, 5
A+B*sqrt(i+C)	0	0.926	0	Inf	-Inf	3.72761300E-13	6, 7
A+B*sqrt(i)	0	0.759		Inf	-Inf	4.88670200E-12	7, 4
A+B*i+C*(i-1)	0.055	0.172	0	Inf	-Inf	1.81568000E-11	6, 5
A+B*i-C*(i-1)	0.649	0.246	0.185	Inf	-Inf	3.03828600E-11	5, 7
A+B*i+C*sqrt(i-1)	0.134	0.119	0.263	Inf	-Inf	5.65225500E-11	4, 6
A+B/i	1.093	0.355		Inf	-Inf	2.52483500E-10	4, 4
A+B*log10(i)^4	0.622	1.216		Inf	-Inf	2.88680900E-10	5, 5
A+B*sqrt(i)+C*sqrt(i-1)	0.04	0.287	0.325	Inf	-Inf	1.82726700E-09	3, 4

RESULTS FOR THE HYPERBOLIC SINE FUNCTION

Using Power A+B* i

```
Fitting sinh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                    ____________________    __    _____    _____

    (Intercept)     0.166740756325319     0      Inf        0
    x1              0.134797155585806     0      Inf        0
    x2                              0     0      NaN       NaN
    x3              -1.43294340393639     0     -Inf        0
    x4               2.05053643208901     0      Inf        0
    x5              -1.07277125638913     0     -Inf        0
    x6              0.199122212329364     0      Inf        0
    x7               7.74999069351594     0      Inf        0
    x8              -21.8531279453803     0     -Inf        0
    x9               31.9586222393037     0      Inf        0
    x10             -27.1123453972227     0     -Inf        0
    x11              13.3852872137688     0      Inf        0
    x12             -3.57365372892862     0     -Inf        0
    x13             0.399745028936055     0      Inf        0
Number of observations: 501, Error degrees of freedom: 488
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF          MeanSq            F      pValue
                   ____________________    ___    ____________________    ___    _____

    Total          30.8337526139365      500    0.061667505227873
    Model          30.8337526139365       12     2.56947938449471      Inf       0
    Residual                       0      488                    0
A = 0.230535, B = 0.500000
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 5.000000
```



```
Ymin = 0.000000 and Ymax = 74.203211
List of powers for Q(x): [0.730535, 1.230535, 1.730535, 2.230535, 2.730535, 3.230535]
List of powers for D(x): [0.730535, 1.230535, 1.730535, 2.230535, 2.730535, 3.230535,
3.730535]
Fitting sinh(x) in range (0.000000, 5.000000)
MSS of errors squared = 5.141943e-12
Corrected MSS of errors squared = 7.271806e-12
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting sinh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    0.0389054721944989    0      Inf       0
    x1             0.0144527058768499    0      Inf       0
    x2             -0.347386724164426    0     -Inf       0
    x3                 1.104836981978    0      Inf       0
    x4             -0.810730069203833    0     -Inf       0
    x5              0.128690002456706    0      Inf       0
    x6              -6.46159195969493    0     -Inf       0
    x7               28.3450659500986    0      Inf       0
    x8                              0    0      NaN      NaN
    x9              -61.5252428320074    0     -Inf       0
    x10                             0    0      NaN      NaN
    x11              40.5130004724692    0      Inf       0
Number of observations: 501, Error degrees of freedom: 491
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF            MeanSq            F
pValue
                ___________________    ___    ____________________    ___
______
    Total       30.8337526139325    500    0.0616675052278649
    Model       30.8337526139325      9      3.42597251265916    Inf
0
    Residual                    0    491                     0
A = 0.097720, B = 0.569612
order Q(x) = 4.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 5.000000
Ymin = 0.000000 and Ymax = 74.203211
List of powers for Q(x): [0.667332, 0.382526, 0.287591, 0.240123]
List of powers for D(x): [0.667332, 0.382526, 0.287591, 0.240123,
0.211643, 0.192655, 0.179093]
Fitting sinh(x) in range (0.000000, 5.000000)
MSS of errors squared = 2.886364e-09
Corrected MSS of errors squared = 4.081935e-09
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```


Using Power A+B*sqrt(i)

```
Fitting sinh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)      0.060513697532925      0      Inf        0
    x1              -0.579933254245904      0     -Inf        0
    x2               0.891529407080723      0      Inf        0
    x3                             0       0      NaN       NaN
    x4              -0.468916515446231      0     -Inf        0
    x5                             0       0      NaN       NaN
    x6                             0       0      NaN       NaN
    x7              0.0968605765420687      0      Inf        0
    x8                26.7786438270311      0      Inf        0
    x9               -108.247900957719      0     -Inf        0
    x10               173.835792435848      0      Inf        0
    x11              -125.036824385593      0     -Inf        0
    x12               33.6702351689717      0      Inf        0
Number of observations: 501, Error degrees of freedom: 491
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F      pValue
                  _______________    ___    ___________________    ___    _______
    Total         30.8337526139417    500    0.0616675052278833
    Model         30.8337526139417      9      3.42597251266019    Inf        0
    Residual                     0    491                     0
A = 0.000000, B = 0.177683
order Q(x) = 7.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 5.000000
Ymin = 0.000000 and Ymax = 74.203211
List of powers for Q(x): [0.177683, 0.251282, 0.307756, 0.355366, 0.397311, 0.435232,
0.470105]
List of powers for D(x): [0.177683, 0.251282, 0.307756, 0.355366, 0.397311]
Fitting sinh(x) in range (0.000000, 5.000000)
MSS of errors squared = 3.037320e-09
Corrected MSS of errors squared = 4.295419e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting sinh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
```



```
Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                   ____________________    __    _____    _______
    (Intercept)    -0.0694324208966136     0     -Inf        0
    x1                742.157559298039     0      Inf        0
    x2               -3912.42079472049     0     -Inf        0
    x3                7426.01316716686     0      Inf        0
    x4               -5526.40669749863     0     -Inf        0
    x5                               0     0      NaN      NaN
    x6                1984.51749591121     0      Inf        0
    x7               -713.485699648625     0     -Inf        0
    x8                8.11454941547142     0      Inf        0
    x9               -12.7646954045271     0     -Inf        0
    x10               5.34454792171233     0      Inf        0
Number of observations: 501, Error degrees of freedom: 491
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                   _________________    ___    ____________________    ___    _______
    Total       30.8337526132098     500    0.0616675052264196
    Model       30.8337526132098       9      3.42597251257886    Inf        0
    Residual                   0     491                     0
A = 0.861913, B = 0.642619, C = 0.629656
order Q(x) = 7.000000, order D(x) = 3.000000
Xmin = 0.000000 and Xmax = 5.000000
Ymin = 0.000000 and Ymax = 74.203211
List of powers for Q(x): [1.682267, 1.903998, 2.086209, 2.244613, 2.386648, 2.516536,
2.636944]
List of powers for D(x): [1.682267, 1.903998, 2.086209]
Fitting sinh(x) in range (0.000000, 5.000000)
MSS of errors squared = 3.110222e-09
Corrected MSS of errors squared = 5.387063e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting sinh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                   ____________________    __    _____    _______
    (Intercept)    -12.0075158696012       0     -Inf        0
    x1                21.4658220071039     0      Inf        0
    x2               -23.5083537142935     0     -Inf        0
    x3                27.7192115722546     0      Inf        0
```



```
    x4              -21.7083373576969    0     -Inf       0
    x5               10.6974087382334    0      Inf       0
    x6              -3.03145127228521    0     -Inf       0
    x7              0.379568967163941    0      Inf       0
    x8               1.29156797248235    0      Inf       0
    x9             -0.366604507060629    0     -Inf       0
    x10            0.0686834636693365    0      Inf       0
Number of observations: 501, Error degrees of freedom: 490
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ________________    ___    ____________________    ___    ______
    Total       30.8337526139339    500    0.0616675052278679
    Model       30.8337526139339     10      3.08337526139339    Inf       0
    Residual                   0    490                      0
A = 0.000000, B = 0.100000, C = 0.270000
order Q(x) = 7.000000, order D(x) = 3.000000
Xmin = 0.000000 and Xmax = 5.000000
Ymin = 0.000000 and Ymax = 74.203211
List of powers for Q(x): [0.100000, 0.470000, 0.840000, 1.210000, 1.580000, 1.950000,
2.320000]
List of powers for D(x): [0.100000, 0.470000, 0.840000]
Fitting sinh(x) in range (0.000000, 5.000000)
MSS of errors squared = 2.513916e-10
Corrected MSS of errors squared = 4.354231e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting sinh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                    Estimate         SE    tStat    pValue
                ________________    ___    _____    ______
    (Intercept)  0.847390141021585    0      Inf       0
    x1          -97.1357923261311     0     -Inf       0
    x2                          0     0      NaN      NaN
    x3           476.370866751263     0      Inf       0
    x4          -578.297853822764     0     -Inf       0
    x5           198.212673457829     0      Inf       0
    x6           377.21542739263      0      Inf       0
    x7          -1021.13398799103     0     -Inf       0
    x8           926.245377168831     0      Inf       0
    x9          -281.324100767157     0     -Inf       0
Number of observations: 501, Error degrees of freedom: 492
```



```
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ________________    ___    ________________    ___    _______
    Total       30.8337526139171    500    0.0616675052278342
    Model       30.8337526139171      8      3.85421907673964    Inf        0
    Residual                   0    492                     0
A = 0.474721, B = 0.103878, C = 0.057713
order Q(x) = 5.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 5.000000
Ymin = 0.000000 and Ymax = 74.203211
List of powers for Q(x): [0.578599, 0.624765, 0.670930, 0.717095, 0.763260]
List of powers for D(x): [0.578599, 0.624765, 0.670930, 0.717095]
Fitting sinh(x) in range (0.000000, 5.000000)
MSS of errors squared = 6.894659e-08
Corrected MSS of errors squared = 1.194190e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting sinh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                   ________________    ___    _____    _______
    (Intercept)     1.63967278802338     0      Inf        0
    x1             -6.90557590499961     0     -Inf        0
    x2              9.92916296273869     0      Inf        0
    x3                             0     0      NaN      NaN
    x4             -11.7732348225659     0     -Inf        0
    x5              9.37636059122518     0      Inf        0
    x6             -2.28870091179198     0     -Inf        0
    x7              5.04651045157779     0      Inf        0
    x8             -11.1131119778492     0     -Inf        0
    x9              11.2889974267275     0      Inf        0
    x10            -5.02942137239259     0     -Inf        0
    x11            0.829340769603731     0      Inf        0
Number of observations: 501, Error degrees of freedom: 490
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ________________    ___    ________________    ___    _______
    Total       30.8337526139327    500    0.0616675052278655
    Model       30.8337526139327     10      3.08337526139327    Inf        0
    Residual                   0    490                     0
```



```
A = 0.147124, B = 0.212988, C = 0.155418
order Q(x) = 6.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 5.000000
Ymin = 0.000000 and Ymax = 74.203211
List of powers for Q(x): [0.360113, 0.728519, 1.005883, 1.268269, 1.522901, 1.772578]
List of powers for D(x): [0.360113, 0.728519, 1.005883, 1.268269, 1.522901]
Fitting sinh(x) in range (0.000000, 5.000000)
MSS of errors squared = 2.545897e-10
Corrected MSS of errors squared = 4.409623e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting sinh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    _____________________    __    _____    ______
    (Intercept)      0.325537600762257    0      Inf       0
    x1               0.478077799726322    0      Inf       0
    x2               -1.03329897390624    0     -Inf       0
    x3               0.428733828490264    0      Inf       0
    x4                5.61636837950563    0      Inf       0
    x5               -14.7519991231338    0     -Inf       0
    x6                18.8632643391274    0      Inf       0
    x7               -12.5545360754387    0     -Inf       0
    x8                4.17658203850266    0      Inf       0
    x9              -0.548729802595446    0     -Inf       0
Number of observations: 501, Error degrees of freedom: 491
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                 _________________    ___    ____________________    ___    ______
    Total        30.8337526139308    500    0.0616675052278616
    Model        30.8337526139308      9      3.42597251265898    Inf       0
    Residual                    0    491                      0
A = 0.512665, B = 0.400000, C = 0.400000
order Q(x) = 3.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 5.000000
Ymin = 0.000000 and Ymax = 74.203211
List of powers for Q(x): [0.912665, 1.478351, 2.005485]
List of powers for D(x): [0.912665, 1.478351, 2.005485, 2.512665, 3.007092, 3.492461]
Fitting sinh(x) in range (0.000000, 5.000000)
MSS of errors squared = 1.688153e-09
Corrected MSS of errors squared = 2.923967e-09
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting sinh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.0346101682763741     0     -Inf        0
    x1                2.41941460053825     0      Inf        0
    x2               -8.64771376124922     0     -Inf        0
    x3                               0     0      NaN      NaN
    x4                23.8722855690384     0      Inf        0
    x5               -25.6234544970861     0     -Inf        0
    x6                8.05163187317098     0      Inf        0
    x7                16.6837517992187     0      Inf        0
    x8               -151.907207549097     0     -Inf        0
    x9                374.115056094587     0      Inf        0
    x10              -324.641273092166     0     -Inf        0
    x11                              0     0      NaN      NaN
    x12               140.470137457169     0      Inf        0
    x13              -53.7580183258745     0     -Inf        0
Number of observations: 501, Error degrees of freedom: 489
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F      pValue
                  _______________    ___    ____________________    ___    ______
    Total        30.833752614008     500    0.0616675052280159
    Model        30.833752614008      11      2.80306841945527    Inf        0
    Residual                   0     489                     0
A = 0.500000, B = 0.233543, C = 0.357879
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 5.000000
Ymin = 0.000000 and Ymax = 74.203211
List of powers for Q(x): [0.733543, 1.188158, 1.410625, 1.586950, 1.737975, 1.872302]
List of powers for D(x): [0.733543, 1.188158, 1.410625, 1.586950, 1.737975, 1.872302,
1.994517]
Fitting sinh(x) in range (0.000000, 5.000000)
MSS of errors squared = 6.393740e-11
Corrected MSS of errors squared = 1.107428e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
```



```
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting sinh(x) in range (0.000000, 5.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    ________________    ___    _____    ______
    (Intercept)    0.156282806922575     0      Inf        0
    x1              60.1990867672937     0      Inf        0
    x2                             0     0      NaN      NaN
    x3             -86.1469249778426     0     -Inf        0
    x4                             0     0      NaN      NaN
    x5              45.4576720619844     0      Inf        0
    x6             -22.9157738306532     0     -Inf        0
    x7              3.52261984747747     0      Inf        0
    x8              556.940636669339     0      Inf        0
    x9                             0     0      NaN      NaN
    x10            -1376.78063105872     0     -Inf        0
    x11             1322.49974124007     0      Inf        0
    x12            -657.624438820489     0     -Inf        0
    x13             175.413331216975     0      Inf        0
    x14            -19.7216019225895     0     -Inf        0
Number of observations: 501, Error degrees of freedom: 489
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F     pValue
                  _______________    ___    ________________    ___    ______
    Total        30.8337526139199    500    0.0616675052278399
    Model        30.8337526139199     11      2.80306841944727    Inf       0
    Residual                    0    489                     0
A = 1.573782, B = 2.533255
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 5.000000
Ymin = 0.000000 and Ymax = 74.203211
List of powers for Q(x): [1.573782, 1.594585, 1.705061, 1.906624, 2.178445, 2.502610,
2.865916]
List of powers for D(x): [1.573782, 1.594585, 1.705061, 1.906624, 2.178445, 2.502610,
2.865916]
Fitting sinh(x) in range (0.000000, 5.000000)
MSS of errors squared = 4.671379e-11
Corrected MSS of errors squared = 6.606328e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the hyperbolic sine function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i	0.23	0.5		Inf	-Inf	5.14194300E-12	6, 7
A+B*log10(i)^4	1.573	2.533		Inf	-Inf	4.67137900E-11	7, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.5	0.233	0.357	Inf	-Inf	6.39374000E-11	6, 7
A+B*i+C*(i-1)	0	0.1	0.27	Inf	-Inf	2.51391600E-10	7, 3
A+B*i+C*sqrt(i-1)	0.147	0.212	0.155	Inf	-Inf	2.54589700E-10	6, 5
A+B*sqrt(i)+C*(i-1)	0.512	0.4	0.4	Inf	-Inf	1.68815300E-09	3, 6
A+B/i	0.097	0.569		Inf	-Inf	2.88636400E-09	4, 7
A+B*sqrt(i)	0	0.177		Inf	-Inf	3.03732000E-09	7, 5
A+B*sqrt(i+C)	0.861	0.642	0.629	Inf	-Inf	3.11022200E-09	7, 3
A+B*i-C*(i-1)	0.474	0.103	0.057	Inf	-Inf	6.89465900E-08	5, 4

RESULTS FOR THE TANGENT FUNCTION

Using Power A+B* i

```
Fitting tan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)       1.22119493759775      0      Inf        0
    x1               -9.41398472969636      0     -Inf        0
    x2                29.6827134180971      0      Inf        0
    x3               -42.4586346894543      0     -Inf        0
    x4                33.4972854383164      0      Inf        0
    x5               -15.6520841049131      0     -Inf        0
    x6                4.10810333646571      0      Inf        0
    x7               -0.469392131735329     0     -Inf        0
    x8               -0.759010399188472     0     -Inf        0
    x9                  1.7884007190154     0      Inf        0
    x10              -0.544591794501931     0     -Inf        0
Number of observations: 101, Error degrees of freedom: 90
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq          DF            MeanSq            F      pValue
                   ________________    ___    ____________________    ___    _______
    Total       7.61910551705052     100    0.0761910551705052
    Model       7.61910551705052      10     0.761910551705052     Inf        0
    Residual                   0      90                     0
A = 0.005646, B = 0.390126
order Q(x) = 7.000000, order D(x) = 3.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.557408
List of powers for Q(x): [0.395772, 0.785899, 1.176025, 1.566151, 1.956278, 2.346404,
2.736530]
```



```
List of powers for D(x): [0.395772, 0.785899, 1.176025]
Fitting tan(x) in range (0.000000, 1.000000)
MSS of errors squared = 5.811414e-13
Corrected MSS of errors squared = 8.218580e-13
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting tan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    0.0212533059465581     0      Inf        0
    x1             0.0454078770530511     0      Inf        0
    x2              0.554030397899078     0      Inf        0
    x3               -4.7337108339356     0     -Inf        0
    x4                              0     0      NaN      NaN
    x5              9.38676545759605     0      Inf        0
    x6                              0     0      NaN      NaN
    x7             -5.00327137073654     0     -Inf        0
    x8             0.0209709854273205    0      Inf        0
    x9             -5.62689863463675     0     -Inf        0
    x10             45.5806913180369     0      Inf        0
    x11            -92.0826958957983     0     -Inf        0
    x12             52.8374573931484     0      Inf        0
Number of observations: 101, Error degrees of freedom: 90
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _______________    ___    ___________________    ___    ______
    Total        7.61910551705192    100    0.0761910551705192
    Model        7.61910551705192     10     0.761910551705192    Inf       0
    Residual                    0     90                      0
A = 0.418201, B = 1.625378
order Q(x) = 7.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.557408
List of powers for Q(x): [2.043579, 1.230890, 0.959994, 0.824546, 0.743277, 0.689098,
0.650398]
List of powers for D(x): [2.043579, 1.230890, 0.959994, 0.824546, 0.743277]
Fitting tan(x) in range (0.000000, 1.000000)
MSS of errors squared = 4.903553e-14
Corrected MSS of errors squared = 6.934671e-14
R-Squared =   1.00000000
```



```
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting tan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)    0.0251506780975498     0      Inf         0
    x1               264.183466267884     0      Inf         0
    x2              -563.716370352249     0     -Inf         0
    x3               300.624126581935     0      Inf         0
    x4              -30.6053854611688     0     -Inf         0
    x5              -102.799928194521     0     -Inf         0
    x6               230.549329432568     0      Inf         0
    x7                             0     0      NaN       NaN
    x8              -97.2603889579172     0     -Inf         0
Number of observations: 101, Error degrees of freedom: 93
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF            MeanSq            F      pValue
                  _________________    ___    ____________________    ___    _______
    Total        7.61910551705121     100    0.0761910551705121
    Model        7.61910551705121       7      1.08844364529303    Inf        0
    Residual                    0      93                     0
A = 1.400000, B = 0.100000
order Q(x) = 3.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.557408
List of powers for Q(x): [1.500000, 1.541421, 1.573205]
List of powers for D(x): [1.500000, 1.541421, 1.573205, 1.600000, 1.623607]
Fitting tan(x) in range (0.000000, 1.000000)
MSS of errors squared = 2.731153e-10
Corrected MSS of errors squared = 3.862433e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting tan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
```


	Estimate	SE	tStat	pValue
(Intercept)	0.0194429380061493	0	Inf	0
x1	5.81008296908619	0	Inf	0
x2	-6.5835122404417	0	-Inf	0
x3	0	0	NaN	NaN
x4	3.44145762489503	0	Inf	0
x5	0	0	NaN	NaN
x6	-2.87611730965545	0	-Inf	0
x7	1.33863603052295	0	Inf	0
x8	-2.41716049426249	0	-Inf	0
x9	2.81003548335816	0	Inf	0
x10	0	0	NaN	NaN
x11	-0.840541205291612	0	-Inf	0
x12	0	0	NaN	NaN
x13	0.709933261381672	0	Inf	0
x14	-0.412257057597328	0	-Inf	0

Number of observations: 101, Error degrees of freedom: 90

R-squared: 1, Adjusted R-Squared: 1

F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	7.6191055170512	100	0.076191055170512		
Model	7.6191055170512	10	0.76191055170512	Inf	0
Residual	0	90	0		

A = 0.000000, B = 1.000000, C = 0.532415

order Q(x) = 7.000000, order D(x) = 7.000000

Xmin = 0.000000 and Xmax = 1.000000

Ymin = 0.000000 and Ymax = 1.557408

List of powers for Q(x): [1.237908, 1.591356, 1.879472, 2.128947, 2.352109, 2.555859, 2.744525]

List of powers for D(x): [1.237908, 1.591356, 1.879472, 2.128947, 2.352109, 2.555859, 2.744525]

Fitting tan(x) in range (0.000000, 1.000000)

MSS of errors squared = 8.023223e-14

Corrected MSS of errors squared = 1.389663e-13

R-Squared = 1.00000000

R-Squared Adjusted = 1.00000000

Particle swarm AICc = -1.000000e+99

AIC = -Inf

AICc = -Inf

Using Power A+B*i+C*(i–1)

Fitting tan(x) in range (0.000000, 1.000000)

Pade-Shammas polynomial power is A+B*i+C*(i-1)

Linear regression model:

y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11

Estimated Coefficients:

	Estimate	SE	tStat	pValue


```
    (Intercept)     0.0435536945786201    0      Inf        0
    x1              -0.591223907261367    0     -Inf        0
    x2                2.08716568485881    0      Inf        0
    x3               -2.0314870417563     0     -Inf        0
    x4               0.589063739811087    0      Inf        0
    x5                3.89801410291627    0      Inf        0
    x6               -6.23031675837789    0     -Inf        0
    x7                3.96286426991333    0      Inf        0
    x8                               0    0      NaN      NaN
    x9               -1.1405962938587     0     -Inf        0
    x10              0.476400765013929    0      Inf        0
    x11             -0.0634382558377565   0     -Inf        0
Number of observations: 101, Error degrees of freedom: 90
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F     pValue
                ________________    ___    ____________________    ___    ______
    Total       7.61910551705124    100    0.0761910551705124
    Model       7.61910551705124     10     0.761910551705124    Inf       0
    Residual                   0     90                      0
A = 0.178126, B = 0.189710, C = 0.079776
order Q(x) = 4.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.557408
List of powers for Q(x): [0.367836, 0.637322, 0.906808, 1.176294]
List of powers for D(x): [0.367836, 0.637322, 0.906808, 1.176294, 1.445780, 1.715266,
1.984752]
Fitting tan(x) in range (0.000000, 1.000000)
MSS of errors squared = 5.074448e-14
Corrected MSS of errors squared = 8.789202e-14
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting tan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)                        0    0      NaN      NaN
    x1              -0.00588389908479305    0     -Inf        0
    x2                0.0191355480848145    0      Inf        0
    x3               -0.0220623097226328    0     -Inf        0
    x4                0.0105364400296416    0      Inf        0
    x5               -0.00174895285844459   0     -Inf        0
```



```
    x6              -0.0756564486282971    0      -Inf        0
    x7                 0.95216786220714    0       Inf        0
    x8                                0    0       NaN      NaN
    x9                0.179422481292871    0       Inf        0
    x10                               0    0       NaN      NaN
    x11             -0.0915637300419178    0      -Inf        0
    x12              0.0356530087216171    0       Inf        0
Number of observations: 101, Error degrees of freedom: 91
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq           DF           MeanSq            F     pValue
                ________________    ___    ____________________    ___    ______
    Total       7.61910551705131    100    0.0761910551705132
    Model       7.61910551705131      9     0.846567279672368    Inf       0
    Residual                   0     91                     0
A = 0.000000, B = 0.100000, C = 0.180259
order Q(x) = 5.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.557408
List of powers for Q(x): [0.100000, 0.019741, -0.060517, -0.140776, -0.221034]
List of powers for D(x): [0.100000, 0.019741, -0.060517, -0.140776, -0.221034, -
0.301293, -0.381551]
Fitting tan(x) in range (0.000000, 1.000000)
MSS of errors squared = 7.437926e-12
Corrected MSS of errors squared = 1.288287e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting tan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)     0.0107071184236761      0      Inf        0
    x1              -0.456672011006603      0     -Inf        0
    x2                3.67096198130633      0      Inf        0
    x3               -4.13774086811415      0     -Inf        0
    x4                1.24844436365224      0      Inf        0
    x5                4.91591026422113      0      Inf        0
    x6               -11.2666388725256      0     -Inf        0
    x7                10.9566196919344      0      Inf        0
    x8               -4.59962737712366      0     -Inf        0
    x9               0.658035709233455      0      Inf        0
Number of observations: 101, Error degrees of freedom: 91
```



```
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ________________    ___    ____________________    ___    ______
    Total       7.61910551705116    100    0.0761910551705116
    Model       7.61910551705116      9     0.846567279672351    Inf       0
    Residual                   0     91                     0
A = 0.340000, B = 0.296849, C = 0.002060
order Q(x) = 4.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.557408
List of powers for Q(x): [0.636849, 0.935758, 1.233460, 1.530964]
List of powers for D(x): [0.636849, 0.935758, 1.233460, 1.530964, 1.828365]
Fitting tan(x) in range (0.000000, 1.000000)
MSS of errors squared = 2.530408e-13
Corrected MSS of errors squared = 4.382795e-13
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting tan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.0221131460475386     0     -Inf       0
    x1               0.328920636801497     0      Inf       0
    x2                1.78361465937294     0      Inf       0
    x3               -1.92060456365052     0     -Inf       0
    x4               0.470987954639113     0      Inf       0
    x5                1.03949421677754     0      Inf       0
    x6               -2.51592491431896     0     -Inf       0
    x7                2.84910544039705     0      Inf       0
    x8               -1.21991052271338     0     -Inf       0
    x9               0.238648948846291     0      Inf       0
    x10             -0.035887537723747     0     -Inf       0
    x11              0.00366882761970095    0      Inf       0
Number of observations: 101, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ________________    ___    ____________________    ___    ______
    Total       7.61910551705138    100    0.0761910551705138
    Model       7.61910551705138     11     0.69264595609558    Inf       0
    Residual                   0     89                     0
```



```
A = 0.238545, B = 0.331191, C = 0.400000
order Q(x) = 4.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.557408
List of powers for Q(x): [0.569735, 1.106919, 1.612183, 2.100926]
List of powers for D(x): [0.569735, 1.106919, 1.612183, 2.100926, 2.579109, 3.049792,
3.514792]
Fitting tan(x) in range (0.000000, 1.000000)
MSS of errors squared = 3.219958e-15
Corrected MSS of errors squared = 5.577131e-15
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting tan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)     -0.278495070029998    0     -Inf       0
    x1                1.05688620145428    0      Inf       0
    x2              -0.378935426529614    0     -Inf       0
    x3             0.00175998031688236    0      Inf       0
    x4                0.80000908831882    0      Inf       0
    x5               -1.17924542066105    0     -Inf       0
    x6                1.54255676059636    0      Inf       0
    x7              -0.564536120122187    0     -Inf       0
Number of observations: 101, Error degrees of freedom: 93
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                 _________________    ___    ____________________    ___    ______
    Total        7.61910551705127     100    0.0761910551705127
    Model        7.61910551705127       7      1.08844364529304    Inf       0
    Residual                    0      93                     0
A = 0.300437, B = 0.100000, C = 0.700000
order Q(x) = 3.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.557408
List of powers for Q(x): [0.400437, 1.141858, 1.463592]
List of powers for D(x): [0.400437, 1.141858, 1.463592, 1.712873]
Fitting tan(x) in range (0.000000, 1.000000)
MSS of errors squared = 9.544665e-10
Corrected MSS of errors squared = 1.653185e-09
R-Squared =   1.00000000
```



```
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting tan(x) in range (0.000000, 1.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)    -0.307951240221355    0     -Inf        0
    x1               23.2236672180267    0      Inf        0
    x2              -26.1724318030885    0     -Inf        0
    x3               3.26231912892453    0      Inf        0
    x4               9.77935969104493    0      Inf        0
    x5                              0    0      NaN      NaN
    x6              -15.1764401650085    0     -Inf        0
    x7               8.31406231549398    0      Inf        0
    x8              -2.16135261923548    0     -Inf        0
    x9              0.238767474038288    0      Inf        0
Number of observations: 101, Error degrees of freedom: 92
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                 _________________    ___    ____________________    ___    _______
    Total        7.61910551705126    100    0.0761910551705126
    Model        7.61910551705126      8     0.952388189631408    Inf       0
    Residual                    0     92                     0
A = 0.312097, B = 2.403959
order Q(x) = 3.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 1.000000
Ymin = 0.000000 and Ymax = 1.557408
List of powers for Q(x): [0.312097, 0.331838, 0.436675]
List of powers for D(x): [0.312097, 0.331838, 0.436675, 0.627951, 0.885898, 1.193517]
Fitting tan(x) in range (0.000000, 1.000000)
MSS of errors squared = 7.046323e-09
Corrected MSS of errors squared = 9.965006e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the tangent function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i)+C*(i-1)	0.238	0.331	0.4	Inf	-Inf	3.21995800E-15	4, 7
A+B/i	0.418	1.625		Inf	-Inf	4.90355300E-14	7, 5
A+B*i+C*(i-1)	0.178	0.189	0.079	Inf	-Inf	5.07444800E-14	4, 7
A+B*sqrt(i+C)	0	1	0.532	Inf	-Inf	8.02322300E-14	7, 7
A+B*i+C*sqrt(i-1)	0.34	0.296	0.002	Inf	-Inf	2.53040800E-13	4, 5
A+B*i	0.005	0.39		Inf	-Inf	5.81141400E-13	7, 3
A+B*i-C*(i-1)	0	0.1	0.18	Inf	-Inf	7.43792600E-12	5, 7
A+B*sqrt(i)	1.4	0.1		Inf	-Inf	2.73115300E-10	3, 5
A+B*sqrt(i)+C*sqrt(i-1)	0.3	0.1	0.7	Inf	-Inf	9.54466500E-10	3, 4
A+B*log10(i)^4	0.312	2.403		Inf	-Inf	7.04632300E-09	3, 6

RESULTS FOR THE HYPERBOLIC TANGENT FUNCTION

Using Power A+B* i

```
Fitting tanh(x) in range (0.000000, 3.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)       0.11293424723102      0      Inf       0
    x1               -0.733127522378225     0     -Inf       0
    x2                1.45736436700364      0      Inf       0
    x3               -1.21267200134335      0     -Inf       0
    x4                0.386143407727928     0      Inf       0
    x5                4.65846039696875      0      Inf       0
    x6               -7.54344660467845      0     -Inf       0
    x7                5.61525044297002      0      Inf       0
    x8               -2.00037645494949      0     -Inf       0
    x9                0.259469722218531     0      Inf       0
Number of observations: 301, Error degrees of freedom: 291
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                ____________________    ___    ____________________    ___    ______
    Total       23.2802959868212        300    0.0776009866227374
    Model       23.2802959868212          9      2.58669955409125    Inf       0
    Residual                     0        291                     0
A = 0.125124, B = 0.397159
order Q(x) = 4.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 3.000000
Ymin = 0.000000 and Ymax = 0.995055
List of powers for Q(x): [0.522284, 0.919443, 1.316603, 1.713762]
List of powers for D(x): [0.522284, 0.919443, 1.316603, 1.713762, 2.110921]
Fitting tanh(x) in range (0.000000, 3.000000)
```



```
MSS of errors squared = 8.409928e-10
Corrected MSS of errors squared = 1.189343e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting tanh(x) in range (0.000000, 3.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)    -0.805085563611522     0     -Inf        0
    x1               1.68154040675821     0      Inf        0
    x2              -53.213895165948      0     -Inf        0
    x3              253.832773190366      0      Inf        0
    x4             -290.799205095913      0     -Inf        0
    x5                             0      0      NaN      NaN
    x6                             0      0      NaN      NaN
    x7               89.3697862265775     0      Inf        0
    x8             -0.744308245073958     0     -Inf        0
    x9               15.398994494148      0      Inf        0
    x10            -46.5522778790317      0     -Inf        0
    x11              32.8316776274296     0      Inf        0
Number of observations: 301, Error degrees of freedom: 291
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ___________________    ___    _______
    Total        23.2802959868157     300    0.0776009866227191
    Model        23.2802959868157       9      2.58669955409064    Inf        0
    Residual                    0     291                     0
A = 0.750835, B = 1.922888
order Q(x) = 7.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 3.000000
Ymin = 0.000000 and Ymax = 0.995055
List of powers for Q(x): [2.673724, 1.712280, 1.391798, 1.231558, 1.135413, 1.071317,
1.025534]
List of powers for D(x): [2.673724, 1.712280, 1.391798, 1.231558]
Fitting tanh(x) in range (0.000000, 3.000000)
MSS of errors squared = 5.987879e-10
Corrected MSS of errors squared = 8.468139e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
```



```
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting tanh(x) in range (0.000000, 3.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -0.00223265768619065    0     -Inf       0
    x1               0.0806295938852522    0      Inf       0
    x2              -0.167256676843509     0     -Inf       0
    x3               0.101524976072688     0      Inf       0
    x4                 29.541190393993     0      Inf       0
    x5              -130.748703657879      0     -Inf       0
    x6               248.062947900471      0      Inf       0
    x7              -239.764153212708      0     -Inf       0
    x8               116.679698845513      0      Inf       0
    x9              -22.7836455052578      0     -Inf       0
Number of observations: 301, Error degrees of freedom: 291
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                 _______________    ___    ____________________    ___    ______
    Total        23.2802959868371    300    0.0776009866227904
    Model        23.2802959868371      9      2.58669955409301    Inf       0
    Residual                    0    291                     0
A = 0.145786, B = 0.894137
order Q(x) = 3.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 3.000000
Ymin = 0.000000 and Ymax = 0.995055
List of powers for Q(x): [1.039923, 1.410287, 1.694477]
List of powers for D(x): [1.039923, 1.410287, 1.694477, 1.934060, 2.145137, 2.335966]
Fitting tanh(x) in range (0.000000, 3.000000)
MSS of errors squared = 4.732130e-10
Corrected MSS of errors squared = 6.692243e-10
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting tanh(x) in range (0.000000, 3.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                      Estimate          SE    tStat    pValue
                  ____________________    __    _____    ______
```



```
    (Intercept)     6.39760366892246    0      Inf       0
    x1             -1961.81596972224    0     -Inf       0
    x2              10930.4610635239    0      Inf       0
    x3             -24138.2669616055    0     -Inf       0
    x4              26332.4996514484    0      Inf       0
    x5             -14205.8243632519    0     -Inf       0
    x6              3036.62567017266    0      Inf       0
    x7              12.2444110010729    0      Inf       0
    x8             -19.6411036998447    0     -Inf       0
    x9              8.31999848998103    0      Inf       0
Number of observations: 301, Error degrees of freedom: 291
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                 _______________    ___    ____________________    ___    ______
    Total        23.2802959874493    300    0.0776009866248309
    Model        23.2802959874493      9      2.58669955416103    Inf       0
    Residual                    0    291                     0
A = 0.230218, B = 0.624199, C = 0.755123
order Q(x) = 6.000000, order D(x) = 3.000000
Xmin = 0.000000 and Xmax = 3.000000
Ymin = 0.000000 and Ymax = 0.995055
List of powers for Q(x): [1.057163, 1.266298, 1.439799, 1.591361, 1.727660, 1.852549]
List of powers for D(x): [1.057163, 1.266298, 1.439799]
Fitting tanh(x) in range (0.000000, 3.000000)
MSS of errors squared = 3.965125e-09
Corrected MSS of errors squared = 6.867798e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting tanh(x) in range (0.000000, 3.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    _____________________    ___    _____    ______
    (Intercept)    -0.334377175443096    0      -Inf       0
    x1               1.98445760864581    0       Inf       0
    x2              -2.74988456750212    0      -Inf       0
    x3               1.10743210740915    0       Inf       0
    x4               5.97126813240499    0       Inf       0
    x5              -9.71722477470136    0      -Inf       0
    x6               6.22541584419443    0       Inf       0
    x7              -1.48708723468847    0      -Inf       0
Number of observations: 301, Error degrees of freedom: 293
```



```
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ________________    ___    ____________________    ___    _______
    Total       23.280295986821     300    0.0776009866227365
    Model       23.280295986821       7      3.32575656954585    Inf        0
    Residual                  0     293                     0
A = 0.287493, B = 0.118598, C = 0.110721
order Q(x) = 3.000000, order D(x) = 4.000000
Xmin = 0.000000 and Xmax = 3.000000
Ymin = 0.000000 and Ymax = 0.995055
List of powers for Q(x): [0.406091, 0.635411, 0.864730]
List of powers for D(x): [0.406091, 0.635411, 0.864730, 1.094049]
Fitting tanh(x) in range (0.000000, 3.000000)
MSS of errors squared = 9.916516e-08
Corrected MSS of errors squared = 1.717591e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting tanh(x) in range (0.000000, 3.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)    0.0689135024150617     0      Inf        0
    x1             -0.911618896570373     0     -Inf        0
    x2                              0     0      NaN       NaN
    x3               3.82851820351611     0      Inf        0
    x4              -4.53689428357068     0     -Inf        0
    x5                1.5517251439454     0      Inf        0
    x6               23.0684328575553     0      Inf        0
    x7              -40.8676595148434     0     -Inf        0
    x8                              0     0      NaN       NaN
    x9               46.5379797000551     0      Inf        0
    x10             -36.3965041425592     0     -Inf        0
    x11              8.65710743009218     0      Inf        0
Number of observations: 301, Error degrees of freedom: 291
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ________________    ___    ____________________    ___    _______
    Total       23.2802959868216    300    0.0776009866227386
    Model       23.2802959868216      9      2.58669955409129    Inf        0
    Residual                  0    291                     0
```



```
A = 0.150167, B = 0.219588, C = 0.115881
order Q(x) = 5.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 3.000000
Ymin = 0.000000 and Ymax = 0.995055
List of powers for Q(x): [0.369755, 0.473463, 0.577170, 0.680878, 0.784586]
List of powers for D(x): [0.369755, 0.473463, 0.577170, 0.680878, 0.784586, 0.888293]
Fitting tanh(x) in range (0.000000, 3.000000)
MSS of errors squared = 7.218134e-10
Corrected MSS of errors squared = 1.250217e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting tanh(x) in range (0.000000, 3.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    _________________    __    _____    ______
    (Intercept)      -0.612444373428     0     -Inf       0
    x1                1.98358874988285   0      Inf       0
    x2               -6.37297067066388   0     -Inf       0
    x3                11.9876421265631   0      Inf       0
    x4               -11.4275047673532   0     -Inf       0
    x5                5.52011700403682   0      Inf       0
    x6               -1.07832508823421   0     -Inf       0
    x7                2.55615519277304   0      Inf       0
    x8               -4.80579680683528   0     -Inf       0
    x9                5.74818606047562   0      Inf       0
    x10              -3.17333518728317   0     -Inf       0
    x11              0.674687760066103   0      Inf       0
Number of observations: 301, Error degrees of freedom: 289
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq            F     pValue
                    _________________    ___    _________________    ___    ______
    Total       23.2802959868206     300    0.0776009866227352
    Model       23.2802959868206      11      2.11639054425642    Inf       0
    Residual                   0     289                     0
A = 0.009610, B = 0.154918, C = 0.174830
order Q(x) = 6.000000, order D(x) = 5.000000
Xmin = 0.000000 and Xmax = 3.000000
Ymin = 0.000000 and Ymax = 0.995055
List of powers for Q(x): [0.164528, 0.494276, 0.721610, 0.932095, 1.133858, 1.330048]
List of powers for D(x): [0.164528, 0.494276, 0.721610, 0.932095, 1.133858]
Fitting tanh(x) in range (0.000000, 3.000000)
```



```
MSS of errors squared = 9.088275e-10
Corrected MSS of errors squared = 1.574135e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting tanh(x) in range (0.000000, 3.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   ____________________    __    _____    ______
    (Intercept)     0.0130535411916556     0      Inf       0
    x1             -0.0733483949703825     0     -Inf       0
    x2               0.160943306092696     0      Inf       0
    x3              -0.15285564028402      0     -Inf       0
    x4              0.0572019512484726     0      Inf       0
    x5                 4.8157129297475     0      Inf       0
    x6              -9.40827906725329      0     -Inf       0
    x7               9.59078175006925      0      Inf       0
    x8              -5.42101593300323      0     -Inf       0
    x9               1.62001808708792      0      Inf       0
    x10            -0.202212529906708      0     -Inf       0
Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F     pValue
                ___________________    ___    ____________________    ___    ______
    Total       23.280295986821     300    0.0776009866227367
    Model       23.280295986821      10       2.3280295986821    Inf       0
    Residual                  0     290                     0
A = 0.236687, B = 0.269499, C = 0.331359
order Q(x) = 4.000000, order D(x) = 6.000000
Xmin = 0.000000 and Xmax = 3.000000
Ymin = 0.000000 and Ymax = 0.995055
List of powers for Q(x): [0.506186, 0.949176, 1.366192, 1.769763]
List of powers for D(x): [0.506186, 0.949176, 1.366192, 1.769763, 2.164742, 2.553619]
Fitting tanh(x) in range (0.000000, 3.000000)
MSS of errors squared = 5.220035e-11
Corrected MSS of errors squared = 9.041366e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)


```
Fitting tanh(x) in range (0.000000, 3.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)    -4.86175057071796e-05    0     -Inf       0
    x1               0.00337710981435268    0      Inf       0
    x2              -0.00609452169076504    0     -Inf       0
    x3               0.00535585835797452    0      Inf       0
    x4                  3.56009356609611    0      Inf       0
    x5                 -18.7214648684462    0     -Inf       0
    x6                  51.2515312182529    0      Inf       0
    x7                 -71.6898234840559    0     -Inf       0
    x8                  55.3073359300734    0      Inf       0
    x9                 -22.4912905218795    0     -Inf       0
    x10                 3.78102833098044    0      Inf       0
Number of observations: 301, Error degrees of freedom: 290
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ___    ____________________    ___    ______
    Total        23.2802959868203    300    0.0776009866227342
    Model        23.2802959868203     10      2.32802959868203    Inf       0
    Residual                    0    290                     0
A = 0.000000, B = 0.469187, C = 0.593534
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 3.000000
Ymin = 0.000000 and Ymax = 0.995055
List of powers for Q(x): [0.469187, 1.257065, 1.652040]
List of powers for D(x): [0.469187, 1.257065, 1.652040, 1.966405, 2.236202, 2.476452,
2.695208]
Fitting tanh(x) in range (0.000000, 3.000000)
MSS of errors squared = 2.083105e-11
Corrected MSS of errors squared = 3.608044e-11
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting tanh(x) in range (0.000000, 3.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
```

```
    (Intercept)    0.062122725342443    0      Inf        0
    x1             -273.925569018731    0     -Inf        0
    x2                             0    0      NaN      NaN
    x3              579.320141489677    0      Inf        0
    x4             -401.151687289074    0     -Inf        0
    x5              95.9002282853393    0      Inf        0
    x6              553.466830086014    0      Inf        0
    x7                             0    0      NaN      NaN
    x8             -826.916960043948    0     -Inf        0
    x9                             0    0      NaN      NaN
    x10             516.520906620277    0      Inf        0
    x11            -292.747962645931    0     -Inf        0
    x12             50.4719498117929    0      Inf        0
Number of observations: 301, Error degrees of freedom: 291
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq           F      pValue
                ________________    ___    ____________________    ___    ______
    Total       23.280295986851    300    0.0776009866228368
    Model       23.280295986851      9      2.58669955409456    Inf       0
    Residual                  0    291                     0
A = 2.200000, B = 1.517210
order Q(x) = 5.000000, order D(x) = 7.000000
Xmin = 0.000000 and Xmax = 3.000000
Ymin = 0.000000 and Ymax = 0.995055
List of powers for Q(x): [2.200000, 2.212459, 2.278625, 2.399345, 2.562143]
List of powers for D(x): [2.200000, 2.212459, 2.278625, 2.399345, 2.562143, 2.756291,
2.973881]
Fitting tanh(x) in range (0.000000, 3.000000)
MSS of errors squared = 9.807783e-10
Corrected MSS of errors squared = 1.387030e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the hyperbolic tangent function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i)+C*sqrt(i-1)	0	0.469	0.593	Inf	-Inf	2.08310500E-11	3, 7
A+B*sqrt(i)+C*(i-1)	0.236	0.269	0.331	Inf	-Inf	5.22003500E-11	4, 6
A+B*sqrt(i)	0.145	0.894		Inf	-Inf	4.73213000E-10	3, 6
A+B/i	0.75	1.922		Inf	-Inf	5.98787900E-10	7, 4
A+B*i-C*(i-1)	0.15	0.219	0.115	Inf	-Inf	7.21813400E-10	5, 6
A+B*i	0.125	0.397		Inf	-Inf	8.40992800E-10	4, 5
A+B*i+C*sqrt(i-1)	0.009	0.154	0.174	Inf	-Inf	9.08827500E-10	6, 5
A+B*log10(i)^4	2.2	1.517		Inf	-Inf	9.80778300E-10	5, 7
A+B*sqrt(i+C)	0.23	0.624	0.755	Inf	-Inf	3.96512500E-09	6, 3
A+B*i+C*(i-1)	0.287	0.118	0.11	Inf	-Inf	9.91651600E-08	3, 4

RESULTS FOR THE TWO-SIDED T-INVERSE DISTRIBUTION

Using Power A+B* i

```
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ___________________    ___    _____    _______
    (Intercept)      0.977925440024234     0       Inf        0
    x1                 -7.7598808794277    0      -Inf        0
    x2               -0.152218661729332    0      -Inf        0
    x3                 16.1533986673985    0       Inf        0
    x4                -8.98264063498965    0      -Inf        0
    x5               -0.293695550491649    0      -Inf        0
    x6               0.0571107523432159    0       Inf        0
    x7                 8.01220618939794    0       Inf        0
    x8                                0    0       NaN      NaN
    x9                -16.6214388129405    0      -Inf        0
    x10                9.60923305697995    0       Inf        0
Number of observations: 981, Error degrees of freedom: 971
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                        SumSq          DF           MeanSq            F      pValue
                    ___________________    ___    ____________________    ___    _____
    Total          8.91039026622893     980    0.00909223496553972
    Model          8.91039026622893       9      0.990043362914325     Inf       0
    Residual                       0    971                          0
A = 0.433302, B = 0.126854
order Q(x) = 6.000000, order D(x) = 4.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 1.983972 and Ymax = 4.302653
List of powers for Q(x): [0.560156, 0.687009, 0.813863, 0.940717, 1.067571, 1.194425]
List of powers for D(x): [0.560156, 0.687009, 0.813863, 0.940717]
```



```
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
MSS of errors squared = 7.128662e-09
Corrected MSS of errors squared = 1.008145e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                   _________________    __    _____    ______
    (Intercept)    0.974054495451247    0      Inf        0
    x1             -5.75560214883135    0     -Inf        0
    x2              18.4462577553416    0      Inf        0
    x3                             0    0      NaN      NaN
    x4                             0    0      NaN      NaN
    x5                             0    0      NaN      NaN
    x6             -13.6647113832511    0     -Inf        0
    x7              5.82205499906867    0      Inf        0
    x8              -19.663343817253    0     -Inf        0
    x9              4.66136490477215    0      Inf        0
    x10                            0    0      NaN      NaN
    x11            -3.85591492156679    0     -Inf        0
    x12                            0    0      NaN      NaN
    x13             14.0358394756436    0      Inf        0
Number of observations: 981, Error degrees of freedom: 972
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq             F      pValue
                 _______________    ___    ____________________    ___    ______
    Total        8.91039026622916    980    0.00909223496553996
    Model        8.91039026622916      8       1.11379878327865    Inf       0
    Residual                    0    972                      0
A = 0.452954, B = 0.370621
order Q(x) = 6.000000, order D(x) = 7.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 1.983972 and Ymax = 4.302653
List of powers for Q(x): [0.823575, 0.638265, 0.576495, 0.545609, 0.527078, 0.514724]
List of powers for D(x): [0.823575, 0.638265, 0.576495, 0.545609, 0.527078, 0.514724,
0.505900]
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
MSS of errors squared = 7.969731e-08
Corrected MSS of errors squared = 1.127090e-07
R-Squared =   1.00000000
```



```
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)      0.978733711683613      0      Inf         0
    x1               -14.5403873904813      0     -Inf         0
    x2                20.8434678703851      0      Inf         0
    x3               -1.24902028247465      0     -Inf         0
    x4               -6.03281184358464      0     -Inf         0
    x5                14.7173165331554      0      Inf         0
    x6               -20.720588488994       0     -Inf         0
    x7                                0     0      NaN       NaN
    x8                8.00025347982193      0      Inf         0
    x9               -1.52339607661437      0     -Inf         0
    x10              0.635809524832536      0      Inf         0
    x11             -0.109386004956704      0     -Inf         0
Number of observations: 981, Error degrees of freedom: 970
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF            MeanSq              F      pValue
                  _________________    ___    ____________________    ___    _______
    Total        8.9103902662277      980    0.00909223496553847
    Model        8.9103902662277       10       0.89103902662277    Inf        0
    Residual                    0     970                       0
A = 0.704685, B = 0.971310
order Q(x) = 4.000000, order D(x) = 7.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 1.983972 and Ymax = 4.302653
List of powers for Q(x): [1.675995, 2.078325, 2.387044, 2.647306]
List of powers for D(x): [1.675995, 2.078325, 2.387044, 2.647306, 2.876601, 3.083900,
3.274531]
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
MSS of errors squared = 2.541188e-09
Corrected MSS of errors squared = 3.593782e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
```

```
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                   _________________    __    _____    ______
    (Intercept)    0.978257579560906    0      Inf       0
    x1             -28.6167108015997    0     -Inf       0
    x2               47.198853234127    0      Inf       0
    x3             -10.7197909790795    0     -Inf       0
    x4             -8.84070272252543    0     -Inf       0
    x5              29.1130227630463    0      Inf       0
    x6              -48.061096385202    0     -Inf       0
    x7              11.0728830741905    0      Inf       0
    x8              8.87523739277901    0      Inf       0
Number of observations: 99, Error degrees of freedom: 90
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq          DF           MeanSq            F     pValue
                   _________________    __    ___________________    ___    ______
    Total          1.43147727432708     98    0.0146069109625212
    Model          1.43147727432708      8     0.178934659290885    Inf       0
    Residual                      0     90                     0
A = 0.882875, B = 0.806839, C = 0.343349
order Q(x) = 4.000000, order D(x) = 4.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 1.983972 and Ymax = 4.302653
List of powers for Q(x): [1.818025, 2.117984, 2.358166, 2.564384]
List of powers for D(x): [1.818025, 2.117984, 2.358166, 2.564384]
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
MSS of errors squared = 1.056260e-07
Corrected MSS of errors squared = 1.829496e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                         Estimate          SE    tStat    pValue
                     _________________    __    _____    ______
    (Intercept)      0.978955852373873    0      Inf       0
    x1               -6.38679040529328    0     -Inf       0
    x2                 8.7397034442777    0      Inf       0
    x3               -3.33187048463136    0     -Inf       0
```


x4	6.53084733266979	0	Inf	0
x5	-8.97545053362909	0	-Inf	0
x6	3.49475453216266	0	Inf	0
x7	-0.0695975163548018	0	-Inf	0
x8	0.0237254301818832	0	Inf	0
x9	-0.0046910488670699	0	-Inf	0
x10	0.000412600473420873	0	Inf	0

Number of observations: 981, Error degrees of freedom: 970
R-squared: 1, Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0

	SumSq	DF	MeanSq	F	pValue
Total	8.91039026622901	980	0.00909223496553981		
Model	8.91039026622901	10	0.891039026622901	Inf	0
Residual	0	970	0		

A = 0.500000, B = 0.166952, C = 0.147464
order Q(x) = 3.000000, order D(x) = 7.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 1.983972 and Ymax = 4.302653
List of powers for Q(x): [0.666952, 0.981368, 1.295784]
List of powers for D(x): [0.666952, 0.981368, 1.295784, 1.610200, 1.924616, 2.239032, 2.553447]
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
MSS of errors squared = 8.649417e-10
Corrected MSS of errors squared = 1.498123e-09
R-Squared = 1.00000000
R-Squared Adjusted = 1.00000000
Particle swarm AICc = -1.000000e+99
AIC = -Inf
AICc = -Inf

Using Power A+B*i–C*(i–1)

Fitting tinv(0.975,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	0.978619838134301	0	Inf	0
x1	-8.12072116676863	0	-Inf	0
x2	11.6932754751954	0	Inf	0
x3	-4.47040287201949	0	-Inf	0
x4	-0.105427527438531	0	-Inf	0
x5	0.0277413422619483	0	Inf	0
x6	-0.00308618993821248	0	-Inf	0
x7	8.32112112808651	0	Inf	0
x8	-12.0550059731198	0	-Inf	0
x9	4.73388539532047	0	Inf	0

Number of observations: 981, Error degrees of freedom: 971


```
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq             F     pValue
                ___________________    ___    ____________________    ___    ______
    Total       8.91039026622902    980    0.00909223496553982
    Model       8.91039026622902      9      0.990043362914336    Inf       0
    Residual                   0    971                      0
A = 0.377419, B = 0.237725, C = 0.000000
order Q(x) = 6.000000, order D(x) = 3.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 1.983972 and Ymax = 4.302653
List of powers for Q(x): [0.615144, 0.852869, 1.090594, 1.328319, 1.566045, 1.803770]
List of powers for D(x): [0.615144, 0.852869, 1.090594]
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
MSS of errors squared = 2.782295e-09
Corrected MSS of errors squared = 4.819077e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    ___    _____    ______
    (Intercept)       0.977868544138532     0      Inf       0
    x1                -4.17529782925789     0     -Inf       0
    x2                 4.01750087664582     0      Inf       0
    x3                0.0915099262536494    0      Inf       0
    x4                -0.918239604677002    0     -Inf       0
    x5                0.00709916312336947   0      Inf       0
    x6               -0.000441651093649261  0     -Inf       0
    x7                 4.28558005558409     0      Inf       0
    x8                -4.17615319154117     0     -Inf       0
    x9                                0     0      NaN      NaN
    x10               0.890573423391392     0      Inf       0
Number of observations: 981, Error degrees of freedom: 971
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq             F     pValue
                ___________________    ___    ____________________    ___    ______
    Total       8.91039026622874    980    0.00909223496553953
    Model       8.91039026622874      9      0.990043362914304    Inf       0
    Residual                   0    971                      0
A = 0.198397, B = 0.207567, C = 0.035557
```



```
order Q(x) = 6.000000, order D(x) = 4.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 1.983972 and Ymax = 4.302653
List of powers for Q(x): [0.405964, 0.649089, 0.871385, 1.090253, 1.307348, 1.523309]
List of powers for D(x): [0.405964, 0.649089, 0.871385, 1.090253]
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
MSS of errors squared = 6.419237e-09
Corrected MSS of errors squared = 1.111844e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______

    (Intercept)      0.978803355169592     0      Inf        0
    x1                 -3.894559903717     0     -Inf        0
    x2                2.23782670968449     0      Inf        0
    x3                1.62011572163585     0      Inf        0
    x4              -0.0131068777017297    0     -Inf        0
    x5               -0.929081253371498    0     -Inf        0
    x6                3.98687852242737     0      Inf        0
    x7               -2.33146146675365     0     -Inf        0
    x8                -1.5960218087652     0     -Inf        0
    x9                               0     0      NaN      NaN
    x10              0.940605877243259     0      Inf        0
Number of observations: 981, Error degrees of freedom: 971
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq             F      pValue
                  _________________    ___    ____________________    ___    ______

    Total       8.91039026622882     980    0.00909223496553961
    Model       8.91039026622882       9      0.990043362914313    Inf       0
    Residual                   0     971                         0
A = 0.335818, B = 0.283906, C = 0.195185
order Q(x) = 5.000000, order D(x) = 5.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 1.983972 and Ymax = 4.302653
List of powers for Q(x): [0.619724, 0.932507, 1.217928, 1.489185, 1.751391]
List of powers for D(x): [0.619724, 0.932507, 1.217928, 1.489185, 1.751391]
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
MSS of errors squared = 3.644616e-09
Corrected MSS of errors squared = 6.312660e-09
```



```
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)      0.979442074561498      0      Inf        0
    x1               -1.97678725430393      0     -Inf        0
    x2              -0.120949862407408      0     -Inf        0
    x3                3.35481522033444      0      Inf        0
    x4               -2.23652290335164      0     -Inf        0
    x5                2.01540923195229      0      Inf        0
    x6               0.114021451550051      0      Inf        0
    x7               -3.38937060750474      0     -Inf        0
    x8                 2.2599412865399      0      Inf        0
Number of observations: 981, Error degrees of freedom: 972
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF            MeanSq            F      pValue
                  _____________    ____    ____________________    ___    _______
    Total        8.9103902662288    980    0.0090922349655396
    Model        8.9103902662288      8       1.1137987832786    Inf        0
    Residual                   0    972                     0
A = 0.177510, B = 0.407747, C = 0.672000
order Q(x) = 4.000000, order D(x) = 4.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 1.983972 and Ymax = 4.302653
List of powers for Q(x): [0.585257, 1.426151, 1.834100, 2.156942]
List of powers for D(x): [0.585257, 1.426151, 1.834100, 2.156942]
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
MSS of errors squared = 8.189139e-08
Corrected MSS of errors squared = 1.418400e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
```



```
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    ______

    (Intercept)      0.980789531848414     0      Inf        0
    x1               -3.87323482240301     0     -Inf        0
    x2                1.37530033408521     0      Inf        0
    x3               -0.436020484430349    0     -Inf        0
    x4                2.40828102329687     0      Inf        0
    x5                                0     0      NaN      NaN
    x6               -0.455116420222932    0     -Inf        0
    x7                2.88738752655732     0      Inf        0
    x8                                0     0      NaN      NaN
    x9                                0     0      NaN      NaN
    x10              -2.33536436248392     0     -Inf        0
    x11              -0.0126451002776082    0     -Inf        0
    x12               0.460622355117639    0      Inf        0
Number of observations: 981, Error degrees of freedom: 971
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq             F     pValue
                ________________    ___    ____________________    ___    ______

    Total       8.91039026622885    980    0.00909223496553964
    Model       8.91039026622885      9      0.990043362914317    Inf       0
    Residual                   0    971                      0
A = 0.396451, B = 2.811796
order Q(x) = 6.000000, order D(x) = 6.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 1.983972 and Ymax = 4.302653
List of powers for Q(x): [0.396451, 0.419541, 0.542165, 0.765890, 1.067599, 1.427407]
List of powers for D(x): [0.396451, 0.419541, 0.542165, 0.765890, 1.067599, 1.427407]
Fitting tinv(0.975,x) in range (2.000000, 100.000000)
MSS of errors squared = 3.150951e-09
Corrected MSS of errors squared = 4.456118e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the two-sided t-invserse function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*i+C*(i-1)	0.5	0.166	0.147	Inf	-Inf	8.64941700E-10	3, 7
A+B*sqrt(i)	0.704	0.971		Inf	-Inf	2.54118800E-09	4, 7
A+B*i-C*(i-1)	0.377	0.237	0	Inf	-Inf	2.78229500E-09	6, 3
A+B*log10(i)^4	0.396	2.811		Inf	-Inf	3.15095100E-09	6, 6
A+B*sqrt(i)+C*(i-1)	0.335	0.283	0.195	Inf	-Inf	3.64461600E-09	5, 5
A+B*i+C*sqrt(i-1)	0.198	0.207	0.035	Inf	-Inf	6.41923700E-09	6, 4
A+B*i	0.433	0.126		Inf	-Inf	7.12866200E-09	6, 4
A+B/i	0.452	0.37		Inf	-Inf	7.96973100E-08	6, 7
A+B*sqrt(i)+C*sqrt(i-1)	0.177	0.407	0.672	Inf	-Inf	8.18913900E-08	4, 4
A+B*sqrt(i+C)	0.882	0.806	0.343	Inf	-Inf	1.05626000E-07	4, 4

RESULTS FOR THE ONE-SIDED T-INVERSE DISTRIBUTION

Using Power A+B* i

```
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                    ___________________    __    _____    ______

    (Intercept)    0.974576938284606     0      Inf       0
    x1             -7.43868503807132     0     -Inf       0
    x2              6.46842786194451     0      Inf       0
    x3              3.37194188695372     0      Inf       0
    x4             -3.38715000170557     0     -Inf       0
    x5             0.010885472045472     0      Inf       0
    x6              7.68086295657792     0      Inf       0
    x7             -6.83415360409598     0     -Inf       0
    x8             -3.19862842811035     0     -Inf       0
    x9              3.35192051590543     0      Inf       0
Number of observations: 981, Error degrees of freedom: 971
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF            MeanSq             F      pValue
                  _____________    ___    ____________________    ___    _______

    Total        9.9434849460635    980    0.0101464132102689
    Model        9.9434849460635      9      1.10483166067372    Inf        0
    Residual                   0    971                     0
A = 0.465704, B = 0.213954
order Q(x) = 5.000000, order D(x) = 4.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 1.660234 and Ymax = 2.919986
List of powers for Q(x): [0.679657, 0.893611, 1.107565, 1.321518, 1.535472]
List of powers for D(x): [0.679657, 0.893611, 1.107565, 1.321518]
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
```



```
MSS of errors squared = 4.780903e-09
Corrected MSS of errors squared = 6.761218e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ___________________    __    _____    _______
    (Intercept)       1.00627563400678     0      Inf         0
    x1               -0.371478418348441    0     -Inf         0
    x2                2.25072288044889     0      Inf         0
    x3                               0     0      NaN       NaN
    x4               -4.32189752846789     0     -Inf         0
    x5                2.25628991893669     0      Inf         0
    x6                               0     0      NaN       NaN
    x7               -0.819913843216153    0     -Inf         0
    x8                0.376100380617057    0      Inf         0
    x9               -2.29808089716075     0     -Inf         0
    x10               0.419065165377475    0      Inf         0
    x11               2.50291602948642     0      Inf         0
Number of observations: 981, Error degrees of freedom: 971
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF           MeanSq           F     pValue
                   _________________    ___    ___________________    ___    _______
    Total       9.94348494606335     980    0.0101464132102687
    Model       9.94348494606335       9      1.10483166067371    Inf        0
    Residual                    0     971                        0
A = 0.011599, B = 1.527944
order Q(x) = 7.000000, order D(x) = 4.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 1.660234 and Ymax = 2.919986
List of powers for Q(x): [1.539543, 0.775571, 0.520914, 0.393585, 0.317188, 0.266256,
0.229877]
List of powers for D(x): [1.539543, 0.775571, 0.520914, 0.393585]
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
MSS of errors squared = 7.083979e-10
Corrected MSS of errors squared = 1.001826e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
```



```
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ________________    __    _____    ______
    (Intercept)    0.974739676151449    0      Inf        0
    x1             -14.2924079591148    0     -Inf        0
    x2              5.30323548721884    0      Inf        0
    x3              15.7193637853167    0      Inf        0
    x4                             0    0      NaN      NaN
    x5                             0    0      NaN      NaN
    x6             -7.70493321037235    0     -Inf        0
    x7              14.1388799418975    0      Inf        0
    x8                             0    0      NaN      NaN
    x9              -33.599256189038    0     -Inf        0
    x10             24.3798615500294    0      Inf        0
    x11            -15.4430984345187    0     -Inf        0
    x12             11.5236142420319    0      Inf        0
Number of observations: 981, Error degrees of freedom: 971
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _____________    ___    ____________________    ___    ______
    Total        9.94348494606388    980    0.0101464132102693
    Model        9.94348494606388      9      1.10483166067376    Inf       0
    Residual                    0    971                     0
A = 0.431861, B = 0.269496
order Q(x) = 6.000000, order D(x) = 6.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 1.660234 and Ymax = 2.919986
List of powers for Q(x): [0.701357, 0.812986, 0.898642, 0.970854, 1.034473, 1.091989]
List of powers for D(x): [0.701357, 0.812986, 0.898642, 0.970854, 1.034473, 1.091989]
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
MSS of errors squared = 4.739322e-09
Corrected MSS of errors squared = 6.702413e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)

```
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
```



```
Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                    ___________________    __    _____    _______
    (Intercept)    0.974395949716885     0       Inf        0
    x1            -55.7198889998043      0      -Inf        0
    x2             152.421518384023      0       Inf        0
    x3            -140.712869283094      0      -Inf        0
    x4             42.9043324073227      0       Inf        0
    x5            0.132448505593002      0       Inf        0
    x6             56.8505280521821      0       Inf        0
    x7            -155.516905072227      0      -Inf        0
    x8             143.772505763573      0       Inf        0
    x9            -44.106097225399       0      -Inf        0
Number of observations: 99, Error degrees of freedom: 89
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F      pValue
                  _____________    ___    ____________________    ___    _______
    Total        1.51844154817144    98    0.0154943015119535
    Model        1.51844154817144     9     0.168715727574605    Inf        0
    Residual                    0    89                     0
A = 0.825544, B = 0.924806, C = 0.484508
order Q(x) = 5.000000, order D(x) = 4.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 1.660234 and Ymax = 2.919986
List of powers for Q(x): [1.952331, 2.283253, 2.551864, 2.783973, 2.991349]
List of powers for D(x): [1.952331, 2.283253, 2.551864, 2.783973]
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
MSS of errors squared = 1.920920e-08
Corrected MSS of errors squared = 3.327132e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                       Estimate          SE     tStat     pValue
                    ___________________    __    _____    _______
    (Intercept)    0.974752214689147     0       Inf        0
    x1            -5.64943472851226      0      -Inf        0
    x2             5.90836620072973      0       Inf        0
    x3                           0       0       NaN      NaN
    x4            -1.23368451581118      0      -Inf        0
    x5             5.80199496157891      0       Inf        0
```



```
    x6                -6.06161083476014    0      -Inf         0
    x7                -0.080560503468396   0      -Inf         0
    x8                  1.43638215960171   0       Inf         0
    x9                -0.136202836845693   0      -Inf         0
    x10               0.0462473722186955   0       Inf         0
    x11              -0.00624990387253868  0      -Inf         0
Number of observations: 981, Error degrees of freedom: 970
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF          MeanSq            F      pValue
                 _________________    ___    _________________    ___    _______
    Total        9.94348494606367     980    0.010146413210269
    Model        9.94348494606367      10    0.994348494606367    Inf        0
    Residual                    0     970                    0
A = 0.339082, B = 0.101213, C = 0.093179
order Q(x) = 4.000000, order D(x) = 7.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 1.660234 and Ymax = 2.919986
List of powers for Q(x): [0.440295, 0.634686, 0.829078, 1.023470]
List of powers for D(x): [0.440295, 0.634686, 0.829078, 1.023470, 1.217862, 1.412254,
1.606646]
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
MSS of errors squared = 1.523131e-09
Corrected MSS of errors squared = 2.638140e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    _________________    ___    _____    _______
    (Intercept)      0.965849048677565    0       Inf        0
    x1               -2.72777666786912    0      -Inf        0
    x2               -0.274933512893827   0      -Inf        0
    x3                 3.01544139804846   0       Inf        0
    x4                                0   0       NaN       NaN
    x5               -0.978581354480028   0      -Inf        0
    x6                 2.91916019229767   0       Inf        0
    x7                                0   0       NaN       NaN
    x8               -2.85202100080057    0      -Inf        0
    x9               -0.0655894227142757  0      -Inf        0
    x10               0.998450775482391   0       Inf        0
Number of observations: 981, Error degrees of freedom: 972
```



```
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ___________________    ___    ____________________    ___    ______
    Total       9.94348494606349    980    0.0101464132102689
    Model       9.94348494606349      8      1.24293561825794    Inf       0
    Residual                    0    972                     0
A = 0.061055, B = 0.178526, C = 0.052932
order Q(x) = 5.000000, order D(x) = 5.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 1.660234 and Ymax = 2.919986
List of powers for Q(x): [0.239581, 0.365176, 0.490770, 0.616364, 0.741959]
List of powers for D(x): [0.239581, 0.365176, 0.490770, 0.616364, 0.741959]
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
MSS of errors squared = 3.218904e-08
Corrected MSS of errors squared = 5.575305e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)       0.975429282069126     0      Inf       0
    x1                 -2.0028120905218     0     -Inf       0
    x2                  1.3030289323208     0      Inf       0
    x3                0.0181730345940597    0      Inf       0
    x4                -0.295878298402934    0     -Inf       0
    x5               0.00233447815406517    0      Inf       0
    x6              -0.000276393415422203   0     -Inf       0
    x7                   2.05339948386544    0      Inf       0
    x8                  -1.34421609752702    0     -Inf       0
    x9                                  0    0      NaN      NaN
    x10                0.290817141261348    0      Inf       0
Number of observations: 981, Error degrees of freedom: 971
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ___________________    ___    ____________________    ___    ______
    Total       9.94348494606361    980    0.0101464132102690
    Model       9.94348494606361      9      1.10483166067373    Inf       0
    Residual                    0    971                     0
A = 0.024946, B = 0.305065, C = 0.221304
```



```
order Q(x) = 6.000000, order D(x) = 4.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 1.660234 and Ymax = 2.919986
List of powers for Q(x): [0.330011, 0.856380, 1.253113, 1.628516, 1.992880, 2.350188]
List of powers for D(x): [0.330011, 0.856380, 1.253113, 1.628516]
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
MSS of errors squared = 6.211117e-09
Corrected MSS of errors squared = 1.075797e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)       0.975845682857271     0      Inf       0
    x1                -2.07941377413434     0     -Inf       0
    x2                0.699234432825696     0      Inf       0
    x3                0.405716860625232     0      Inf       0
    x4               -0.00121584460223002   0     -Inf       0
    x5                 2.12863943688855     0      Inf       0
    x6                -0.725168513685155    0     -Inf       0
    x7                -0.403554605430858    0     -Inf       0
Number of observations: 981, Error degrees of freedom: 973
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _____________    ____    ____________________    ___    ______
    Total       9.94348494606347    980    0.0101464132102689
    Model       9.94348494606347      7      1.42049784943764    Inf       0
    Residual                    0    973                     0
A = 0.218212, B = 0.366481, C = 0.245992
order Q(x) = 4.000000, order D(x) = 3.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 1.660234 and Ymax = 2.919986
List of powers for Q(x): [0.584693, 0.982487, 1.344961, 1.689152]
List of powers for D(x): [0.584693, 0.982487, 1.344961]
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
MSS of errors squared = 1.033988e-07
Corrected MSS of errors squared = 1.790919e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
```



```
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    _____________________    __    _____    ______
    (Intercept)       0.975192149559897    0      Inf        0
    x1                -2.89070596566229    0     -Inf        0
    x2                 2.70149417576639    0      Inf        0
    x3                                0    0      NaN      NaN
    x4               -0.107112673803069    0     -Inf        0
    x5               -0.670334630916117    0     -Inf        0
    x6              -0.00858395826145142    0     -Inf        0
    x7              4.77690098687768e-05    0      Inf        0
    x8                 2.96553965841986    0      Inf        0
    x9                -2.82190592287339    0     -Inf        0
    x10                0.130194022488719    0      Inf        0
    x11                               0    0      NaN      NaN
    x12                0.726173809119057    0      Inf        0
Number of observations: 981, Error degrees of freedom: 970
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______
    Total       9.94348494606347    980    0.0101464132102689
    Model       9.94348494606347     10     0.994348494606347    Inf       0
    Residual                   0    970                      0
A = 0.238966, B = 0.314009, C = 0.419660
order Q(x) = 7.000000, order D(x) = 5.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 1.660234 and Ymax = 2.919986
List of powers for Q(x): [0.552975, 1.102702, 1.376335, 1.593857, 1.780432, 1.946517,
2.097710]
List of powers for D(x): [0.552975, 1.102702, 1.376335, 1.593857, 1.780432]
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
MSS of errors squared = 1.427416e-09
Corrected MSS of errors squared = 2.472357e-09
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
```



```
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      0.974681575493204      0      Inf        0
    x1               -47.2096718412061      0     -Inf        0
    x2               -7.89204619017152      0     -Inf        0
    x3                82.5438843408632      0      Inf        0
    x4               -28.4196333653297      0     -Inf        0
    x5              0.00273281570544083     0      Inf        0
    x6                54.4805403027177      0      Inf        0
    x7                               0      0      NaN      NaN
    x8               -82.1239081148017      0     -Inf        0
    x9                28.6433941447606      0      Inf        0
Number of observations: 981, Error degrees of freedom: 972
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq           F     pValue
                  _________________    ___    ____________________    ___    ______
    Total       9.94348494606288     980    0.0101464132102682
    Model       9.94348494606288       8      1.24293561825786    Inf       0
    Residual                   0     972                     0
A = 1.438150, B = 2.913588
order Q(x) = 5.000000, order D(x) = 4.000000
Xmin = 2.000000 and Xmax = 100.000000
Ymin = 1.660234 and Ymax = 2.919986
List of powers for Q(x): [1.438150, 1.462075, 1.589138, 1.820963, 2.133594]
List of powers for D(x): [1.438150, 1.462075, 1.589138, 1.820963]
Fitting tinv(0.95,x) in range (2.000000, 100.000000)
MSS of errors squared = 3.491829e-08
Corrected MSS of errors squared = 4.938192e-08
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the one-sided t-invserse function. The results are sorted in ascending order using the MSSE values.

Power	*A*	*B*	*C*	*F*	*AICc*	*MSSE*	*Q/D Orders*
A+B/i	0.011	1.527		Inf	-Inf	7.08397900E-10	7, 4
A+B*sqrt(i)+C*sqrt(i-1)	0.238	0.314	0.419	Inf	-Inf	1.42741600E-09	7, 5
A+B*i+C*(i-1)	0.339	0.101	0.093	Inf	-Inf	1.52313100E-09	4, 7
A+B*sqrt(i)	0.431	0.269		Inf	-Inf	4.73932200E-09	6, 6
A+B*i	0.465	0.213		Inf	-Inf	4.78090300E-09	5, 4
A+B*i+C*sqrt(i-1)	0.024	0.305	0.221	Inf	-Inf	6.21111700E-09	6, 4
A+B*sqrt(i+C)	0.825	0.924	0.484	Inf	-Inf	1.92092000E-08	5, 4
A+B*i-C*(i-1)	0.061	0.178	0.052	Inf	-Inf	3.21890400E-08	5, 5
A+B*log10(i)^4	1.438	2.913		Inf	-Inf	3.49182900E-08	5, 4
A+B*sqrt(i)+C*(i-1)	0.218	0.366	0.245	Inf	-Inf	1.03398800E-07	4, 3

RESULTS FOR THE TRIGAMMA FUNCTION

Using Power A+B* i

```
Fitting tigamma(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate           SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)        0.988530559919482    0      Inf       0
    x1                 -6.96659932172144    0     -Inf       0
    x2                  14.3100611039866    0      Inf       0
    x3                 -13.1386208245914    0     -Inf       0
    x4                  5.81748925613752    0      Inf       0
    x5                 -1.01022222836933    0     -Inf       0
    x6              -0.000638003827680559    0     -Inf       0
    x7                  7.0417780630343     0      Inf       0
    x8                 -14.4585359135273    0     -Inf       0
    x9                  13.2720268279964    0      Inf       0
    x10                -5.87768417971282    0     -Inf       0
    x11                 1.02241493139378    0      Inf       0
Number of observations: 991, Error degrees of freedom: 979
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq              F      pValue
                 _____________      ___    ____________________     ___    _______
    Total        5.90929128627541    990    0.00596898109724789
    Model        5.90929128627541     11       0.53720829875231    Inf        0
    Residual                    0    979                      0
A = 0.214240, B = 0.468041
order Q(x) = 6.000000, order D(x) = 5.000000
Xmin = 1.000000 and Xmax = 100.000000
Ymin = 0.010050 and Ymax = 1.644935
```



```
List of powers for Q(x): [0.682281, 1.150322, 1.618363, 2.086404, 2.554445, 3.022486]
List of powers for D(x): [0.682281, 1.150322, 1.618363, 2.086404, 2.554445]
Fitting tigamma(x) in range (1.000000, 100.000000)
MSS of errors squared = 5.333991e-07
Corrected MSS of errors squared = 7.543403e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B/i

```
Fitting trigamma(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B/i
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9
Estimated Coefficients:
                       Estimate         SE    tStat    pValue
                    ________________    __    _____    ______
    (Intercept)    0.987176150003022    0      Inf       0
    x1              6.96633671993163    0      Inf       0
    x2             -124.761700754922    0     -Inf       0
    x3              328.409542204392    0      Inf       0
    x4             -214.532030680731    0     -Inf       0
    x5              2.93067720899878    0      Inf       0
    x6             -7.00790912066826    0     -Inf       0
    x7              125.265441491015    0      Inf       0
    x8             -327.983452211156    0     -Inf       0
    x9              210.725919416023    0      Inf       0
Number of observations: 991, Error degrees of freedom: 981
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _______________    ___    ____________________    ___    ______
    Total        5.90929128626777    990    0.00596898109724017
    Model        5.90929128626777      9      0.656587920696419    Inf       0
    Residual                    0    981                      0
A = 1.287525, B = 1.260495
order Q(x) = 5.000000, order D(x) = 4.000000
Xmin = 1.000000 and Xmax = 100.000000
Ymin = 0.010050 and Ymax = 1.644935
List of powers for Q(x): [2.548020, 1.917773, 1.707690, 1.602649, 1.539624]
List of powers for D(x): [2.548020, 1.917773, 1.707690, 1.602649]
Fitting trigamma(x) in range (1.000000, 100.000000)
MSS of errors squared = 6.561190e-06
Corrected MSS of errors squared = 9.278923e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
```



```
AICc =  -Inf
```

Using Power A+B*sqrt(i)

```
Fitting trigamma(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12 + x13 + x14
Estimated Coefficients:
                        Estimate         SE    tStat    pValue
                    ________________    ___    _____    ______
    (Intercept)     0.987558777501172    0      Inf        0
    x1              -22.7302037366899    0     -Inf        0
    x2               69.4419246647453    0      Inf        0
    x3              -65.6978423396493    0     -Inf        0
    x4              -1.08433960338455    0     -Inf        0
    x5               24.2640977446109    0      Inf        0
    x6              -0.212165050771357    0     -Inf        0
    x7              -4.96903047857251    0     -Inf        0
    x8               23.0190961657927    0      Inf        0
    x9              -70.4651240266913    0     -Inf        0
    x10              67.1832214171652    0      Inf        0
    x11                             0    0      NaN      NaN
    x12             -23.7672229884678    0     -Inf        0
    x13                             0    0      NaN      NaN
    x14              5.03002944329855    0      Inf        0
Number of observations: 991, Error degrees of freedom: 978
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq            F     pValue
                  _________________    ___    ____________________    ___    ______
    Total        5.90929128627045     990    0.00596898109724287
    Model        5.90929128627045      12      0.492440940522537    Inf       0
    Residual                    0     978                      0
A = 0.105809, B = 1.000000
order Q(x) = 7.000000, order D(x) = 7.000000
Xmin = 1.000000 and Xmax = 100.000000
Ymin = 0.010050 and Ymax = 1.644935
List of powers for Q(x): [1.105809, 1.520023, 1.837860, 2.105809, 2.341877, 2.555299,
2.751560]
List of powers for D(x): [1.105809, 1.520023, 1.837860, 2.105809, 2.341877, 2.555299,
2.751560]
Fitting trigamma(x) in range (1.000000, 100.000000)
MSS of errors squared = 1.481588e-06
Corrected MSS of errors squared = 2.095281e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i+C)


```
Fitting trigamma(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i+C)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    _____________________    __    _____    ______
    (Intercept)       0.987425305712803     0      Inf       0
    x1                -10.8320082026609     0     -Inf       0
    x2                 17.9076202747772     0      Inf       0
    x3                -8.05602407229627     0     -Inf       0
    x4               -0.0069503079663157    0     -Inf       0
    x5                 10.9351051463023     0      Inf       0
    x6                -18.0718725113184     0     -Inf       0
    x7                 8.13673586597451     0      Inf       0
Number of observations: 100, Error degrees of freedom: 92
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                      SumSq          DF          MeanSq           F     pValue
                    _____________________    __    _____________________    ___    ______
    Total       1.20244531658334     99    0.0121459122887206
    Model       1.20244531658334      7     0.171777902369049    Inf       0
    Residual                     0    92                     0
A = 0.521337, B = 1.000000, C = 0.000000
order Q(x) = 4.000000, order D(x) = 3.000000
Xmin = 1.000000 and Xmax = 100.000000
Ymin = 0.010050 and Ymax = 1.644935
List of powers for Q(x): [1.521337, 1.935550, 2.253388, 2.521337]
List of powers for D(x): [1.521337, 1.935550, 2.253388]
Fitting trigamma(x) in range (1.000000, 100.000000)
MSS of errors squared = 4.728030e-07
Corrected MSS of errors squared = 8.189188e-07
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*(i–1)

```
Fitting trigamma(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11 + x12
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    _____________________    __    _____    ______
    (Intercept)       0.987686344612683     0      Inf       0
    x1                -4.00520483421105     0     -Inf       0
    x2                 4.76120093817875     0      Inf       0
    x3                                0     0      NaN      NaN
```



```
    x4              -3.93921839874502    0      -Inf         0
    x5               2.82402791286125    0       Inf         0
    x6              -0.628491942007977   0      -Inf         0
    x7               4.05541105921332    0       Inf         0
    x8              -4.82853172907624    0      -Inf         0
    x9              0.0259547174697319   0       Inf         0
    x10              3.95518483485184    0       Inf         0
    x11             -2.84028646759446    0      -Inf         0
    x12              0.632267574787581   0       Inf         0
Number of observations: 991, Error degrees of freedom: 979
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ________________    ___    ____________________    ___    ______
    Total       5.90929128627477    990    0.00596898109724724
    Model       5.90929128627477     11      0.537208298752252    Inf       0
    Residual                   0    979                      0
A = 0.235559, B = 0.268394, C = 0.185262
order Q(x) = 6.000000, order D(x) = 6.000000
Xmin = 1.000000 and Xmax = 100.000000
Ymin = 0.010050 and Ymax = 1.644935
List of powers for Q(x): [0.503953, 0.957608, 1.411263, 1.864918, 2.318574, 2.772229]
List of powers for D(x): [0.503953, 0.957608, 1.411263, 1.864918, 2.318574, 2.772229]
Fitting trigamma(x) in range (1.000000, 100.000000)
MSS of errors squared = 2.166712e-06
Corrected MSS of errors squared = 3.752856e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i–C*(i–1)

```
Fitting trigamma(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i-C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8
Estimated Coefficients:
                       Estimate          SE    tStat    pValue
                   _________________    ___    _____    ______
    (Intercept)    0.985316236824104    0       Inf        0
    x1              -75.285647256925    0      -Inf        0
    x2              124.796785856731    0       Inf        0
    x3             -9.47091773333107    0      -Inf        0
    x4             -41.0255363519996    0      -Inf        0
    x5              73.1648147144116    0       Inf        0
    x6             -116.985599216624    0      -Inf        0
    x7                             0    0       NaN      NaN
    x8              44.8207841264166    0       Inf        0
Number of observations: 991, Error degrees of freedom: 983
```



```
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ___________________    ___    ____________________    ___    ______
    Total       5.90929128627444    990    0.0059689810972469
    Model       5.90929128627444      7     0.844184469467777    Inf       0
    Residual                   0    983                     0
A = 0.226650, B = 0.218870, C = 0.244367
order Q(x) = 4.000000, order D(x) = 4.000000
Xmin = 1.000000 and Xmax = 100.000000
Ymin = 0.010050 and Ymax = 1.644935
List of powers for Q(x): [0.445520, 0.420022, 0.394525, 0.369027]
List of powers for D(x): [0.445520, 0.420022, 0.394525, 0.369027]
Fitting trigamma(x) in range (1.000000, 100.000000)
MSS of errors squared = 9.236342e-07
Corrected MSS of errors squared = 1.599781e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*i+C*sqrt(i–1)

```
Fitting trigamma(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*i+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                        Estimate          SE     tStat     pValue
                    ____________________    __    _____    _______
    (Intercept)         0.9885117621761     0      Inf       0
    x1                 -3.90266903576285    0     -Inf       0
    x2                  8.17386674223695    0      Inf       0
    x3                 -7.52955991483448    0     -Inf       0
    x4                  2.26985096471752    0      Inf       0
    x5                  3.94503797198475    0      Inf       0
    x6                 -8.25350914785322    0     -Inf       0
    x7                  7.58595231952848    0      Inf       0
    x8                 -2.25684715448098    0     -Inf       0
    x9                -0.0332793039098402    0     -Inf       0
    x10                0.0152360727218907    0      Inf       0
    x11               -0.00259101733890077    0     -Inf       0
Number of observations: 991, Error degrees of freedom: 979
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                    SumSq          DF           MeanSq            F      pValue
                ___________________    ___    ____________________    ___    ______
    Total       5.90929128627605    990    0.00596898109724853
    Model       5.90929128627605     11      0.537208298752368    Inf       0
    Residual                   0    979                     0
```



```
A = 0.319430, B = 0.196069, C = 0.360000
order Q(x) = 4.000000, order D(x) = 7.000000
Xmin = 1.000000 and Xmax = 100.000000
Ymin = 0.010050 and Ymax = 1.644935
List of powers for Q(x): [0.515499, 1.071568, 1.416754, 1.727245]
List of powers for D(x): [0.515499, 1.071568, 1.416754, 1.727245, 2.019776, 2.300829,
2.573730]
Fitting trigamma(x) in range (1.000000, 100.000000)
MSS of errors squared = 1.230221e-06
Corrected MSS of errors squared = 2.130805e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*(i–1)

```
Fitting trigamma(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10
Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                      ____________________    __    _____    ______
    (Intercept)        0.988149883335406      0      Inf       0
    x1                -5.14059461807912       0     -Inf       0
    x2                 8.48775852339281       0      Inf       0
    x3                -5.73199227193116       0     -Inf       0
    x4                 1.39375312575659       0      Inf       0
    x5                0.00329458980308936     0      Inf       0
    x6               -0.000368013768730772    0     -Inf       0
    x7                 5.19796598825886       0      Inf       0
    x8                -8.58040163046018       0     -Inf       0
    x9                 5.79792804888304       0      Inf       0
    x10               -1.41549301615136       0     -Inf       0
Number of observations: 991, Error degrees of freedom: 980
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF           MeanSq             F     pValue
                  _______________    ___    ____________________    ___    ______
    Total        5.90929128627611    990    0.00596898109724859
    Model        5.90929128627611     10      0.590929128627611    Inf       0
    Residual                    0    980                      0
A = 0.407904, B = 0.294561, C = 0.397217
order Q(x) = 6.000000, order D(x) = 4.000000
Xmin = 1.000000 and Xmax = 100.000000
Ymin = 0.010050 and Ymax = 1.644935
List of powers for Q(x): [0.702465, 1.221693, 1.712533, 2.188678, 2.655432, 3.115515]
List of powers for D(x): [0.702465, 1.221693, 1.712533, 2.188678]
Fitting trigamma(x) in range (1.000000, 100.000000)
```



```
MSS of errors squared = 8.811397e-07
Corrected MSS of errors squared = 1.526179e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*sqrt(i)+C*sqrt(i–1)

```
Fitting trigamma(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*sqrt(i)+C*sqrt(i-1)
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8
Estimated Coefficients:
                        Estimate          SE    tStat    pValue
                    ____________________    __    _____    ______
    (Intercept)      0.987718909374131    0      Inf       0
    x1               -7.06313314756173    0     -Inf       0
    x2                10.8470736788237    0      Inf       0
    x3               -4.72274875618448    0     -Inf       0
    x4              -0.0613311535465363    0     -Inf       0
    x5               0.0124284863565541    0      Inf       0
    x6                 7.15630804184149    0      Inf       0
    x7               -11.0286040920423    0     -Inf       0
    x8                4.87229203962178    0      Inf       0
Number of observations: 991, Error degrees of freedom: 982
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                     SumSq          DF            MeanSq             F     pValue
                  _________________    ___    ____________________    ___    ______
    Total       5.90929128627485     990    0.00596898109724732
    Model       5.90929128627485       8      0.738661410784356    Inf       0
    Residual                    0     982                         0
A = 0.000000, B = 0.689142, C = 0.000000
order Q(x) = 5.000000, order D(x) = 3.000000
Xmin = 1.000000 and Xmax = 100.000000
Ymin = 0.010050 and Ymax = 1.644935
List of powers for Q(x): [0.689142, 0.974595, 1.193630, 1.378285, 1.540969]
List of powers for D(x): [0.689142, 0.974595, 1.193630]
Fitting trigamma(x) in range (1.000000, 100.000000)
MSS of errors squared = 1.134882e-06
Corrected MSS of errors squared = 1.965673e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Using Power A+B*log10(i)^4

```
Fitting trigamma(x) in range (1.000000, 100.000000)
Pade-Shammas polynomial power is A+B*log10(i)^4
```



```
Linear regression model:
    y ~ 1 + x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 + x9 + x10 + x11
Estimated Coefficients:
                          Estimate           SE    tStat    pValue
                      ____________________    __    _____    ______
    (Intercept)          0.9870890056812      0      Inf        0
    x1                  -62.6395828216557     0     -Inf        0
    x2                  -8.96044046679891     0     -Inf        0
    x3                   128.459829134651     0      Inf        0
    x4                  -70.8017995058409     0     -Inf        0
    x5                   12.9549055013056     0      Inf        0
    x6                -2.09586080387331e-07   0     -Inf        0
    x7                   70.2839601744122     0      Inf        0
    x8                                  0     0      NaN      NaN
    x9                  -127.210558061799     0     -Inf        0
    x10                  70.9420700531012     0      Inf        0
    x11                 -13.0154724847337     0     -Inf        0
Number of observations: 991, Error degrees of freedom: 980
R-squared: 1,  Adjusted R-Squared: 1
F-statistic vs. constant model: Inf, p-value = 0
                       SumSq          DF           MeanSq            F     pValue
                    _____________    ___    ____________________    ___    ______
    Total        5.90929128626666    990    0.00596898109723905
    Model        5.90929128626666     10      0.590929128626666    Inf       0
    Residual                    0    980                      0
A = 0.992613, B = 2.626351
order Q(x) = 6.000000, order D(x) = 5.000000
Xmin = 1.000000 and Xmax = 100.000000
Ymin = 0.010050 and Ymax = 1.644935
List of powers for Q(x): [0.992613, 1.014180, 1.128716, 1.337687, 1.619497, 1.955575]
List of powers for D(x): [0.992613, 1.014180, 1.128716, 1.337687, 1.619497]
Fitting trigamma(x) in range (1.000000, 100.000000)
MSS of errors squared = 1.057468e-06
Corrected MSS of errors squared = 1.495486e-06
R-Squared =   1.00000000
R-Squared Adjusted =   1.00000000
Particle swarm AICc =  -1.000000e+99
AIC =  -Inf
AICc =  -Inf
```

Comments

The following table shows the summary results for the ten models that fit the trigamma function. The results are sorted in ascending order using the MSSE values.

Power	A	B	C	F	AICc	MSSE	Q/D Orders
A+B*sqrt(i+C)	0.521	1	0	Inf	-Inf	4.72803000E-07	4, 3
A+B*i	0.214	0.468		Inf	-Inf	5.33399100E-07	6, 5
A+B*sqrt(i)+C*(i-1)	0.407	0.294	0.397	Inf	-Inf	8.81139700E-07	6, 4
A+B*i-C*(i-1)	0.226	0.218	0.244	Inf	-Inf	9.23634200E-07	4, 4
A+B*log10(i)^4	0.992	2.626		Inf	-Inf	1.05746800E-06	6, 5
A+B*sqrt(i)+C*sqrt(i-1)	0	0.689	0	Inf	-Inf	1.13488200E-06	5, 3
A+B*i+C*sqrt(i-1)	0.319	0.196	0.36	Inf	-Inf	1.23022100E-06	4, 7
A+B*sqrt(i)	0.105	1		Inf	-Inf	1.48158800E-06	7, 7
A+B*i+C*(i-1)	0.235	0.268	0.185	Inf	-Inf	2.16671200E-06	6, 6
A+B/i	1.287	1.26		Inf	-Inf	6.56119000E-06	5, 4

CONCLUSIONS

Overall, the Pade-Shammas polynomials did well in fitting many tested equations. The Pade-Shammas polynomials struggled in fitting the following equations:

- The cosine integral.
- The Bessel functions.
- The Fresnel cosine and sine integrals.

So how do the various Shammas Polynomial powers compare with each other. The next table shows the frequency of these powers in attaining the top 3 spots. The counts colored in red indicate the top of their placement.

Power	Top 1	Top 2	Top 3
A+B*i	0	1	1
A+B/i	4	6	6
A+B*sqrt(i)	1	3	0
A+B*sqrt(i+C)	1	6	4
A+B*i+C*(i–1)	5	0	3
A+B*i–C*(i–1)	9	2	6
A+B*i+C*sqrt(i–1)	1	1	3
A+B*sqrt(i)+C*(i–1)	4	3	2
A+B*sqrt(i)+C*sqrt(i–1)	4	4	5
A+B*log10(i)^4	3	6	2

The top Shammas polynomial power is A+B*i–C*(i–1). The second ranked top powers are A+B/I, A+B*sqrt(i+C), and A+B*log10(i)^4. The third ranked top powers are A+B*i–C*(i–1) and A+B/i.

Overall, the power A+B*i–C*(i–1) is the top ranked followed by A+B/i.

NEXT

The next study I recommend for you to look at is Part IB. In Part IB the Pade-Shammas polynomials perform curve fitting on the same set of functions, but without normalizing the x and y data. Thus, Part IB allows us to compare the effect of normalizing the data on the results of model fitting. This comparison poses the question “Does bypassing data normalization have a hefty cost on rouding errors and results?”

DOCUMENT HISTORY

Date	*Version*	*Comments*
10/4/2020	1.00.00	Initial release.

