# Half-Function Value Quadratic Integration 

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## Introduction

Numerical analysis offers a wealth of algorithms to perform numerical integration. Among the simplest algorithms are the trapezoidal integration and Simpson's Rule. The latter has two versions, namely, the $1 / 3$ rule and the $3 / 8$ rule. The firrst rule is based on a quadratic polynomial that integrates a function $f(x)$ from a to $b$. The basic Simpson's Rule formula is:
$\int_{a}^{b} f(x) d x=\frac{(b-a)}{3} *\left(f(a)+4 * f\left(\frac{a+b}{2}\right)+f(b)\right)$
Equation 1 samples the function $f(x)$ at $a$, $b$, and the midpoint between them. Simpson's Rule is very simple to implement and yields reasonably good answers. The competition for this algorithm includes Romberg's method and the GaussLegendre quadrature that perform numerical integration for finite limits.

This paper asks the question, "What is a numerical integration algorithm that implements an integral by replacing $\mathrm{f}((\mathrm{a}+\mathrm{b}) / 2)$ with $\mathrm{f}(\mathrm{m})=(\mathrm{f}(\mathrm{a})+\mathrm{f}(\mathrm{b})) / 2$. In other words, the new method locates a value of $x=m$ such that the function value at $m$ is the average between $f(a)$ and $f(b)$. The logic behind this integration algorithm is to integrate an interpolated quadratic polynomial based on $\mathrm{a}, \mathrm{m}$, and b . I will call this new algorithm the Half-Function Value Quadratic Integration, or HFVQI algorithm for short.

## The HFVQI Algorithm

Let's derive the equations for the HFVQI algorithm. The first step is to calculate c as the midpoint between the integral limits $a$ and $b$ :
$\mathrm{c}=(\mathrm{a}+\mathrm{b}) / 2$
We then calculate $f \odot$. Next, we calculate $m$, such that $f(m)=(f(a)+f(b)) / 2$, we perform an inverse Lagrangian interpolation using ( $\mathrm{a}, \mathrm{f}(\mathrm{a})$, (c, $\mathrm{f}(\mathrm{c})$, and $(\mathrm{b}, \mathrm{f}(\mathrm{b}))$ :

$$
\begin{align*}
\mathrm{m}= & \mathrm{a} *[(\mathrm{f}(\mathrm{~m})-\mathrm{f}(\mathrm{c})) *(\mathrm{f}(\mathrm{~m})-\mathrm{f}(\mathrm{~b}))] /[(\mathrm{f}(\mathrm{a})-\mathrm{f}(\mathrm{c})) *(\mathrm{f}(\mathrm{a})-\mathrm{f}(\mathrm{~b}))]+ \\
& \mathrm{c} *[(\mathrm{f}(\mathrm{~m})-\mathrm{f}(\mathrm{a})) *(\mathrm{f}(\mathrm{~m})-\mathrm{f}(\mathrm{~b}))] /[\mathrm{f}(\mathrm{c})-\mathrm{f}(\mathrm{a})) \mathrm{v} \mathrm{f}(\mathrm{c})-\mathrm{f}(\mathrm{~b}))]+ \\
& \mathrm{b} *[(\mathrm{f}(\mathrm{~m})-\mathrm{f}(\mathrm{a})) *(\mathrm{f}(\mathrm{~m})-\mathrm{f}(\mathrm{~b}))] /[(\mathrm{f}(\mathrm{~b})-\mathrm{f}(\mathrm{a})) \mathrm{v}(\mathrm{f}(\mathrm{~b})-\mathrm{f}(\mathrm{c}))] \tag{3}
\end{align*}
$$

After calculating $m$, we calculate the actual value of $f(m)$ and proceed to obtain the quadratic polynomial that we will use for the integration. Again, using Lagrangian interpolation we get:

$$
\begin{align*}
& \mathrm{P}(\mathrm{x})= \mathrm{f}(\mathrm{a}) *(\mathrm{x}-\mathrm{m}) *(\mathrm{x}-\mathrm{b}) /[(\mathrm{a}-\mathrm{m}) *(\mathrm{a}-\mathrm{b})]+ \\
& \mathrm{f}(\mathrm{~m}) *(\mathrm{x}-\mathrm{a}) *(\mathrm{x}-\mathrm{b}) /[(\mathrm{m}-\mathrm{a}) *(\mathrm{~m}-\mathrm{b})]+ \\
& \mathrm{f}(\mathrm{~b}) *(\mathrm{x}-\mathrm{a}) *(\mathrm{x}-\mathrm{m}) /[(\mathrm{b}-\mathrm{a}) *(\mathrm{~b}-\mathrm{m})] \tag{4}
\end{align*}
$$

We define the following factors:

$$
\begin{align*}
& \mathrm{Ca}=\mathrm{f}(\mathrm{a}) /\left[(\mathrm{a}-\mathrm{m})^{*}(\mathrm{a}-\mathrm{b})\right]  \tag{5a}\\
& \mathrm{Cm}=\mathrm{f}(\mathrm{~m}) /\left[(\mathrm{m}-\mathrm{a})^{*}(\mathrm{~m}-\mathrm{b})\right]  \tag{5b}\\
& \mathrm{Cb}=\mathrm{f}(\mathrm{~b}) /\left[(\mathrm{b}-\mathrm{a})^{*}(\mathrm{~b}-\mathrm{m})\right] \tag{5c}
\end{align*}
$$

And,

$$
\begin{align*}
& \mathrm{K} 1=(\mathrm{Ca}+\mathrm{Cb}+\mathrm{Cb})  \tag{6a}\\
& \mathrm{K} 2=(\mathrm{Ca} *(\mathrm{~m}+\mathrm{b})+\mathrm{Cm} *(\mathrm{a}+\mathrm{b})+\mathrm{Cb} *(\mathrm{~m}+\mathrm{a}))  \tag{6b}\\
& \mathrm{K} 3=\left(\mathrm{Ca} * \mathrm{mb}+\mathrm{Cm} * \mathrm{ab}+\mathrm{Cb} * \mathrm{~m}^{*} \mathrm{a}\right) \tag{6c}
\end{align*}
$$

Using equations $5 \mathrm{a}, 5 \mathrm{~b}, 5 \mathrm{c}, 6 \mathrm{a}, 6 \mathrm{~b}$, and 6 c , we transform equation 1 into a simpler form:
$P(x)=K 1^{*} x^{\wedge} 2-K 2 * x+K 3$
Integrating equation 7 yields the following equation:
$\int f(x) d x=\mathrm{K} 1 * \mathrm{X}^{\wedge} 3 / 3-\mathrm{K} 2 * \mathrm{x}^{\wedge} 2 / 2+\mathrm{K} 3 * \mathrm{x}+$ Konst
Applying the integral limits ( $a, b$ ) to equation 8 , we get

$$
\begin{align*}
\int_{a}^{b} f(x) d x & =\mathrm{K} 1 *\left(\mathrm{~b}^{3}-\mathrm{a}^{3}\right) / 3-\mathrm{K} 2 *\left(\mathrm{~b}^{2}-\mathrm{a}^{2}\right) / 2+\mathrm{K} 3 *(\mathrm{~b}-\mathrm{a}) \\
& =(\mathrm{b}-\mathrm{a})^{*}\left[\mathrm{~K} 1 / 3^{*}\left(\mathrm{~b}^{2}+\mathrm{a}^{*} \mathrm{~b}+\mathrm{a}^{2}\right)-\mathrm{K} 2 / 2 *(\mathrm{~b}+\mathrm{a})+\mathrm{K} 3\right] \tag{9}
\end{align*}
$$

Equation 9 is the formula we use to perform numerical integration for the HFVQI algorithm. For the sake of increased accuracy, we can apply both Simpson's Rule and equation 9 to subintervals of (a, b). In fact, we can use subintervals with any
numerical integration method with defined limits. In the case of Simpson's Rule, we call the interval division method the Composite Simpson's Rule.

## The Excel VBA Code for the Integration Algorithms

Listing 1 shows a VBA function that implements Simpson's Rule.

```
Function SimpsonArea(ByVal A As Double, ByVal B As Double, ByVal
N As Long)
    Dim Sum As Double
    Dim h As Double, I As Long, k As Long
    If (N Mod 2) = 0 Then N = N + 1
    h = (B - A) / N
    Sum = f(A) + f(B)
    k = 4
    For I = 1 To N - 1
        Sum = Sum + k * f(A + I * h)
        k = 6-k
    Next
    SimpsonArea = Sum * h / 3
End Function
```

Listing 1. The Simpson's Rule function.

```
Function HVFQI_Area(ByVal A As Double, ByVal B As Double, ByVal
N As Long)
    Dim Sum As Double
    Dim Fa As Double, Fb As Double, Fc As Double, Fm As Double
    Dim Ca As Double, Cb As Double, Cm As Double
    Dim K1 As Double, K2 As Double, K3 As Double
    Dim Incr As Double, m As Double, C As Double
    Dim I As Long
    Incr = (B - A) / N
    Sum = 0
    For I = 1 To N
        B = A + Incr
        C=(A + B) / 2
        Fa=f(A)
        Fb}=\textrm{f}(\textrm{B}
        FC=f(C)
        Fm = (Fa + Fb) / 2
```

```
    m=A* *(Fm - Fc) * (Fm - Fb) ) / ((Fa - Fc) * (Fa - Fb)) +
    C * ((Fm-Fa) * (Fm - Fb)) / ( (Fc - Fa) * (Fc - Fb)) +
    B * ((Fm - Fa) * (Fm - Fc)) / ((Fb - Fa) * (Fb - Fc))
    Fm=f(m)
    Ca = Fa / (A - m) / (A - B)
    Cm = Fm/(m-A) / (m-B)
    Cb}=\textrm{Fb}/(\textrm{B}-\textrm{A})/(\textrm{B}-\textrm{m}
    K1 = (Ca + Cm + Cb)
    K2 = (Ca * (m+B) + Cm* (A + B) + Cb * (m + A))
    K3 = (Ca*m* B + Cm*A* B + Cb*m*A)
    Sum = Sum + Incr * (K1 / 3 * (B^ ^ + A * B + A ^ 2) - K2 /
2*(B + A) + K3)
    A = B
    Next I
    HVFQI_Area = Sum
End Function
Listing 2. The HVFQI algorithm function
```

Listing 2 shows a VBA function that implements of the HVFQI algorithm.

## Testing the Integral of $f(x)=1 / x$

A simple integral is $f(x)=1 / x$ which gives an integral function $I(x)=\ln (x)+C$. Table 1 shows the results of integrating $f(x)$ between 1 and 5 for $N=5$.

|  | HFVQI | Simpson 1 | Simpson 2 |
| :--- | :--- | :--- | :--- |
| N | 5 | 5 | 88 |
| Integral | 1.606402047 | 1.558430415 | 1.606428327 |
| Actual Integral | 1.609437912 | 1.609437912 | 1.609437912 |
| \%Err | 0.188628905 | -3.169274014 | 0.186996049 |

Table 1. Results using $N=5$.
The data under the HFVQI column show the calculated integral, actual integral, and percent error. The third column shows similar results for Simpson's rule. Notice that the error in the latter method is over 20 time greater than the percent error for the HFVQI algorithm. The last column shows the high number of N used with Simpson's rule to achieve a comparable percent error with the HFVQI algorithm.

The ratio of the two percent errors is over 15 ! I will be using Table 1 as template for other similar calculations in this paper. I will also refer to N in the second and fourth columns as N_HVFQI and N_Sipmson, respectively.

Table 2 shows the results of integrating $\mathrm{f}(\mathrm{x})$ between 1 and 5 for $\mathrm{N}=10$.

|  | HFVQI | Simpson 1 | Simpson 2 |
| :--- | :--- | :--- | :--- |
| N | 10 | 10 | 1300 |
| Integral | 1.609232799 | 1.584774355 | 1.609232879 |
| Actual Integral | 1.609437912 | 1.609437912 | 1.609437912 |
| \%Err | 0.01274443 | -1.532432994 | 0.012739451 |

Table 2. Results using $N=10$.
In Table 2, notice that the error in the latter method is 3 orders of magnitude greater than the percent error for the HFVQI algorithm. The last column shows the high number of N used with Simpson's rule to achieve a comparable percent error with the HFVQI algorithm. The ratio of the two percent errors is over 100!

Table 3 shows the results of integrating $\mathrm{f}(\mathrm{x})$ between 1 and 5 for $\mathrm{N}=15$.

|  | HFVQI | Simpson 1 | Simpson 2 |
| :--- | :--- | :--- | :--- |
| N | 15 | 15 | 6500 |
| Integral | 1.609396587 | 1.591332647 | 1.609396891 |
| Actual Integral | 1.609437912 | 1.609437912 | 1.609437912 |
| \%Err | 0.002567686 | -1.12494341 | 0.002548831 |

Table 3. Results using $N=15$.
In Table 3, notice that the error in the latter method is 3 orders of magnitude greater than the percent error for the HFVQI algorithm. The last column shows the high number of N used with Simpson's rule to achieve a comparable percent error with the HFVQI algorithm. The ratio of the two percent errors is over 430!

Table 4 shows the results of integrating $\mathrm{f}(\mathrm{x})$ between 1 and 5 for $\mathrm{N}=20$.

|  | HFVQI | Simpson 1 | Simpson 2 |
| :--- | :--- | :--- | :--- |
| N | 20 | 20 | 20000 |
| Integral | 1.609424737 | 1.596538305 | 1.60942458 |
| Actual Integral | 1.609437912 | 1.609437912 | 1.609437912 |
| \%Err | 0.000818607 | -0.801497657 | 0.000828422 |

Table 4. Results using $N=20$.

In Table 4, notice that the error in the latter method is 3 orders of magnitude greater than the percent error for the HFVQI algorithm. The last column shows the high number of N used with Simpson's rule to achieve a comparable percent error with the HFVQI algorithm. The ratio of the two percent errors is about 1000 !

Notice that the percent error in Simpson's rule that uses the same number of divisions, remains 3 orders of magnitudes compared with that of the HFVQI algorithm. By contrast, the number of divisions needed by Simpson's Rule to match the percent error of the HFVQI algorithm has increased in whopping paces!

Table 5 shows the results of integrating $\mathrm{f}(\mathrm{x})$ between 1 and 5 for $\mathrm{N}=25$.

|  | HFVQI | Simpson 1 | Simpson 2 |
| :--- | :--- | :--- | :--- |
| N | 25 | 25 | 50000 |
| Integral | 1.609432496 | 1.598621273 | 1.609432579 |
| Actual Integral | 1.609437912 | 1.609437912 | 1.609437912 |
| \%Err | 0.000336524 | -0.672075586 | 0.000331375 |

Table 5. Results using $N=25$.
In Table 5, notice that the error in the latter method is 3 orders of magnitude greater than the percent error for the HFVQI algorithm. The last column shows the high number of N used with Simpson's rule to achieve a comparable percent error with the HFVQI algorithm. The ratio of the two percent errors is about 2000!

Table 6 shows the results of integrating $f(x)$ between 1 and 5 for $\mathrm{N}=30$.

| HFVQI |  | Simpson 1 | Simpson 2 |
| :--- | :--- | :--- | :--- |
| N | 30 | 30 | 102000 |
| Integral | 1.609435295 | 1.600733686 | 1.609435298 |
| Actual Integral | 1.609437912 | 1.609437912 | 1.609437912 |
| \%Err | 0.000162617 | -0.540823974 | 0.00016244 |

Table 6. Results using $N=30$.
In Table 6, notice that the error in the latter method is 3 orders of magnitude greater than the percent error for the HFVQI algorithm. The last column shows the high number of N used with Simpson's rule to achieve a comparable percent error with the HFVQI algorithm. The ratio of the two percent errors is about 3400 !

Table 7 shows the results of integrating $\mathrm{f}(\mathrm{x})$ between 1 and 5 for $\mathrm{N}=35$.

|  | HFVQI | Simpson 1 | Simpson 2 |
| :--- | :--- | :--- | :--- |
| N | 35 | 35 | 190000 |
| Integral | 1.609436498 | 1.601737323 | 1.609436509 |
| Actual Integral | 1.609437912 | 1.609437912 | 1.609437912 |
| \%Err | $8.78848 \mathrm{E}-05$ | -0.478464546 | $8.72046 \mathrm{E}-05$ |

Table 7. Results using $N=35$.
In Table 7, notice that the error in the latter method is 4 orders of magnitude greater than the percent error for the HFVQI algorithm. The last column shows the high number of N used with Simpson's rule to achieve a comparable percent error with the HFVQI algorithm. The ratio of the two percent errors is about 5400 !

Figure 1 shows the graph for N for Simpson's Rule vs N for the HVFQI algorithm, where the first N allows Simpson's Rule to generate comparable Error to the HVFQI algorithm.

## Simpson



Figure 1. Plot of N_Simpson vs N_HFVQI.
Figure 1 shows that here is a fairly exponential relationship between the values of N, for Simpson's Rule, and N, for the HFVQI algorithm. Table 8 shows the results of the regression of the following log-linear model:

$$
\begin{equation*}
\ln \left(\mathrm{N} \_ \text {Simp }\right)=4.434289+0.241413 * \text { N_HFVQI } \tag{10}
\end{equation*}
$$

| SUMMARY OUTPUT |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Regression Statistics |  |  |  |  |  |
| Multiple R | 0.964392 |  |  |  |  |
| R Square | 0.930052 |  |  |  |  |
| Adjusted $\quad R$ Square | 0.916062 |  |  |  |  |
| Standard Error | 0.783357 |  |  |  |  |
| Observations | 7 |  |  |  |  |
|  |  |  |  |  |  |
| ANOVA |  |  |  |  |  |
|  | $d f$ | SS | MS | $F$ | Significance <br> F |
| Regression | 1 | 40.79603 | 40.79603 | 66.4812 | 0.000451 |
| Residual | 5 | 3.068238 | 0.613648 |  |  |
| Total | 6 | 43.86427 |  |  |  |
|  |  |  |  |  |  |
|  | Coefficients | Standard Error | $t$ Stat | P-value |  |
| Intercept | 4.434289 | 0.662057 | 6.697743 | 0.001122 |  |
| Slope | 0.241413 | 0.029608 | 8.153601 | 0.000451 |  |

Table 8. The results of the regression of the model in equation 10.
We can improve the fit between N (Simpson) and N(HFVQI) using the following log-quadratic model:
$\ln \left(\mathrm{N} \_\right.$Simp $)=2.255521+0.531915 *$ N_HFVQI $-0.00726 *$ N_HFVQI^2
The quadratic term in equation 11 performs a correction needed to improve the curve fit. Table 9 shows the results of using the model in equation 11.

## SUMMARY OUTPUT

|  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Regression Statistics |  |  |  |  |  |
| Multiple R | 0.996584 |  |  |  |  |
| R Square | 0.99318 |  |  |  |  |
| Adjusted $\quad$ R Square | 0.989771 |  |  |  |  |
| Standard Error | 0.273467 |  |  |  |  |
| Observations | 7 |  |  |  |  |
|  |  |  |  |  |  |
| ANOVA |  |  |  |  |  |
|  | $d f$ | SS | MS | F | Significance F |
| Regression | 2 | 43.56514 | 21.78257 | 291.272 | 4.65E-05 |
| Residual | 4 | 0.299137 | 0.074784 |  |  |
| Total | 6 | 43.86427 |  |  |  |
|  |  |  |  |  |  |
|  | Coefficients | Standard Error | $t$ Stat | P-value |  |
| Intercept | 2.255521 | 0.426168 | 5.292565 | 0.006118 |  |
| Slope1 | 0.531915 | 0.048846 | 10.88953 | 0.000404 |  |
| Slope2 | -0.00726 | 0.001194 | -6.08505 | 0.003687 |  |

Table 9. The results of the regression of the model in equation 11.
Figure 2 shows the plot of equation 11.

Ln(Simpson)


Figure 2. Plot model in equation 11

## Testing other Integrals

Let's look at the test case of integration of $f(x)=\exp (x)$ with an integral $\mathrm{I}(\mathrm{x})=\exp (\mathrm{x})+\mathrm{C}$. Table 10 shows the results of integrating $\mathrm{f}(\mathrm{x})$ between 0 and 1 for N $=10$.

|  | HFVQI | Simpson 1 | Simpson 2 |
| :--- | :--- | :--- | :--- |
| N | 10 | 10 | 1700000 |
| Integral | 1.718281291 | 1.639651917 | 1.718281295 |
| Actual Integral | 1.718281828 | 1.718281828 | 1.718281828 |
| \%Err | $3.12662 \mathrm{E}-05$ | -4.57607772 | $3.10191 \mathrm{E}-05$ |

Table 10. Results for $f(x)=\exp (x)$ integral from 0 to 1 , using $N=10$.
The results of Table 10 shows how superior the HFVQI algorithm is to Simpson's Rule. The ratio between N_HFVQI and N_Simpson is over 150,000! These results show that HFVQI excels by leaps and bounds over the legacy Simpson's Rule for monotonic functions that exhibit a quick change in function value.

Let's look at the test case of integration of $f(x)=\sqrt{x}$ with an integral $I(x)-(2 / 3) * x^{3 / 2}$. Table 11 shows the results of integrating $f(x)$ between 1 and 2 for $N=10$.

| HFVQI | Simpson 1 | Simpson 2 |  |
| :--- | :--- | :--- | :--- |
| N | 10 | 10 | 1000000 |
| Integral | 1.218951427 | 1.176583269 | 1.218950945 |


| Actual Integral | 1.218951416 | 1.218951416 | 1.218951416 |
| :--- | :--- | :--- | :--- |
| $\%$ Err | $-8.76989 \mathrm{E}-07$ | -3.475786359 | $-3.86729 \mathrm{E}-05$ |

Table 11. Results for $f(x)=\sqrt{\mathrm{x}}$ integral from 1 to 2 , using $N=10$.
The results of Table 11 shows how better the HFVQI algorithm is to Simpson's Rule. Even with $\mathrm{N}=1$ million, the Simpson's Rule produces a percent error that is 2 orders of magnitude higher than the HFVQI algorithm.

Let's examine the test case of integration of $f(x)=x^{3}$ with an integral $I(X)=x^{4} / 4+C$. Table 12 shows the results of integrating $f(x)$ between 0 and 2 for $N=10$.

| HFVQI | Simpson 1 | Simpson 2 |  |
| :--- | :--- | :--- | :--- |
| N | 10 | 10 | 14000 |
| Integral | 3.999615714 | 3.581176149 | 3.999619116 |
| Actual Integral | 4 | 4 | 4 |
| \%Err | 0.009607142 | -10.47059627 | -0.009522109 |

Table 12. Results for $f(x)=\mathrm{x}^{3}$ integral from 0 to 2 , using $N=10$.
The results of Table 12 shows how superior the HFVQI algorithm is to Simpson's Rule. The ratio of N_HFVQI to N_SIMPSON that achieves comparable percent error is 1400 . The percent error of the HFVQI algorithm is 3 orders of magnitude better than the percent error of the Simpson's Rule for $\mathrm{N}=10$.

Let's consider the test case of integration of $f(x)=\sinh (x)$ with an integral $\mathrm{I}(\mathrm{x})=\cosh (\mathrm{x})+C$. Table 13 shows the results of integrating $\mathrm{f}(\mathrm{x})$ between 1 and 2 for $\mathrm{N}=10$.

|  | HFVQI | Simpson 1 | Simpson 2 |
| :--- | :--- | :--- | :--- |
| N | 10 | 10 | 500000 |
| Integral | 2.219114363 | 2.114389558 | 2.219112638 |
| Actual Integral | 2.219115056 | 2.219115056 | 2.219115056 |
| \%Err | $3.12606 \mathrm{E}-05$ | -4.719245971 | -0.000108958 |

Table 13. Results for $f(x)=\sinh (\mathrm{x})$ integral from 1 to 2 , using $N=10$.
The results of Table 12shows that the HFVQI algorithm is superior is to Simpson's Rule. The percent error of the HFVQI algorithm is 5 orders of magnitude better than the percent error of the Simpson's Rule for $\mathrm{N}=10$.

## Conclusion

The new Half-Function Value Quadratic Integration algorithm is able to compete very effectively against the legacy Simpson's Rule. While the new algorithm required more calculations per integration step, the calculations yield more accurate results. In other words, one integration step is worth tens of thousands small Simpson Rule steps! Such a superiority is true for monotonic function that have a high curvature (i.e. high second derivative).

Document History

## Date

Version
Comment

| June 3, 2018 | 0.9 .0 | Initial release. |
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