A New Look at Multivariable Interpolation

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Introduction

Interpolation using a single independent variable usually involves using legacy algorithm such as the Lagrangian Interpolation, the Newton Divided Difference method, and the Newton Difference method. These methods work with a varying number of *anchor points* used to implicitly construct and use the polynomials to calculate the value for the interpolated dependent variable.

When it comes to multiple variables, the matter quickly becomes more complex. The algorithms I found, on the internet, for interpolating with two independent variables, call them X and Y, have the following features:

- The interpolation is linear.
- The anchor points required four (X,Y) points forming a square on the X-Y plane.

The algorithms for three independent variables, call them X, Y, and Z, have similar features:

- The interpolation is linear.
- The anchor points required eight (X,Y,Z) points forming a cube in the X-Y-Z space.

The appendix in this article contains the summary of the equations used for linear interpolation using two and three independent variables. In the case of the latter, the appendix shows three interpolation versions.

The questions that came to mind are:

1. What if the points used to interpolate with two or three independent variables do not satisfy forming a square or a cube, respectively?
2. How does one conduct quadratic interpolation with two or three independent variables?

3. How does one conduct interpolation with two or three independent variables while using custom non-polynomial models?

A New Approach

After a brief mathematical wrangling with the above problems, I quickly realized that using matrices would simply the solutions.

You may recall the Vandermonde matrix that can be used to calculate the coefficients of a regular polynomial. This matrix has n rows and m columns. Here is a sample Vandermonde matrix that provides values for fifth-order polynomials:

$$\mathbf{A} = \begin{bmatrix} 1 & X_1 & X_1^2 & X_1^3 & X_1^4 & X_1^5 \\ 1 & X_2 & X_2^2 & X_2^3 & X_2^4 & X_2^5 \\ 1 & X_3 & X_3^2 & X_3^3 & X_3^4 & X_3^5 \\ 1 & X_4 & X_4^2 & X_4^3 & X_4^4 & X_4^5 \\ 1 & X_5 & X_5^2 & X_5^3 & X_5^4 & X_5^5 \\ 1 & X_6 & X_6^2 & X_6^3 & X_6^4 & X_6^5 \end{bmatrix}$$

The constant vector **b** has the values of the polynomials at the points X_1, X_2, X_3, X_4, X_5 and X_6. The polynomial coefficients are calculated by solving for the unknown coefficients in the solution vector **c** in the following matrix equation:

$\mathbf{A}\ \mathbf{c} = \mathbf{b}$

I will define a new variant of the Vandermonde matrix and call it the *Vandermonde-Shammas* matrix. This matrix is always a square matrix with the number of rows (or columns) equal to the number of coefficients used in the interpolation model. The first column is typically populated with 1 to reflect the presence of a constant term in the interpolation model. Should you decide to use a model without a constant term, then the first column need not be full of ones. In this case, eliminating the columns of ones also requires that you reduce the number of rows by one to maintain a square matrix. The examples in this article systematically include the constant term in the interpolation model.

The model I want to use to interpolate values for two independent variables is:

$F(X,Y) = C_1 + C_2 X + C_3 Y$

To use the Vandermonde-Shammas matrix for the linear interpolation with two variables X and Y, we have the matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & X_1 & Y_1 \\ 1 & X_2 & Y_2 \\ 1 & X_3 & Y_3 \end{bmatrix}$$

As you can see, matrix **A** contains values for two variables, making it different from regular Vandermonde matrices.

The constant vector **b** has the function values at (X_1,Y_1), (X_2,Y_2), and (X_3,Y_3). That vector is:

$$\mathbf{b} = \begin{bmatrix} F(X_1,Y_1) \\ F(X_2,Y_2) \\ F(X_3,Y_3) \end{bmatrix}$$

The following matrix equation calculates the coefficients in the solution vector **c** for the above linear model:

$$\mathbf{c} = \mathbf{A}^{-1}\,\mathbf{b}$$

Notice that the new approach requires three points for interpolation instead of four! Moreover, these points need not be aligned in any particular way, as long as they generate a nonsingular matrix **A**. In addition, once you have the solution vector **c**, you can easily perform additional interpolation using the same anchor points. There is no need to recalculate the values in vector **c**.

In the case of three independent variables, the model I want to use for linear interpolation is:

$$F(X,Y,Z) = C_1 + C_2\,X + C_3\,Y + C_4\,Z$$

To use the Vandermonde-Shammas matrix for the linear interpolation with three variables X, Y, and Z, we have the coefficient matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & X_1 & Y_1 & Z_1 \\ 1 & X_2 & Y_2 & Z_2 \\ 1 & X_3 & Y_3 & Z_3 \\ 1 & X_4 & Y_4 & Z_4 \end{bmatrix}$$

The above coefficient matrix **A** has four rows and four columns, because the target model has four coefficients. Moreover, it has values for three variables all raised to power one. Again, matrix **A** is not your typical Vandermonde matrix.

The constant vector **b** has the function values at (X_1,Y_1,Z_1) through (X_4,Y_4,Z_4) and is:

$$\mathbf{b} = \begin{bmatrix} F(X_1,Y_1,Z_1) \\ F(X_2,Y_2,Z_2) \\ F(X_3,Y_3,Z_3) \\ F(X_4,Y_4,Z_4) \end{bmatrix}$$

As before, the following matrix equation calculates the values in the solution vector **c** for the above linear model:

$$\mathbf{c} = \mathbf{A}^{-1}\,\mathbf{b}$$

Notice that the new approach requires four points for interpolation instead of eight used by the trilinear cube model! As before, once you calculate the values in the solution vector **c**, you can easily perform additional interpolation using the same anchor points.

Quadratic Interpolation with Two Independent Variables

In the case of quadratic interpolation with two independent variables, the model used is:

$$F(X,Y) = C_1 + C_2\,X + C_3\,X^2 + C_4\,Y + C_5\,Y^2 + C_6\,X\,Y$$

The Vandermonde-Shammas matrix for the bi-quadratic interpolation is:

$$\mathbf{A} = \begin{bmatrix} 1 & X_1 & X_1^2 & Y_1 & Y_1^2 & X_1Y_1 \\ 1 & X_2 & X_2^2 & Y_2 & Y_2^2 & X_2Y_2 \\ 1 & X_3 & X_3^2 & Y_3 & Y_3^2 & X_3Y_3 \\ 1 & X_4 & X_4^2 & Y_4 & Y_4^2 & X_4Y_4 \\ 1 & X_5 & X_5^2 & Y_5 & Y_5^2 & X_5Y_5 \\ 1 & X_6 & X_6^2 & Y_6 & Y_6^2 & X_6Y_6 \end{bmatrix}$$

The above matrix **A** has six rows and six columns, because the target model has six coefficients.

The constant vector **b** has the function values at (X_1,Y_1) through (X_6,Y_6) and is:

$$\mathbf{b} = \begin{bmatrix} F(X_1,Y_1) \\ F(X_2,Y_2) \\ F(X_3,Y_3) \\ F(X_4,Y_4) \\ F(X_5,Y_5) \\ F(X_6,Y_6) \end{bmatrix}$$

As before, the following matrix equation calculates the coefficients for the above linear model:

$$\mathbf{c} = \mathbf{A}^{-1}\,\mathbf{b}$$

Quadratic Interpolation with Three Independent Variables

In the case of quadratic interpolation with three independent variables, the model used is:

$$F(X,Y,Z) = C_1 + C_2\,X + C_3\,X^2 + C_4\,Y + C_5\,Y^2 + C_6\,Z + C_7\,Z^2 +$$
$$C_8\,X\,Y + C_9\,X\,Z + + C_{10}\,Y\,Z + + C_{11}\,X\,Y\,Z$$

The Vandermonde-Shammas matrix for the bi-quadratic interpolation is:

$$\mathbf{A} = \begin{bmatrix} 1 & X_1 & X_1^2 & Y_1 & Y_1^2 & Z_1 & Z_1^2 & X_1Y_1 & X_1Z_1 & Y_1Z_1 & X_1Y_1Z_1 \\ \ldots & & & & & \ldots & & & & & \ldots \\ 1 & X_{11} & X_{11}^2 & Y_{11} & Y_{11}^2 & Z_{11} & Z_{11}^2 & X_{11}Y_{11} & X_{11}Z_{11} & Y_{11}Z_{11} & X_{11}Y_{11}Z_{11} \end{bmatrix}$$

The above matrix **A** has eleven rows and eleven columns, because the target model has eleven coefficients.

The constant vector **b** has the function values at (X_1,Y_1,Z_1) through (X_{11},Y_{11},Z_{11}) and is:

$$\mathbf{b} = \begin{bmatrix} F(X_1,Y_1,Z_1) \\ \ldots \\ F(X_{11},Y_{11},Z_{11}) \end{bmatrix}$$

As before, the following matrix equation calculates the coefficients for the above linear model:

$$\mathbf{c} = \mathbf{A}^{-1}\,\mathbf{b}$$

Beyond Quadratic Interpolation

The new approach for interpolation with multivariable has opened a new door that allows you to select the number of variables that appear in your problem, select the model that you feel to be at least adequate for the interpolation, and proceed from there. The approach I am advocating is equivalent to performing a multiple linearized regression with a minimum number of data points required for the target model. The statistical coefficient of determination will always be 1, since we are using the minimum number of data points required. How adequate are the models you are using? That is something that may be known only to you based on theoretical or empirical basis. To investigate your models you need to use them with more than the minimum number of required data points, perform multiple linearized regression, and then obtain regression statistics that you use to infer how adequate a model is. If you do not have a generous number of additional points, you can device a scheme where you rotate the number of anchor points and use the excluded points for testing the goodness of the interpolation model.

I will end this section by giving you two examples of using the Vandermonde-Shammas matrix to support interpolation with custom (i.e. non-polynomial) models. The models I present are essentially arbitrary and I am using them for the sake of giving examples.

The first example uses the following model:

$$F(X,Y) = C_1 + C_2 \sin(X) + C_3 \sin(2X) + C_4\, X\, Y$$

The Vandermonde-Shammas matrix is:

$$\mathbf{A} = \begin{bmatrix} 1 & \sin(X_1) & \sin(2*X_1) & X_1Y_1 \\ 1 & \sin(X_2) & \sin(2*X_2) & X_2Y_2 \\ 1 & \sin(X_3) & \sin(2*X_3) & X_3Y_3 \\ 1 & \sin(X_4) & \sin(2*X_4) & X_4Y_4 \end{bmatrix}$$

The coefficient matrix **A** has four rows and four columns, because the target model has four coefficients. The constant vector **b** has the function values at (X_1,Y_1) through (X_4,Y_4). The solution for vector C employs the matrix calculations presented earlier. Once you have the values for solution vector **c** you can interpolate for different values of (X,Y).

The second example uses the following model:

$F(X,Y,Z) = C_1 + C_2/X + C_3\ Y/Z + C_4 \ln(X)/Z$

The Vandermonde-Shammas matrix is:

$$\mathbf{A} = \begin{bmatrix} 1 & 1/X_1 & Y_1/Z_1 & \ln(X_1)/Z_1 \\ 1 & 1/X_2 & Y_2/Z_2 & \ln(X_2)/Z_2 \\ 1 & 1/X_3 & Y_3/Z_3 & \ln(X_3)/Z_3 \\ 1 & 1/X_4 & Y_4/Z_4 & \ln(X_4)/Z_4 \end{bmatrix}$$

The matrix **A** has four rows and four columns, because the target model has four coefficients. The constant vector **b** has the function values at (X_1,Y_1,Z_1) through (X_4,Y_4,Z_4). The solution vector **c** employs the matrix calculations presented earlier. Once you have the values for vector **c** you can interpolate for different values of (X,Y,Z).

Bonus for Single-Variable Interpolation

The approach used with multi-variable interpolation also works very well with single variables that employ non-polynomial models. Here is an example. Assume that I want to interpolate data using the following non-polynomial model:

$F(X) = C_1 + C_2/X + C_3\ X + C_4 \ln(X)$

The Vandermonde-Shammas matrix is:

$$\mathbf{A} = \begin{bmatrix} 1 & 1/X_1 & X_1 & \ln(X_1) \\ 1 & 1/X_2 & X_2 & \ln(X_2) \\ 1 & 1/X_3 & X_3 & \ln(X_3) \\ 1 & 1/X_4 & X_4 & \ln(X_4) \end{bmatrix}$$

The above matrix has four rows and four columns, because the interpolation model has four coefficients.

The constant vector **b** has the function values at X_1 through X_4 and is:

$$\mathbf{B} = \begin{bmatrix} F(X_1) \\ F(X_2) \\ F(X_3) \\ F(X_4) \end{bmatrix}$$

Calculating the values for solution vector c follows a path that should be familiar by now.

Summary

This article presents a new approach for multivariable interpolation that empowers you to work with two or more variables and allows you to select a model that you deem appropriate to perform the interpolation. The benefit of this new approach can be also applied to single variable interpolation that uses custom models. In addition, since you calculate the coefficients of the interpolation model, you can reuse these coefficients to interpolate for multiple values.

Appendix

Here is a summary of linear interpolation methods that I found in the following references:

1. *Computational Color Technology* by Henry R. Kang. Published by SPIE Publications in 2006.
2. *A Novel Algorithm for Color Space Conversion Model from CMYK to LAB* by Juan-li Hu, Jia-bing Deng, and Shan-shan Zou. Published by ACADEMY PUBLISHER in 2010.

The next subsections present the equations for the bilinear and trilinear interpolation found in the above references.

Bilinear Interpolation

Given the four points (X1, Y1), (X1,Y2), (X2,Y1) and (X2,Y2) that form a square on the X-Y plane. These points have the function values $F_{1,1}$, $F_{1,2}$, $F_{2,1}$, and $F_{2,2}$, respectively. To calculate F(X,Y) use the following equation for interpolation:

$$F(X,Y) = F_{1,1} + (F_{2,1} - F_{1,1}) * R_x + (F_{1,2} - F_{1,1}) * R_y + (F_{2,2} - F_{1,2} - F_{2,1} + F_{1,1}) * R_x * R_y$$

Where,

$R_x = (X - X_1)/(X_2 - X_1)$

$R_y = (Y - Y_1)/(Y_2 - Y_1)$

Trilinear Interpolation

Given the eight cube-forming points (X_1,Y_1,Z_1), (X_1,Y_1,Z_2), (X_1,Y_2,Z_1), (X_1,Y_2,Z_2), (X_2,Y_1,Z_1), (X_2,Y_1,Z_2), (X_2,Y_2,Z_1), and (X_2,Y_2,Z_2). These points have the function values $F_{1,1,1}$, $F_{1,1,2}$, $F_{1,2,1}$, $F_{1,2,2}$, $F_{2,1,1}$, $F_{2,1,2}$, $F_{2,2,1}$, and $F_{2,2,2}$, respectively. To calculate F(X,Y,Z) use the following equation for interpolation:

$$F(X,Y,Z) = C_1 + C_2 R_x + C_3 R_y + C_4 R_z + C_5 R_x Ry + C_6 R_y R_z + C_7 R_x Rz + C_8 R_x R_y R_z$$

Where,

$R_x = (X - X_1)/(X_2 - X_1)$

$R_y = (Y - Y_1)/(Y_2 - Y_1)$

$R_z = (Z - Z_1)/(Z_2 - Z_1)$

And the coefficients C_1 through C_8 are given by:

$C_1 = F_{1,1,1}$

$C_2 = F_{2,1,1} - F_{1,1,1}$

$C_3 = F_{1,2,1} - F_{1,1,1}$

$C_4 = F_{1,1,2} - F_{1,1,1}$

$C_5 = F_{2,2,1} - F_{1,2,1} - F_{2,1,1} + F_{1,1,1}$

$C_6 = F_{1,2,2} - F_{1,1,2} - F_{1,2,1} + F_{1,1,1}$

$C_7 = F_{2,1,2} - F_{1,1,2} - F_{2,1,1} + F_{1,1,1}$

$C_8 = F_{2,2,2} - F_{1,2,2} - F_{2,1,2} - F_{2,2,1} + F_{2,1,1} + F_{1,1,2} + F_{1,2,1} - F_{1,1,1}$

Prism Interpolation

Given the eight cube-forming points (X_1,Y_1,Z_1), (X_1,Y_1,Z_2), (X_1,Y_2,Z_1), (X_1,Y_2,Z_2), (X_2,Y_1,Z_1), (X_2,Y_1,Z_2), (X_2,Y_2,Z_1), and (X_2,Y_2,Z_2). These points have the function values $F_{1,1,1}$, $F_{1,1,2}$, $F_{1,2,1}$, $F_{1,2,2}$, $F_{2,1,1}$, $F_{2,1,2}$, $F_{2,2,1}$, and $F_{2,2,2}$, respectively. Since a cube contains two symmetrical prisms there are two versions of the actual interpolating function. To calculate F(X,Y,Z) use the following equation for interpolation you need to first calculate:

$R_x = (X - X_1)/(X_2 - X_1)$

$R_y = (Y - Y_1)/(Y_2 - Y_1)$

$R_z = (Z - Z_1)/(Z_2 - Z_1)$

If $(X - X_1) > (Y - Y_1)$ Then $F(X,Y,Z) = P_1(X,Y,Z)$ Else $F(X,Y,Z) = P_2(X,Y,Z)$

Where,

$$P_1(X,Y,Z) = F_{1,1,1} + (F_{2,1,1} - F_{1,1,1}) * R_x + (F_{2,2,1} - F_{2,1,1}) * R_y + (F_{1,1,2} - F_{1,1,1}) * R_Z + (F_{2,1,2} - F_{1,1,2} - F_{2,1,1} + F_{1,1,1}) * R_x * R_z + (F_{2,2,2} - F_{2,1,2} - F_{2,2,1} + F_{2z,1,1}) * R_y * R_z$$

$$P_2(X,Y,Z) = F_{1,1,1} + (F_{2,2,1} - F_{1,2,1}) * R_x + (F_{1,2,1} - F_{1,1,1}) * R_y + (F_{1,1,2} - F_{1,1,1}) * R_Z + (F_{2,2,2} - F_{1,2,2} - F_{2,2,1} + F_{1,2,1}) * R_x * R_z + (F_{1,2,2} - F_{1,1,2} - F_{1,2,1} + F_{1,1,1}) * R_y * R_z$$

Pyramid Interpolation

Given the eight cube-forming points (X_1,Y_1,Z_1), (X_1,Y_1,Z_2), (X_1,Y_2,Z_1), (X_1,Y_2,Z_2), (X_2,Y_1,Z_1), (X_2,Y_1,Z_2), (X_2,Y_2,Z_1), and (X_2,Y_2,Z_2). These points have the function values $F_{1,1,1}$, $F_{1,1,2}$, $F_{1,2,1}$, $F_{1,2,2}$, $F_{2,1,1}$, $F_{2,1,2}$, $F_{2,2,1}$, and $F_{2,2,2}$, respectively. Since a cube contains three pyramids, there are three versions of the actual interpolating function. To calculate F(X,Y,Z) use the following equation for interpolation you need to first calculate:

$R_x = (X - X_1)/(X_2 - X_1)$

$R_y = (Y - Y_1)/(Y_2 - Y_1)$

$R_z = (Z - Z_1)/(Z_2 - Z_1)$

If $(Y - Y_1) > (X - X_1)$ And $(Z - Z_1) > (X - X_1)$ Then

$F(X,Y,Z) = P_1(X,Y,Z)$

Else If $(X - X_1) > (Y - Y_1)$ And $(Z - Z_1) > (Y - Y_1)$ Then

$F(X,Y,Z) = P_2(X,Y,Z)$

Else

$$F(X,Y,Z) = P_3(X,Y,Z)$$

Where in the case of calculating $P_1(X,Y,Z)$ we have:

$C_1 = F_{2,2,2} – F_{1,2,2}$

$C_2 = F_{1,2,1} – F_{1,1,1}$

$C_3 = F_{1,1,2} – F_{1,1,1}$

$C_4 = F_{1,2,2} – F_{1,1,2} – F_{1,2,1} + F_{1,1,1}$

$P_1(X,Y,Z) = F_{1,1,1} + C_1 R_x + C_2 R_y + C_3 R_z + C_4 R_y R_z$

In the case of calculating $P_2(X,Y,Z)$ we have:

$C_1 = F_{2,1,1} – F_{1,1,1}$

$C_2 = F_{2,2,2} – F_{2,1,2}$

$C_3 = F_{1,1,2} – F_{1,1,1}$

$C_4 = F_{2,1,2} – F_{1,1,2} – F_{2,1,1} + F_{1,1,1}$

$P_2(X,Y,Z) = F_{1,1,1} + C_1 R_x + C_2 R_y + C_3 R_z + C_4 R_x R_z$

And finally, in the case of calculating $P_3(X,Y,Z)$ we have:

$C_1 = F_{2,1,1} – F_{1,1,1}$

$C_2 = F_{2,2,2} – F_{2,1,2}$

$C_3 = F_{1,1,2} – F_{1,1,1}$

$C_4 = F_{2,1,2} – F_{1,1,2} – F_{2,1,1} + F_{1,1,1}$

$P_3(X,Y,Z) = F_{1,1,1} + C_1 R_x + C_2 R_y + C_3 R_z + C_4 R_x R_y$