New Modification of Newton Method with Third-Order Convergence for Solving Nonlinear Equations of Type $f(0)=0$

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Abstract   The objective of this paper is to define a new Newton-type method for finding simple roots of nonlinear equations. It is proved that the new method has the convergence order of three requiring only two function evaluations per iteration. Kung and Traub conjectured that the multipoint iteration methods, without memory based on n evaluations, could achieve maximum convergence order 2^{n-1}, but the new method produces convergence order of three, which is better than expected maximum convergence order of two. Therefore, we show that the conjecture fails for a particular set of nonlinear equations. In fact, we have found that the new method is only usable for zero roots. However, it is observed that our proposed method is very competitive with the third-order methods, thus the new method is very effective in high precision computations.

Keywords   Newton method, Nonlinear equations, Kung-Traub's conjecture, Optimal order of convergence, Efficiency index, One-point method

1. Introduction

Finding the root of nonlinear equations is one of important problem in science and engineering [4, 6, 9]. In this paper, we present a new one-point third-order iterative method to find a simple root α of the nonlinear equation $f(x)=0$, where $f:D\subset\mathbb{R}\rightarrow\mathbb{R}$ for an open interval D is a scalar function. Many higher order variants of the Newton method have been developed based on the Kung and Traub conjecture [3]. Here we present a new iterative method which has a better efficiency index than the second and third order methods given in [2, 4-7, 10]. Hence, the proposed third-order method is significantly better when compared with the established methods.

The classical Newton method for finding simple roots is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

which converges quadratically [4, 6, 9]. For the purpose of this paper, we improve the classical Newton method and construct a new third-order iterative method for finding simple roots of nonlinear equations. The prime motive of this

study is to develop a new class of one-point method for finding simple roots of nonlinear equations. The third-order method presented in this paper only uses two evaluations of the function per iteration. Kung and Traub conjectured that the multipoint iteration methods, without memory based on n evaluations, could achieve optimal convergence order 2^{n-1}. In fact, we have obtained a higher order of convergence than the optimal order of convergence suggested by Kung and Traub conjecture [3]. We demonstrate that the Kung and Traub conjecture fails for a particular case, that is when the simple root of a nonlinear equation is equal to zero.

The outline of this paper is as follows: Some basic definitions relevant to the present work are stated in the section 2. In section 3 the new one-point third-order iterative method is constructed and proved. In section 4, four well-established two-point third-order methods are stated, it will demonstrate the effectiveness of the new one-point third-order iterative method. Finally, in section 5, numerical comparisons are made to demonstrate the performance of the new one-point third method.

2. Preliminaries

In order to establish the order of convergence of the iterative methods, some of the definitions are stated:

Definition 1 Let $f(x)$ be a real function with a simple root α and let $\{x_n\}$ be a sequence of real numbers that

converge towards α. The order of convergence p is given by

$$\lim_{n \to \infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} = \lambda \neq 0 \qquad (2)$$

where $p \in \mathbb{R}^+$ and λ is the asymptotic error constant, [1, 4, 6, 9].

Definition 2 Let $e_k = x_k - \alpha$ be the error in the kth iteration, then the relation

$$e_{k+1} = \zeta e_k^p + \mathrm{O}\left(e_k^{p+1}\right), \qquad (3)$$

is the error equation. If the error equation exists then p is the order of convergence of the iterative method, [1, 4, 6, 9].

Definition 3 Let r be the number of function evaluations of the iterative method. The efficiency of the iterative method is measured by the concept of efficiency index and defined as

$$\sqrt[r]{p} \qquad (4)$$

where p is the order of the method, [6].

Definition 4 (Kung and Traub conjecture) Let $x_{n+1} = g(x_n)$ define an iterative function without memory with k-evaluations. Then

$$p(g) \leq p_{opt} = 2^{k-1} \qquad (5)$$

where p_{opt} is the maximum order [3].

Definition 5 Suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α of (1). Then the computational order of convergence may be approximated by

$$\mathrm{COC} \approx \frac{\ln|\delta_n \div \delta_{n-1}|}{\ln|\delta_{n-1} \div \delta_{n-2}|}, \qquad (6)$$

where $\delta_i = f(x_i) \div f'(x_i)$ [8].

3. Construction of the One-Point Method and Analysis of Convergence

In this section we define a new one point third-order method for finding simple roots of a nonlinear equation. In fact, the new iterative method is the improvement of the classical Newton method, given by (1). The one-point third-order Newton-type method is expressed by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - (2x_n)^{-1}\left[x_n^2 - \left(\frac{f(x_n)}{f'(x_n)}\right)^2\right], \qquad (7)$$

where x_0 is the initial guess and provided that denominators of (7) are not equal to zero. Now, we shall verify the convergence property of the new one-point

third-order iterative method (7).

Theorem 1

Let $\alpha \in D$ be a simple zero of a sufficiently smooth function $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D. If the initial guess x_0 is sufficiently close to α, then the convergence order of the new one-point iterative method defined by (7) is three.

Proof

Let α be a simple root of $f(x)$, i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, and the error is expressed as

$$e = x - \alpha .$$

Using the Taylor series expansion and taking into account $f(\alpha) = 0$, we have

$$f(x_n) = f'(\alpha)\left(e_n + \sum_{i=2}^{m} c_i e_n^i + \mathrm{O}\left(e_n^{m+1}\right)\right). \qquad (8)$$

$$f'(x_n) = f'(\alpha)\left(1 + \sum_{i=2}^{m} i c_i e_n^{i-1} + \mathrm{O}\left(e_n^{m}\right)\right). \qquad (9)$$

where

$$c_k = \frac{f^{(k)}(\alpha)}{k! f'(\alpha)} \qquad k \geq 2 . \qquad (10)$$

Dividing (8) by (9), we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2\left(c_2^2 - c_3\right)e_n^3 + \cdots. \qquad (11)$$

and

$$\left(\frac{f(x_n)}{f'(x_n)}\right)^2 = e_n^2 - 2c_2 e_n^3 + \cdots. \qquad (12)$$

Substituting (11) and (12) in (7), we obtain

$$e_{n+1} = e_n - \frac{f(x_n)}{f'(x_n)} - (2e_n)^{-1}\left[e_n^2 - \left(\frac{f(x_n)}{f'(x_n)}\right)^2\right], \qquad (13)$$

Simplifying (13), yields

$$e_{n+1} = 2^{-1} c_2^2 e_n^3. \qquad (14)$$

The expression (14) establishes the asymptotic error constant has third-order convergence for the Newton-type method defined by (7).

4. The Established Methods

For the purpose of comparison, four two-point third-order methods presented in [2, 5, 7, 10] are considered. Since these methods are well established, we state the essential formulas used to calculate the simple root of nonlinear equations and

thus compare the effectiveness of the new one-point third-order method.

4.1. The Weerakoon and Fernando Method

Weerakoon et al. [10] developed the third-order Newton-type method, given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad (15)$$

$$x_{n+1} = x_n - \left(\frac{2f(x_n)}{f'(x_n) + f'(y_n)} \right). \qquad (16)$$

4.2. The Homeier Method

Homeier [2] presented a third-order Newton-type method, given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad (17)$$

$$x_{n+1} = x_n - \left(\frac{f(x_n)}{2} \right) \left(\frac{1}{f'(x_n)} + \frac{1}{f'(y_n)} \right). \qquad (18)$$

4.3. The Thukral Method

Thukral [7] developed a third-order variants of the Newton-type method, given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad (19)$$

$$x_{n+1} = x_n - \left(\frac{f(x_n)^2 + f'(y_n)^2}{f'(x_n)f'(y_n) + f'(y_n)^2} \right) \left(\frac{f(x_n)}{f'(x_n)} \right), \qquad (20)$$

4.4. The Potra-Ptak Method

Potra et al. [5] developed a third-order variants of the Newton-type method, given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad (21)$$

$$x_{n+1} = x_n - \left(\frac{f(x_n) + f(y_n)}{f'(x_n)} \right). \qquad (22)$$

5. Numerical Examples

The present one-point third-order method given by (7) is employed to solve nonlinear equations with simple roots. To demonstrate the performance of the new one-point third-order method, ten particular nonlinear equations are used. The consistency and stability of results by examining the convergence of the new iterative methods are determined. The findings are generalised by illustrating the effectiveness of the new method for determining the simple roots of a nonlinear equation. Consequently, estimates are given of the approximate solutions produced by the methods considered and list the errors obtained by each of the methods. The numerical computations listed in the tables were performed on an algebraic system called Maple. In fact, the errors displayed are of absolute value and insignificant approximations by the various methods have been omitted in the following tables.

Table 1. Test functions with simple root $\alpha = 0$

Functions	*Initial Guess*
$f_1(x) = \exp(x)\sin(x) + \ln(1 + x^2)$	$x_0 = 5^{-1}$
$f_2(x) = \sin(x)\exp(x) + \cos(x)\ln(1 + x)$	$x_0 = 10^{-1}$
$f_3(x) = \exp(\sin(x)) - 5^{-1}x - 1$	$x_0 = 1$
$f_4(x) = (x + 1)\exp(\sin(x)) - x^2 \exp(\cos(x)) - 1$	$x_0 = 2^{-1}$
$f_5(x) = \sin(x) + \cos(x) + \tan(x) - 1$	$x_0 = 3^{-1}$
$f_6(x) = \exp(-x) - \cos(x)$	$x_0 = 4^{-1}$
$f_7(x) = \ln(1 + x^2) + \exp(x^2 - 3x)\sin(x)$	$x_0 = -5^{-1}$
$f_8(x) = x^3 + \ln(1 + x)$	$x_0 = 7^{-1}$
$f_9(x) = \sin(x) - 3^{-1}x$	$x_0 = -2^{-1}$
$f_{10}(x) = (x - 10)^6 - 10^6$	$x_0 = -4^{-1}$

Table 2. Performance of various iterative methods

f_i	(1)	(16)	(18)	(20)	(22)	(7)
f_1	0.248e-4	0.194e-14	0.146e-24	0.703e-18	0.595e-12	0.379e-19
f_2	0.645e-12	0.148e-39	0.858e-45	0.214e-48	0.181e-38	0.346e-46
f_3	0.450e-5	0.427e-6	0.208e-11	0.157e-4	0.523e-16	0.691e-27
f_4	0.233e-9	0.118e-12	0.234e-15	0.534e-15	0.412e-26	0.439e-43
f_5	0.407e-8	0.784e-25	0.130e-28	0.295e-35	0.220e-25	0.834e-34
f_6	0.954e-4	0.840e-13	0.158e-23	0.376e-12	0.326e-6	0.156e-15
f_7	0.321e-3	0.336e-9	0.255e-13	0.279e-24	0.241e-8	0.687e-15
f_8	0.887e-11	0.628e-24	0.853e-25	0.598e-26	0.333e-34	0.367e-44
f_9	0.455e-11	0.429e-19	0.421e-18	0.261e-17	0.979e-47	0.172e-61
f_{10}	0.736e-9	0.107e-30	0.773e-39	0.244e-34	0.240e-28	0.538e-36

Table 3. COC of various iterative methods

f_i	(1)	(16)	(18)	(20)	(22)	(7)
f_1	1.9995	3.0004	3.0119	2.7902	2.9927	2.9993
f_2	2.0002	3.0000	3.0000	2.9697	3.0000	3.0000
f_3	1.0170	2.9764	2.9499	1.8030	3.0019	2.9999
f_4	2.0008	3.0009	3.0149	2.7213	2.9999	3.0000
f_5	2.0028	2.9999	3.0000	29.445	2.9998	3.0000
f_6	2.0062	2.9973	3.0062	2.6008	3.0348	3.0009
f_7	2.1848	3.0498	3.0177	2.8825	3.0651	3.0168
f_8	2.0023	2.9996	2.9999	3.0005	2.9999	3.0000
f_9	2.5555	2.8248	2.8077	2.7924	5.0000	5.0000
f_{10}	2.0007	3.0000	3.0000	2.9412	3.0000	3.0000

The new one-point third-order method requires two function evaluations and has the order of convergence three. To determine the efficiency index of the new method, definition 3 shall be used. Hence, the efficiency index of the new iterative method given by (7) is $\sqrt[2]{3} \approx 1.7132$. and the efficiency index of the two-point third-order methods considered in this paper is given by (16), (18), (20), (22) is $\sqrt[3]{3} \approx 1.4423$. It is shown that the efficiency index of the new one-point third-order method is much better than the other similar methods. The test functions and the initial guess x_0 are displayed in table 1. The difference between the simple root α and the approximation x_n for test functions with initial guess x_0 are displayed in tables. In fact, x_n is calculated by using the same total number of function

evaluations for all methods. Furthermore, the computational order of convergence approximations (COC) are displayed in tables. From the tables we observe that the COCs perfectly coincides with the theoretical result. However, this is the case when initial guess is reasonably close to the sought zeros.

6. Remarks and Conclusions

In this paper, the performance of the new one-point third-order Newton-type method has been demonstrated. The prime motive of presenting the new iterative method was to improve the classical Newton method. The effectiveness of the new third-order method by showing the accuracy of the simple root of a nonlinear equation have been examined. The main purpose of demonstrating the new Newton-type

method for several types of nonlinear equations was purely to illustrate the accuracy of the approximate solution, the stability of the convergence, the consistency of the results and to determine the efficiency of the new iterative method. It has been shown numerically and verified that the new Newton-type method has a convergence order of three. The main advantages are; very high computational efficiency, new method is not limited to the Kung and Traub conjecture, better efficiency index, one point iteration method and very competitive with the two-point third-order methods. Empirically, we have found a drawback of the new one-point third-order method that is only usable when the simple root is zero, hence further investigation is essential.

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