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A third-order Newton type method for nonlinear equations based on modified homotopy perturbation method

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Abstract

In this paper, we present a new iterative method for solving nonlinear algebric equations by using modified homotopy perturbation method. We also discuss the convergence criteria of the present method. To assess its validity and accuracy, the method is applied to solve several test problems.
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Keywords: Modified homotopy perturbation method; Nonlinear algebraic equations; Iterative methods

1. Introduction

The development of numerical techniques for solving nonlinear algebraic equations is a subject of considerable interest. There are many papers that deal with nonlinear algebraic equations, e.g., Golbabai and Javidi [1], Chun [2], Noor and Noor [3,4], Basto et al. [5], Abbasbandy [6], Babolian and Biazar [7], Jafari and Gejji [8], He [9]. A more extensive list of references as well as a survey on progress made on this class of problems may be found in Noor [10].

In the recent paper, a numerical method based on modified homotopy perturbation method is proposed for solving nonlinear equation $f(x) = 0$. The proposed method is applied to solve test problems in order to assess its validity and accuracy.

2. Modified homotopy perturbation method

The application of homotopy perturbation method in linear and nonlinear problems has been devoted by scientists and engineers, because this method is to continuously deform a simple problem which is easy to solve

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into the under study problem which is difficult to solve. This method was proposed first by He in 1997 and systematical description in 2000 which is, in fact, a coupling of the traditional perturbation method and homotopy in topology [11]. This method was further developed and improved by He and applied to nonlinear oscillators with discontinuities [12], nonlinear wave equations [13], asymptotology [14], boundary value problem [15], limit cycle and bifurcation of nonlinear problems [16] and many other subjects. Thus He's method is a universal one which can solve various kinds of nonlinear equations. After that many researchers applied the method to various linear and nonlinear problems: Abbasbandy [17], Ariel et al. [18], Ganji and Sadighi [19], Rafei and Ganji [20], Siddiqui et al. [21], Ghasemi et al. [22], El-Shahed [23], Javidi and Golbabai [24].

Consider the nonlinear algebraic equation

$$f(x) = 0, \ x \in R. \tag{1}$$

We assume that r is a simple zero of Eq. (1) and λ is an initial guess sufficiently close to r.

Using the Taylor's series around λ for Eq. (1), we have

$$f(\lambda) + (x-\lambda)f'(\lambda) + \frac{1}{2}(x-\lambda)^2 f''(\lambda) = 0. \tag{2}$$

We can rewrite Eq. (2) in the following form

$$x = c + N(x), \tag{3}$$

where

$$c = \lambda - \frac{f(\lambda)}{f'(\lambda)} \tag{4}$$

and

$$N(x) = -\frac{1}{2}(x-\lambda)^2 \frac{f''(\lambda)}{f'(\lambda)}. \tag{5}$$

To illustrate basic ideas of modified homotopy perturbation method, we construct a homotopy $\Theta : (R \times [0,1]) \times R \to R$ for Eq. (3), which satisfies

$$\Theta(\varpi, \beta, \theta) = \varpi - c - \beta N(\varpi) + \beta(1-\beta)\theta = 0, \quad \theta, \varpi \in R, \quad \beta \in [0,1], \tag{6}$$

where θ is an unknown real number and β is embedding parameter.

It is obvious that

$$\Theta(\varpi, 0, \theta) = \varpi - c = 0, \tag{7}$$

$$\Theta(\varpi, 1, \theta) = \varpi - c - N(\varpi) = 0. \tag{8}$$

The embedding parameter β monotonically increases from zero to unit as trivial problem $\Theta(\varpi, 0, \theta) = \varpi - c = 0$ is continuously deformed to original problem $\Theta(\varpi, 1, \theta) = \varpi - c - N(\varpi) = 0$. The modified HPM uses the homotopy parameter β as an expanding parameter to obtain [10–20]:

$$\varpi = x_0 + \beta x_1 + \beta^2 x_2 + \cdots \tag{9}$$

The approximate solution of Eq. (1), therefore, can be readily obtained:

$$r = \lim_{\beta \to 1} \varpi = x_0 + x_1 + x_2 + \cdots \tag{10}$$

The convergence of the series (10) has been proved by He in this paper [25].

For the application of modified HPM to (1) we can write (3) as follows by expanding $N(\varpi)$ into a Taylor series around x_0:

$$\varpi - c - \beta \left\{ N(x_0) + (\varpi - x_0)\frac{N'(x_0)}{1!} + (\varpi - x_0)^2 \frac{N''(x_0)}{2!} + \cdots \right\} + \beta(1-\beta)\theta = 0. \tag{11}$$

Substitution of (9) into (11) yields

$$x_0 + \beta x_1 + \beta^2 x_2 + \cdots - c - \beta\Big\{N(x_0) + (x_0 + \beta x_1 + \beta^2 x_2 + \cdots - x_0)\frac{N'(x_0)}{1!} + (x_0 + \beta x_1 + \beta^2 x_2 + \cdots - x_0)^2\frac{N''(x_0)}{2!} + \cdots\Big\} - \beta(1-\beta)\theta = 0. \tag{12}$$

By equating the terms with identical powers of β, we have

$$\beta^0 : x_0 - c = 0, \tag{13}$$

$$\beta^1 : x_1 - N(x_0) - \theta = 0, \tag{14}$$

$$\beta^2 : x_2 - x_1 N'(x_0) + \theta = 0, \tag{15}$$

$$\beta^3 : x_3 - x_2 N'(x_0) + \frac{1}{2}x_1^2 N''(x_0) = 0. \tag{16}$$

We try to find parameter θ, such that

$$x_2 = 0. \tag{17}$$

Hence by substituting $x_1 = N(x_0) + \theta$ from (14) into (15), we have

$$x_2 - (N(x_0) + \theta)N'(x_0) + \theta = 0. \tag{18}$$

Setting $x_2 = 0$ into (18) and solve it, we have

$$\theta = \frac{N(x_0)N'(x_0)}{1 - N'(x_0)}. \tag{19}$$

By substituting (19) into (14), we have

$$x_1 = N(x_0) + \frac{N(x_0)N'(x_0)}{1 - N'(x_0)} = \frac{N(x_0)}{1 - N'(x_0)}. \tag{20}$$

Now by substituting (13)–(16) into (10), we can obtain the zero of Eq. (1) as follows:

$$\begin{aligned} r &= x_0 + x_1 + x_2 + x_3 + \cdots = c + \frac{N(x_0)}{1 - N'(x_0)} - \frac{1}{2}\left(\frac{N(x_0)}{1 - N'(x_0)}\right)^2 N''(x_0) + \cdots \\ &= \lambda - \frac{f(\lambda)}{f'(\lambda)} - \frac{f^2(\lambda)f''(\lambda)}{2[f'^3(\lambda) - f(\lambda)f'(\lambda)f''(\lambda)]} + \frac{1}{2}\left(\frac{f^2(\lambda)f''(\lambda)}{2[f'^3(\lambda) - f(\lambda)f'(\lambda)f''(\lambda)]}\right)^2 \frac{f''(\lambda)}{f'(\lambda)} + \cdots \end{aligned} \tag{21}$$

This formulations allows us to suggest the following iterative method for solving nonlinear Eq. (1).

Algorithm 2.1. For a given ω_0, calculate the approximation solution ω_{n+1} by the iterative scheme

$$\omega_{n+1} = \omega_n - \frac{f(\omega_n)}{f'(\omega_n)} - \frac{f^2(\omega_n)f''(\omega_n)}{2[f'^3(\omega_n) - f(\omega_n)f'(\omega_n)f''(\omega_n)]}. \tag{22}$$

Algorithm 2.2. For a given ω_0, calculate the approximation solution ω_{n+1} by the iterative scheme

$$\omega_{n+1} = \omega_n - \frac{f(\omega_n)}{f'(\omega_n)} - \frac{f^2(\omega_n)f''(\omega_n)}{2[f'^3(\omega_n) - f(\omega_n)f'(\omega_n)f''(\omega_n)]} + \frac{1}{2}\left(\frac{f^2(\omega_n)f''(\omega_n)}{2[f'^3(\omega_n) - f(\omega_n)f'(\omega_n)f''(\omega_n)]}\right)^2 \frac{f''(\omega_n)}{f'(\omega_n)}. \tag{23}$$

We consider the convergence of Algorithm 2.1.

Definition 1. Let $e_n = \omega_n - r$ be the truncation error in the nth iterate. If there exists a number $k \geqslant 1$ and a constant $c \neq 0$ such that

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^k} = c, \tag{24}$$

then k is called the order of convergence of the method.

Theorem 2.1. *Consider the nonlinear equation $f(x) = 0$. Suppose f is sufficiently differentiable. Then for the iterative method defined by Eq. (22), the convergence is at least of order 3.*

Proof 1. Let r be a simple zero of f. Since f is sufficiently differentiable, by expanding $f(\omega_n), f'(\omega_n)$ and $f''(\omega_n)$ around r, we get

$$\begin{aligned} f(\omega_n) &= f'(r)\left[e_n + d_2e_n^2 + d_3e_n^3 + d_4e_n^4 + d_5e_n^5 \ \cdots\right], \\ f'(\omega_n) &= f'(r)\left[1 + 2d_2e_n + 3d_3e_n^2 + 4d_4e_n^3 + 5d_5e_n^4 + 6d_6e_n^5 \ \cdots\right], \\ f''(\omega_n) &= f'(r)\left[2d_2 + 6d_3e_n + 12d_4e_n^2 + 20d_5e_n^3 + 30d_6e_n^4 + 42d_7e_n^5 \ \cdots\right], \end{aligned} \tag{25}$$

where $d_n = \frac{1}{n!}\frac{f^{(n)}(r)}{f'(r)}$, $n = 1, 2, 3, \ldots$ and $e_n = \omega_n - r$. We can rewrite (22) as follows

$$e_{n+1} = e_n - \frac{f(\omega_n)}{f'(\omega_n)} - \frac{f^2(\omega_n)f''(\omega_n)}{2[f'^3(\omega_n) - f(\omega_n)f'(\omega_n)f''(\omega_n)]}. \tag{26}$$

Now from (25) and (26), we have

$$e_n - \frac{f(\omega_n)}{f'(\omega_n)} = e_n^2\frac{d_2 + 2d_3e_n + 3d_4e_n^2 + 4d_5e_n^3 + \cdots}{1 + 2d_2e_n + 3d_3e_n^2 + 4d_4e_n^3 + \cdots} \tag{27}$$

and

$$\frac{f^2(\omega_n)f''(\omega_n)}{2[f'^3(\omega_n) - f(\omega_n)f'(\omega_n)f''(\omega_n)]} = e_n^2\frac{2d_2 + (6d_3 + 4d_2^2)e_n + (12d_4 + 12d_2d_4)e_n^2 + (20d_5 + 24d_2d_4)e_n^3 + \cdots}{2 + d_2e_n + (12d_2^2 + 6d_3)e_n^2 + (16d_4 + 24d_2d_3 - 4d_3 + 8d_2^2)e_n^3 + \cdots} \tag{28}$$

Combining all the above terms, we have

$$\lim_{n\to\infty}\frac{|e_{n+1}|}{|e_n|^3} = d_3 = \frac{1}{6}\frac{f'''(r)}{f'(r)}, \tag{29}$$

which shows that Algorithm 2.1 is at least a third order convergent method, the required result. □

3. Test problems

We present some examples to illustrate the efficiency of the developed method in this paper. We compare the method of Babolian (BM) [7], the method of Abbasbandy (AM) [6], the method of Basto (BAM) [5] the method of Javidi (JM) [26] and the method of Golbabai and Javidi (GJM), introduced in this present paper by using Algorithm 2.1. Also, Newton–Raphson method (NR) and Adomian's method (ADM) are performed for comparison purposes.

Example 1. $x^3 + 4x^2 + 8x + 8 = 0$ with $x_0 = -1$. The exact solution prospected is $x = -2$. In Table 1, we list the results obtained by modified homotopy perturbation method. As we see from this table, it is clear that the result obtained by the present method is very superior to that obtained other methods.

Table 1
Numerical results for Example 1

Method	Number iterations	Obtained solution
NR	1	−2.000000000
ADM	–	Slow convergence
BM	–	Divergence
AM	2	−2.003987741
BAM	3	−2.000100903
JM	4	−1.9999225382
GJM	1	−2

Table 2
Numerical results for Example 1

Method	Number iterations	Obtained solution
NR	3	2.120028239
ADM	6	2.120013306
BM	4	2.120016168
AM	2	2.120028239
BAM	2	2.120028239
JM	2	2.120028278
JM	3	2.120028239
GJM	2	2.12002823898764

Example 2. $x - 2 - e^{-x} = 0$ with $x_0 = 2$. The exact solution prospected is $x = 2.12002823898764$. In Table 2, we list the results obtained by modified homotopy perturbation method. As we see from this table, it is clear that the result obtained by the present method is very superior to that obtained other methods.

Example 3. $x^2 - (1-x)^5 = 0$ with $x_0 = 0.2$. The exact solution prospected is $x = 0.34595481584824$. The numerical results given in Table 3.

Example 4. $e^x - 3x^2 = 0$ with $x_0 = 0$ for Adomian's, Javidi's and Babolian's methods and $x_0 = 0.5$ for the remainder. The exact solution prospected is $x = 0.91000757248871$. With $x_0 = 0$, the exact solution prospected for Newton–Raphson, Abbasbandy and new iterative method (14), is $x = -0.45896226753695$. The numerical results given in Table 4.

Table 3
Numerical results for Example 3

Method	Number iterations	Obtained solution
NR	3	0.345953774
ADM	10	0.340622225
BM	5	0.346021366
AM	2	0.345954646
BAM	2	0.345952189
JM	2	0.346930007
JM	4	0.345954816
GJM	2	0.34595218921176
GJM	3	0.34595481584824

Table 4
Numerical results for Example 4

Method	Number iterations	Obtained solution
NR, $x_0 = 0.5$	4	0.910007662
NR, $x_0 = 0$	5	−0.458962274
ADM, $x_0 = 0$	10	0.904938647
BM, $x_0 = 0$	6	0.9032577054
AM, $x_0 = 0.5$	4	0.910007573
AM, $x_0 = 0$	5	−0.458964191
BAM, $x_0 = 0.5$	3	0.910007573
BAM, $x_0 = 0$	2	−0.458992962
JM, $x_0 = 0$	5	−0.458938090
JM, $x_0 = 0$	6	−0.458962268
GJM, $x_0 = 0$	2	−0.45899296202335
GJM, $x_0 = 0$	3	−0.45896226753695
GJM, $x_0 = 0.5$	2	0.91001094056187
GJM, $x_0 = 0.5$	3	0.91000757248871

Table 5
Numerical results for Example 5

Method	Number iterations	Obtained solution
JM	3	−1.40310806911862
JM	4	−1.40449014569029
JM	5	−1.40449164821357
JM	6	−1.40449164821534
GJM	1	−1.44545149431052
GJM	2	−1.40448794117752
GJM	3	−1.40449164821534
GJM	4	−1.40449164821534

Example 5. $\sin^2(x) - x^2 + 1 = 0$ with $x_0 = -1$. The exact solution prospected is $x = -1.40449164821534$. The numerical results given in Table 5.

4. Conclusions

Modified homotopy perturbation method is applied to numerical solution for solving nonlinear algebraic equations. Comparison of the result obtained by the present method with that obtained by different methods [5–7,26] and Newton–Raphson method (NR) and Adomian's method (ADM) reveals that the present method is very effective and convenient.

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