

# A Six-order Modification of Newton's Method For Solving Nonlinear Equations

Manoj Kumar Singh and S. R. Singh

**Abstract**—A new six order variant of Newton's method has been developed by extending a third order method of Ozban [14] and its convergence properties have been discussed. The order of convergence of the proposed method is six. Starting with a suitably chosen  $x_0$ , the method generates a sequence of iterates converging to the root. The convergence analysis is provided to establish its sixth order of convergence. In terms of computational cost, it requires evaluations of only two functions and two first order derivatives per iteration. This implies that efficiency index of our method is 1.5651. Proposed method is comparable with the methods of Parhi, and Gupta [15] and that of Kou and Li [7]. It does not require the evaluation of the second order derivative of the given function as required in the family of Chebyshev–Halley type methods. The efficiency of the method is tested on a number of numerical examples. It is observed that our method takes less number of iterations than Newton's method and the other third order variants of Newton's method. On comparison with the sixth order methods, it behaves either similarly or better for the examples considered. Copyright © 2011 Yang's Scientific Research Institute, LLC. All rights reserved.

**Index Terms**—Newton's method, Iteration function, Order of convergence, Function evaluations, Efficiency index.

## I. INTRODUCTION

IN Science and Engineering, many of the nonlinear and transcendental problems of the form  $f(x) = 0$ , are complex in nature. Since it is not always possible to obtain its exact solution by usual algebraic process, therefore numerical iterative methods such as Newton, secant methods are often used to obtain the approximate solution of such problems. Though these methods are very effective, but there are some limitations that they do not give the result as fast as one want, and take several iterations. These methods can also be used to find local maxima [http://en.wikipedia.org/wiki/Maxima\\_and\\_minima](http://en.wikipedia.org/wiki/Maxima_and_minima) and local minima of functions, as these extrema are the roots of the derivative function. There are many methods developed on the improvement of quadratically convergent Newton's method, so as to get a superior convergence order than Newton. This paper is concerned with the iterative methods for finding a simple root  $\alpha$ , i.e.  $f(\alpha) = 0$ , and  $f'(\alpha) \neq 0$  of  $f(x) = 0$ , where  $f : R \rightarrow R$ , be the continuously differentiable real function. Here Symbols used have their usual meanings.

We consider the problem of finding a real zero of a function  $f : I \subset R \rightarrow R$ . This zero can be determined as a fixed point

of some iteration function  $g$  by means of the one-point iteration method

$$x_{n+1} = g(x_n), n = 0, 1, \dots$$

where  $x_0$  is the starting value. The best known and the most widely used example of these types of methods is the classical Newton's method given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, \dots \quad (1)$$

It converges quadratically to simple zeros and linearly to multiple zeros. In the literature, some of its modifications have been introduced in order to accelerate it or to get a method with a higher order of convergence. If we consider the definition of efficiency index as  $p^{1/m}$ , where  $p$  is the order of the method and  $m$  is the number of functions evaluation required by the method (units of work per iteration) then the efficiency index of this method is 1.414. A number of ways are considered by many researchers to improve the local order of convergence of Newton's method at the expense of additional evaluations of functions, derivatives and changes in the points of iterations.

All these modifications are in the direction of increasing the local order of convergence with the view of increasing their efficiency indices. For example, the method developed by Weerakoon [18], called as trapezoidal Newton's or arithmetic mean Newton's method, suggests for some other variants of Newton's method. Frontini and Sormani [3] developed new modifications of Newton's method to produce iterative method with order of convergence of three and efficiency index of 1.442. With the same efficiency index, Ozban [14] also gave some new variants of Newton's method, and Chen [1] described some new iterative formulae having third order convergence.

Ostrowski [13] developed both third and fourth order methods each requiring evaluations of two functions and one derivative per iteration. Maheshwari [11] developed a fourth order iterative method for solving non linear equations. Recently a number of sixth order methods are also appearing as the extensions of above mentioned methods to solve  $f(x) = 0$ . Sharma and Guha [16] developed a one parameter family of sixth order methods based on Ostrowski fourth order multipoint method. Each family required three evaluations of the given function and one evaluation of the derivative per iteration. Chun [2] presented a one parameter family of variants of Jarratt's fourth order method for solving nonlinear equations. It show there that the order of convergence of each family member is improved from order four to six even though it adds one evaluation of the function at the point

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iterated by Jarratt's method per iteration. Kou et al. [6]–[9] presented a family of new variants of Chebyshev–Halley methods and also an improvement of Jarratt method. These new methods have sixth order of convergence although they only add one evaluation of the function at the point iterated by Chebyshev–Halley method and Jarratt method. Parhi and Gupta [15] developed a sixth order method for nonlinear equations, by extending a third order method of Weerakoon and Fernando [18], requires evaluations of two functions and two first derivative per iteration.

Our method also requires evaluations of two functions and two first derivatives per iteration, is comparable with the methods of Kou and Li [7] and that of Parhi and Gupta [15]. It does not require the evaluation of the second order derivative of the given function as required in Chebyshev–Halley type methods [6], [9]. The efficiency of the method is tested on a number of numerical examples. It is observed that the proposed method takes less number of iterations than taken by Newton's method and the other third order variant of Newton's method. On comparison with the other sixth order methods, it behaves either similarly or better for the examples considered.

## II. DEFINITIONS

**Definition II.1.** [5], [12], [17] If the sequence  $\{x_n/n \geq 0\}$  tends to a limit  $\alpha$  in such a way that

$$\lim_{x_n \rightarrow \alpha} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} = C. \quad (2)$$

For some  $C \neq 0$  and  $p \geq 1$ , then the order of convergence of the sequence is said to be  $p$ , and is known as asymptotic error constant.

When  $p = 1$ , the convergence is linear, and it is called first order convergence.

While for  $p = 2$  and  $p = 3$  the sequence is said to converge quadratically and cubically, respectively.

The value of  $p$  is called the order of convergence of the method which produces the sequence  $\{x_n/n \geq 0\}$ . Let  $e_n = x_n - \alpha$  Then the relation  $e_{n+1} = Ce_n^p + O(e_n^{p+1})$  is called the error equation for the method,  $p$  being the order of convergence.

**Definition II.2.** [12], [17] Efficiency index is simply defined as  $p^{1/m}$  where  $p$  is the order of the method and  $m$  is the number of functions evaluations required by the method (units of work per iteration). Therefore the efficiency index of Newton's method is 1.414 and that of third order method is 1.442 and efficiency index of method proposed by me is 1.5651.

## III. DESCRIPTION OF METHODS

Let  $\alpha$  be a simple zero of a sufficiently differentiable function  $f$  and consider the numerical solution of the equation  $f(x) = 0$ , then it can be written as

$$f(x) = f(x_n) + \int_{x_n}^x f'(t) dt. \quad (3)$$

Suppose, we interpolate  $f'$  on the interval  $[x_n, x]$  by constant  $f'(x_n)$ , then  $(x - x_n)f'(x_n)$  provides an approximation for the integral in Eq. (3) and by taking  $x = \alpha$  we obtain

$$0 \approx f(x_n) + (\alpha - x_n)f'(x_n).$$

And hence, a new approximation  $x_{n+1}$  to  $\alpha$  is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \dots$$

Which is Newton's method for  $n = 0, 1, \dots$ . On the other hand, if we approximate the indefinite integral in Eq. (3) by the trapezoidal rule and take  $x = \alpha$ , we obtain

$$0 \approx f(x_n) + 1/2(\alpha - x_n)(f'(x_n) + f'(\alpha)).$$

And therefore, a new approximation  $x_{n+1}$  to  $\alpha$  is given by

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_n + 1)}. \quad (4)$$

If the value of Newton's method is used on the right-hand side of the above equation to overcome the implicity problem, then

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(z_{n+1})},$$

where  $z_{n+1} = x_n - f(x_n)/f'(x_n)$  is obtained which is, for  $n = 0, 1, 2, \dots$ , the trapezoidal Newton's method of Weerakoon [18]. Let us rewrite Eq. (4) as

$$x_{n+1} = x_n - \frac{f(x_n)}{(f'(x_n) + f'(z_{n+1}))/2}, \quad n = 0, 1, 2, \dots$$

And further to re-write as

$$z_n = x_n - \frac{f(x_n)}{(f'(x_n) + f'(y_n))/2}, \quad n = 0, 1, 2, \dots \quad (5)$$

So this variant of Newton's method can be viewed as obtained by using arithmetic mean of  $f'(x_n)$  and  $f'(y_n)$  instead of  $f'(x_n)$  in Newton's method defined by Eq. (1), which was called as *arithmetic mean Newton's method*.

### New Variant of Newton's Method

In Eq. (5) if we use the harmonic mean instead of arithmetic mean we get

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad (6)$$

$$z_n = x_n - \frac{f(x_n)}{2} \left( \frac{1}{f'(x_n)} + \frac{1}{f'(y_n)} \right). \quad (7)$$

Again

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \quad n = 0, 1, \dots \quad (8)$$

Now using the linear interpolation on two points  $(x_n, f'(x_n))$  and we get  $(y_n, f'(y_n))$  We get

$$f'(x) \approx \frac{x - x_n}{y_n - x_n} f'(y_n) + \frac{x - y_n}{x_n - y_n} f'(x_n). \quad (9)$$

Thus an approximation to  $f'(z_n)$  is given by

$$f'(z_n) \approx \frac{z_n - x_n}{y_n - x_n} f'(y_n) + \frac{z_n - y_n}{x_n - y_n} f'(x_n).$$

$$f'(z_n) \approx \frac{2f'(x_n)f'(y_n) - f'(x_n)^2 + f'(y_n)^2}{2f'(y_n)}. \quad (10)$$

From Eqs. (9)(11) our proposed method can be written as follows

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{f(x_n)}{2} \left( \frac{1}{f'(x_n)} + \frac{1}{f'(y_n)} \right) \\ x_{n+1} &= z_n - \frac{f(z_n)[2f'(y_n)]}{[2f'(x_n)f'(y_n) - f'(x_n)^2 + f'(y_n)^2]}. \end{aligned} \quad (11)$$

Clearly this method requires evaluations of two functions  $f$  and two derivative  $f'$  and no second order derivative  $f''$  of  $f$ .

#### IV. CONVERGENCE ANALYSIS

**Theorem IV.1.** Let  $\alpha \in I$  be a simple zero of a sufficiently differentiable function  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  for an open interval  $I$ . If  $x_0$  is sufficiently close to  $\alpha$ , then the methods defined by Eq. (11) has six order convergence.

*Proof.* Let  $\alpha \in I$  be a simple zero of  $f$ . Since  $f$  is sufficiently differentiable, by expanding  $f(x_n)$  and  $f'(x_n)$  about  $\alpha$  we get

$$\begin{aligned} f(x_n) &= f(\alpha + e_n) = f'(\alpha)e_n + f''(\alpha)\frac{e_n^2}{2!} + f'''(\alpha)\frac{e_n^3}{3!} + O(e_n^4) \\ &= f'(\alpha)[e_n + C_2e_n^2 + C_3e_n^3 + O(e_n^4)] \end{aligned} \quad (12)$$

where  $C_j = (1/j!)f^{(j)}(\alpha)/f'(\alpha)$ .  $\square$

$$\begin{aligned} f'(x_n) &= f'(\alpha + e_n) = f'(\alpha) + f''(\alpha)e_n + f'''(\alpha)\frac{e_n^2}{2!} + O(e_n^4) \\ &= f'(\alpha)[1 + 2C_2e_n + 3C_3e_n^2 + 4C_4e_n^3 + O(e_n^4)] \\ \frac{f(x_n)}{f'(x_n)} &= [e_n + C_2e_n^2 + C_3e_n^3 + O(e_n^4)][1 + 2C_2e_n + 3C_3e_n^2 + 4C_4e_n^3 + O(e_n^4)]^{-1} \\ &= [e_n - C_2e_n^2 + (2C_2^2 - 2C_3)e_n^3 + O(e_n^4)]. \end{aligned} \quad (13)$$

Newton method is

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ y_n &= \alpha + C_2e_n^2 + (2C_3 - 2C_2^2)e_n^3 + O(e_n^4). \end{aligned}$$

That is Newton's method converges quadratically.

Again we have

$$\begin{aligned} y_n &= \alpha + C_2e_n^2 + (2C_3 - 2C_2^2)e_n^3 + O(e_n^4), \\ f'(y_n) &= f'(\alpha)[1 + \{2C_2e_n^2 + 4C_2(C_3 - C_2^2)e_n^3 + O(e_n^4)\}]. \end{aligned} \quad (14) \quad (15)$$

Harmonic mean Newton's method

$$z_n = x_n - \frac{f(x_n)}{2} \left( \frac{1}{f'(x_n)} + \frac{1}{f'(y_n)} \right)$$

We have

$$e_{n+1} = \frac{1}{2}C_3e_n^3 + O(e_n^4). \quad (16)$$

Again

$$f'(x_n)f'(y_n) = f'(\alpha)^2[1 + 2C_2e_n + (2C_2^2 + 3C_3)e_n^2 + 4(C_2C_3 + C_4)e_n^3 + O(e_n^4)]$$

$$f'(x_n)^2 = f'(\alpha)^2[1 + 4C_2e_n + (4C_2^2 + 6C_3)e_n^2 + (8C_4 + 12C_2C_3)e_n^3 + O(e_n^4)] \quad (17)$$

$$f'(y_n)^2 = f'(\alpha)^2[1 + 4C_2^2e_n^2 + (8C_2C_3 - 8C_2^3)e_n^3 + O(e_n^4)] \quad (18)$$

$$\begin{aligned} f'(z_n) &\approx \frac{2f'(x_n)f'(y_n) - f'(x_n)^2 + f'(y_n)^2}{2f'(y_n)} \\ f'(z_n) &\approx f'(\alpha)[1 - (2C_2C_3)e_n^3 + O(e_n^4)]. \end{aligned} \quad (19)$$

Hence from

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}. \quad (20)$$

We get Eq. (21) in the next page.

On solving we have

$$\begin{aligned} x_{n+1} &= \alpha - \left(\frac{3}{4}C_2C_3^2\right)e_n^6 + O(e_n^7), \\ e_{n+1} &= -\left(\frac{3}{4}C_2C_3^2\right)e_n^6 + O(e_n^7). \end{aligned} \quad (22)$$

Hence the proposed method defined by Eq. (12) has six order convergence and when second order derivative or third order derivative will be zero then the order of convergence increase up to seven or eight.

**Example - 1** Consider the equation  $f(x) = x^3 + 4x^2 - 10$ .

We start with  $x_n = 2$ . The results obtained by Newton iteration, Maheshwari iteration and present iteration method is shown in Table I.

**Example - 2** Consider the equation,  $f(x) = x^3 - e^{-x}$ .

We start with  $x_n = 1$ . The results obtained by Newton iteration, Maheshwari iteration and present iteration method is shown in Table I and Fig. 1.

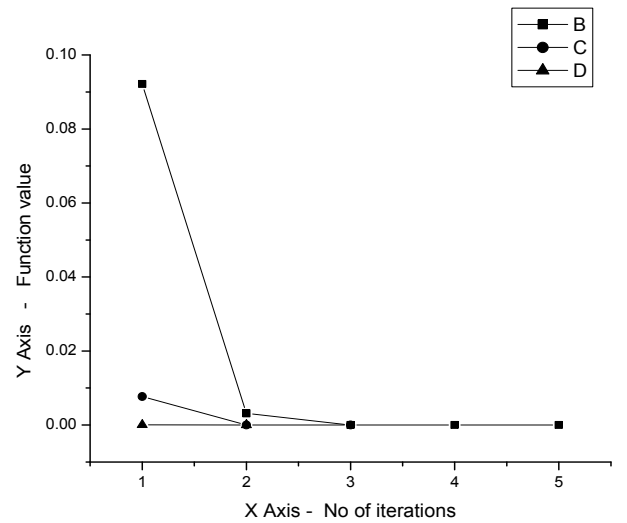


Fig. 1: B: Newton method, C: Maheshwari method, D: Proposed method.

**Example - 3** Consider the equation  $f(x) = \sin(x)$ .

$$\begin{aligned}
x_{n+1} &= \alpha + \frac{1}{2}C_3e_n^3 + O(e_n^4) - \frac{f(\alpha + \frac{1}{2}C_3e_n^3 + O(e_n^4))}{f'(\alpha)[1 - (2C_2C_3)e_n^3 + O(e_n^4)]} \\
&= \alpha + \frac{1}{2}C_3e_n^3 + O(e_n^4) - \frac{f(\alpha) + \{\frac{1}{2}C_3e_n^3 + O(e_n^4)\}f'(\alpha) + \{\frac{1}{2}C_3e_n^3 + O(e_n^4)\}^2\frac{f''(\alpha)}{2!}}{f'(\alpha)[1 - (2C_2C_3)e_n^3 + O(e_n^4)]} \\
&= \alpha + \frac{1}{2}C_3e_n^3 + O(e_n^4) - \frac{[f'(\alpha) \cdot \{\frac{1}{2}C_3e_n^3 + O(e_n^4)\} + \{\frac{1}{2}C_3e_n^3 + O(e_n^4)\}^2C_2]}{f'(\alpha)[1 - (2C_2C_3)e_n^3 + O(e_n^4)]}.
\end{aligned} \tag{21}$$

We start with  $x_n = 1.5$ . The results obtained by Newton iteration, Maheshwari iteration and present iteration method is shown in Table I.

**Example - 4** Consider the equation  $f(x) = \sin(x) - 0.5x$ .

We start with  $x_n = 1.6$ . The results obtained by Newton iteration, Maheshwari iteration and present iteration method is shown in Table I.

TABLE I: COMPARISON WITH NEWTON AND MAHESHWARI METHOD

Example 1		
Newton iteration for solving $f(x) = x^3 + 4x^2 - 10$		
$n$	$x_n$	$f(x_n)$
1	1.5000000000000000	2.3750000000000000
2	1.3733333333333333	0.134345481481482
3	1.365262014874627	5.284611795151051e-004
4	1.365230013916147	8.290548692002631e-009
5	1.365230013414097	0.0000000000000000
Maheshwari iteration for solving $f(x) = x^3 + 4x^2 - 10$		
$n$	$x_n$	$f(x_n)$
1	1.383460112875796	0.303737459638828
2	1.365230059872621	7.671881707693729e-007
3	1.365230013414097	0.0000000000000000
Proposed iteration for solving $f(x) = x^3 + 4x^2 - 10$		
$n$	$x_n$	$f(x_n)$
1	1.365293603219183	0.001050116565054
2	1.365230013414097	0.0000000000000000
Example 2		
Newton iteration for solving $f(x) = x^3 - e^{-x}$		
$n$	$x_n$	$f(x_n)$
1	0.812309030097381	0.092166771534313
2	0.774276548985500	0.003144824978613
3	0.772884756209622	4.050085547491200e-006
4	0.772882959152202	6.742550962002269e-012
5	0.772882959149210	0.0000000000000000
Maheshwari iteration for solving $f(x) = x^3 - e^{-x}$		
$n$	$x_n$	$f(x_n)$
1	0.776281164264182	0.007682769461876
2	0.772882959509889	8.128708817167762e-010
3	0.772882959149210	0.0000000000000000
Proposed iteration for solving $f(x) = x^3 - e^{-x}$		
$n$	$x_n$	$f(x_n)$
1	0.772890464773999	1.691572695283750e-005
2	0.772882959149210	0.0000000000000000

Example 3

Newton iteration for solving, $f(x) = \sin(x)$		
$n$	$x_n$	$f(x_n)$
1	-12.601419947171721	-0.035042157161019
2	12.566356255118672	1.435924050063516e-005
3	-12.566370614359174	-1.286498119741309e-015
Maheshwari iteration for solving, $f(x) = \sin(x)$		
$n$	$x_n$	$f(x_n)$
1	-12.140250122502410	0.413341334176534
2	-12.567623187401553	-0.001252572714845
3	-12.566370614359172	4.898587196589413e-016
Proposed iteration for solving $f(x) = \sin(x)$		
$n$	$x_n$	$f(x_n)$
1	-6.287519145437536	-0.004333824691493
2	-6.283185307179586	2.449293598294706e-016

Example 4

Newton iteration for solving $f(x) = \sin(x) - 0.5x$		
$n$	$x_n$	$f(x_n)$
1	1.977123551007066	-0.069983138933437
2	1.898950910895084	-0.002836729003200
3	1.895501147295299	-5.635111423818451e-006
4	1.895494267061370	-2.243205621255129e-011
5	1.895494267033981	0.0000000000000000
Maheshwari iteration for solving $f(x) = \sin(x) - 0.5x$		
$n$	$x_n$	$f(x_n)$
1	1.925586494937329	-0.025073843759850
2	1.895494852455987	-4.794739714153451e-007
3	1.895494267033981	0.0000000000000000
Proposed iteration for solving $f(x) = \sin(x) - 0.5x$		
$n$	$x_n$	$f(x_n)$
1	1.895869187902239	-3.071352434207419e-004
2	1.895494267033981	0.0000000000000000

In Table II, we give the number of iterations (N) and the number of function evaluations (NOFE) required to satisfy the stopping criterion, F denotes that method fails and D denotes for divergence. PM denotes proposed method. CN denotes for classical Newton method, AN-Arithmetic mean Newton method [18], HN- Harmonic mean Newton method, GN- Geometric mean Newton method [10], MN-Mid-point Newton method.

In Table III, we give the number of iterations (N) and the number of function evaluations (NOFE) required satisfying the stopping criterion, F denotes that method fails, D denotes for divergence. PM denotes proposed method. Kou and Li is the method [7]. An improvement of Jarratt method. Gupta denotes for [15] Parhi and Gupta, A sixth order method for nonlinear

TABLE II: COMPARISON WITH THIRD ORDER METHOD

$F(x)$	$x_0$	N						NOFE					
		CN	AN	HN	GN	MN	PM	CN	AN	HN	GN	MN	PM
(a)	1	5	3	3	3	3	2	10	9	9	9	9	8
	2	5	3	3	3	3	2	10	9	9	9	9	8
(b)	-3	6	5	3	7	3	2	12	15	9	21	9	8
	-2	5	5	3	6	3	2	10	15	9	18	9	8
(c)	2	5	5	3	F	4	2	10	15	9	-	12	8
	-3	18	D	17	D	D	12	54	-	51	-	-	8
(d)	2	4	3	3	D	3	2	08	9	9	-	9	8
	2.5	5	3	3	D	3	2	10	9	9	-	9	8
(e)	3	5	4	3	D	4	2	10	12	9	-	12	8
	0.5	5	4	3	13	3	2	10	12	9	39	9	8
(f)	1	5	3	3	15	3	2	10	9	9	45	9	8
	2	7	4	4	20	4	3	14	12	12	60	12	12
(g)	0.3	3	3	3	29	3	2	6	9	9	87	9	8
	0.5	5	3	3	29	3	2	10	9	9	87	9	8
(h)	0.55	5	3	4	26	4	2	10	9	12	53	12	8
	0.1	7	6	6	188	4	4	18	12	9	564	12	12
(i)	2.5	6	4	3	31	4	2	21	15	12	93	12	8
	3.5	7	5	4	37	5	3	21	18	18	111	15	12
(j)	0.5	43	D	21	D	D	13	86	-	63	-	-	52
	0.8	9	D	5	10	6	3	18	-	15	30	18	12
(k)	2	5	4	4	D	3	2	10	12	12	-	9	8
	3	6	4	4	D	4	2	12	12	12	-	12	8
(l)	4	7	5	5	D	5	3	14	14	15	15	-	12
	1	232	D	79	D	D	57	696	-	237	-	-	40
(m)	3.5	14	9	8	D	9	5	42	27	24	-	27	24
	4.5	25	17	14	D	16	9	75	51	42	-	48	40

equations.

In Table II and Table III the following test functions have been used

- (a)  $x^3 + 4x^2 - 10$ ,  $\alpha = 1.365230013414097$ ,
- (b)  $\sin^2 x - x^2 + 1$ ,  $\alpha = -1.404491648215341$ ,
- (c)  $x^3 - 10$ ,  $\alpha = 2.154434690031884$
- (d)  $x^3 - e^{-x}$ ,  $\alpha = 0.772882959149210$
- (e)  $x \sin(1/x) - 0.2e^{-x}$ ,  $\alpha = 0.363715708657122$
- (f)  $(x - 1)^3 - 1$ ,  $\alpha = 2$ ,
- (g)  $(x - 2)^{23} - 1$ ,  $\alpha = 3$ ,
- (h)  $x^2 - e^x - 3x + 2$ ;  $\alpha = 0.257530285439861$ .

Hence from the Tables 1, II, III, the main observations are as follows:

- 1) Present method takes lesser number of iterations than the others.
- 2) Example shows that the present method requires lesser number of functional evaluations, as compared to other methods.
- 3) The results in the last column of the selected examples, shows the absolute deviation of  $f(x)$  in each step with respect to its targeted value, i.e. zero. It can be observed from these examples that the value of absolute error is least for the present method. [Table I, example 3].

Thus, the present method is not only faster but the cost effecting parameters obtain in examples show that it has minimum cost among all the methods taken here.

## V. NUMERICAL RESULTS AND CONCLUSIONS

In this section, we present the results of some numerical tests to compare the efficiencies of the proposed method. We employed CN method, Maheshwari method, some third order methods — AN method of Fernando et al. [2], HN, GN, MN methods and some six order methods [7] and [15]. Numerical computations reported here have been carried out in MATLAB and the stopping criterion has been taken as  $|x_{n+1} - \alpha| + |f(x_{n+1})| < 10^{-14}$ .

Thus the proposed, sixth order method for finding simple real roots of nonlinear equations, is free from second order derivative of the given function, as required in the family of Chebyshev–Halley type methods. Method can also be used to find local maxima [http://en.wikipedia.org/wiki/Maxima\\_and\\_minima](http://en.wikipedia.org/wiki/Maxima_and_minima) and local minima of functions as these extrema are the roots of the derivative function. Our method requires evaluations of two functions and two first order derivatives per iteration. The convergence analysis of the method is performed in much simpler way to show that the order of convergence of the method is six. The high order convergence is also corroborated by numerical tests. Method has the efficiency index equal to 1.5651, which is better to Newton's method with efficiency index equal to

TABLE III: COMPARISON WITH SIX ORDER METHOD

$F(x)$	$x_0$	N			NOFE		
		Kou and Li	Gupta	PM	Kou and Li	Gupta	PM
(a)	1	2	2	2	8	8	8
	2	2	2	2	8	8	8
(b)	-3	2	15	2	8	60	8
	-2	2	30	2	8	120	8
	2	2	2	2	8	8	8
	3	2	3	2	8	12	8
(c)	2	2	2	2	8	8	8
	2.5	2	2	2	8	8	8
	3	2	2	2	8	8	8
(d)	0.5	2	2	2	8	8	8
	1	2	2	2	8	8	8
	2	3	3	3	8	12	12
(e)	0.3	2	2	2	8	8	8
	0.5	2	2	2	8	8	8
	0.55	2	2	2	8	8	8
(f)	0.1	5	4	3	20	16	12
	2.5	2	2	2	8	8	8
	3.5	3	3	3	12	12	12
(g)	0.5	12	13	13	48	52	52
	0.8	3	3	3	12	12	12
(h)	2	2	2	2	8	8	8
	3	2	2	2	8	8	8
	4	3	3	3	12	12	12
(i)	1	180	57	57	720	228	40
	3.5	5	5	5	20	20	24
	4.5	8	9	9	32	36	40

1.414 and the classical third order methods (1.442), such as Weerakoon and Fernando method, Chebyshev's method, Halley's method and Super-Halley method, fifth order method (1.495) of Kou, Li, Wang [8] and Grau, Noguera [4] and six order method (1.565) of Parhi and Gupta [15] and Kou and Li [7]. The method is tested on a number of numerical examples. On comparing our results with those obtained by Newton's method (NM) third order methods and fourth order methods, it is found that our method is most effective as it converges to the root much faster. When compared with the sixth order methods of Parhi, Gupta [15] and Kou and Li [7], our method behaves either similarly or better on the examples considered.

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