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Power means based modification of Newton's method for solving nonlinear equations with cubic convergence

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ABSTRACT

A class of Newton-type method known as trapezoidal power means Newton method for solving nonlinear equation is proposed in this paper. The new class of method incorporates power means in the trapezoidal integration rule along with midpoint evaluation, thus replacing the denominator term in the classical Newton's method. The order of convergence of these methods is shown to be three. The efficiency index of the proposed method is found to be 1.442. Numerical examples and their results are provided to compare the efficiency of the new methods with few other similar methods.

Keywords: Iterative Method; Newton's Method; Non-linear equation; Power mean; Third order convergence.

1. Introduction

Let us consider the problem finding a simple root of a single non-linear equation of the form:

$$f(x) = 0 \tag{1.1}$$

where $f : I \subseteq R \to R$ for an open interval I is a scalar function. Solution of non-linear equation (1.1) has been given much attention because of its application in many branches of science and engineering. It is well known that Newton's method for solving (1.1) has quadratic convergence. Cubic convergence without second derivative was first established in [11] using arithmetic mean Newton's method. This trend continued in [1, 7, 10 and 13] using either arithmetic mean or midpoint Newton's method. Nedzhibov [9] gave many classes of iterative methods using different quadrature rules for solving (1.1). Modifications in the Newton's method based on harmonic mean or geometric mean are suggested in [1, 5, 7, 8, 10 and 13]. Different formulation of Simpson integration of Newton's theorem are proposed in [1, 2, 3, 6 and 13]. Xiaojian [12] gave a class of Newton's methods based on power means by using the trapezoid formula and this idea was extended to Simpson's formula using harmonic mean by Jayakumar et al [6]. Herceg et al [4] considered a family of six sets of means based modifications of Newton's method. In all the above papers, the proposed methods used only the first derivative of $f(x)$ and established third order convergence.

In this paper, the idea of power means [12] is used along with midpoint evaluation of the derivative of f in the trapezoidal formula [9], thus modifying the classical Newton's method. In Section 2, some definitions for our study are given. Section 3 presents the derivation of the new method and Section 4 its analysis of convergence. Finally, Section 5 gives numerical results and discussion.

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2. Some Definitions

Definition 2.1 [11]: Let $\alpha \in \Re$, $x_n = 0, 1, 2, ...$ Then the sequence $\{x_n\}$ is said to converge to the root α, if $\lim_{n\to\infty} |x_n - \alpha| = 0$. If, in addition, there exist a constant $C \geq 0$, an integer $n_0 \geq 0$ and $q \geq 0$, such that for all $n > n_0$, $|x_{n+1} - \alpha| \leq C|x_n - \alpha|^q$, then $\{x_n\}$ is said to converge to α with order at least q. If q = 2 or 3, the convergence is said to be quadratic or cubic, respectively. When $e_n = x_n - \alpha$ is the error in the n^{th} iterate, the relation $e_{n+1} = Ce_n^q + O(e_n^{q+1})$ is called the error equation. The value of q thus obtained is called the order of this method.
Definition 2.2 [1]: The Efficiency Index of an iterative method is given by

$$EI = q^{\frac{1}{d}}, \tag{2.1}$$

where q is the order of the method and d is the total number of new function evaluations (the values of f and its derivatives) per iteration.
Definition 2.3 [11]: Let α be a root of the function (1.1) and suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α. Then, the computational order of convergence (COC) denoted by ρ can be approximated using the formula

$$\rho = \frac{\ln |(x_{n+1} - \alpha)/(x_n - \alpha)|}{\ln |(x_n - \alpha)/(x_{n-1} - \alpha)|}.$$

Definition 2.4 [12]: Let a and b be positive scalars. For a finite real number p, the p-power mean of a and b is defined as

$$m_p = \left(\frac{a^p + b^p}{2}\right)^{1/p}.$$

Setting $p = 1, -1, 2, -2$ and $1/2$, we get several particular cases of well-known means as given below:

$$m_1 = \left(\frac{a+b}{2}\right), \; m_{-1} = \left(\frac{a^{-1} + b^{-1}}{2}\right)^{-1}, \; m_2 = \left(\frac{a^2 + b^2}{2}\right)^{1/2},$$

$$m_{-2} = \left(\frac{a^{-2} + b^{-2}}{2}\right)^{-1/2}, \; m_{1/2} = \left(\frac{\sqrt{a} + \sqrt{b}}{2}\right)^2,$$

which are respectively called arithmetic mean, harmonic mean, root mean square, inverse root inverse-square mean and square-mean root of a and b. We also include the case p = 0. That is, $m_0(a, b) = \lim_{p\to 0} m_p(a, b) = \sqrt{ab}$, which is called Geometric mean of a and b.

3. Derivation of the method

Let α be a simple root of a sufficiently differentiable function $f(x)$. Consider the numerical solution of the equation $f(x) = 0$. From the Newton's theorem, we have

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt. \tag{3.1}$$

Let us substitute the integral in (3.1) with the quadrature formula of the trapeziums [9]:

$$\int_{x_n}^{x} f'(t)dt \approx \frac{x - x_n}{2m}\left(f'(x) + 2\sum_{i-1}^{m-1} f'\left(x_n + i\frac{x - x_n}{m}\right) + f'(x_n)\right). \tag{3.2}$$

Simplifying and computing the iteration value for $x = x_{n+1}$ we obtain from equation (3.2)

$$x_{n+1} = x_n - \frac{2mf(x_n)}{\left(f'(x_{n+1}) + 2\sum_{i-1}^{m-1} f'\left(x_n - \frac{if(x_n)}{mf'(x_n)}\right) + f'(x_n)\right)}. \tag{3.3}$$

As equation (3.3) is implicit, x_{n+1} is replaced with x^*_{n+1}, where

$$x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 1, 2, 3... \tag{3.4}$$

For m = 1 in (3.3), we get the method of [11]:

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x^*_{n+1})}.$$

For m = 2 in (3.3), we get one of the methods of found in [9]:

$$x_{n+1} = x_n - \frac{4f(x_n)}{f'(x_n) + 2f'(\frac{x_n + x^*_{n+1}}{2}) + f'(x^*_{n+1})}. \tag{3.5}$$

To obtain the present method, we rewrite equation (3.5) as

$$x_{n+1} = x_n - \frac{2f(x_n)}{\frac{f'(x_n) + f'(x^*_{n+1})}{2} + f'(\frac{x_n + x^*_{n+1}}{2})}. \tag{3.6}$$

Replacing arithmetic mean with p-power means in the denominator of the above equation, we obtain a new class of method called Trapezoidal Power Means Newton's Method $(TPMN)$:

$$x_{n+1} = x_n - \frac{2f(x_n)}{sign(f'(x_n))\left(\frac{f'(x_n)^p + f'(x^*_{n+1})^p}{2}\right)^{1/p} + f'(\frac{x_n + x^*_{n+1}}{2})}, n = 1, 2, 3.... \tag{3.7}$$

where p is a finite real number and x^*_{n+1} is calculated from (3.4). The efficiency index of the current method is found to be $EI = 1.442$ which is greater than the efficiency index of Newton's method (1.414).

4. Analysis of Convergence

Theorem 4.1: Let $\alpha \in I$, be a simple zero of a sufficiently differentiable function $f : I \subseteq R \rightarrow R$ for an open interval I. If x_0 is sufficiently close to α then the method (3.7) has third order convergence.
Proof.
Let α be a simple zero of the function $f(x) = 0$ (That is, $f(\alpha) = 0$ and $f'(\alpha) \neq 0$). By expanding $f(x_n)$ and $f'(x_n)$ by Taylor series about α, we obtain

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)] \tag{4.1}$$

and

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + O(e_n^4)], \tag{4.2}$$

where $c_k = \frac{f^{(k)}(\alpha)}{k! f'(\alpha)}$, $k = 2, 3, 4...$ From equation (3.4), we get

$$x^*_{n+1} = \alpha + c_2 e_n^2 - 2(c_2^2 - c_3)e_n^3 + O(e_n^4). \tag{4.3}$$

Expanding $f'(x^*_{n+1})$ by Tailor's series about α and using (4.3), we get

$$f'(x^*_{n+1}) = f'(\alpha)[1 + 2c_2^2 e_n^2 + 4(c_2c_3 - c_2^3)e_n^3 + O(e_n^4)]. \tag{4.4}$$

Further, we have

$$\frac{x^*_{n+1} + x_n}{2} = \alpha + \frac{1}{2}e_n + \frac{1}{2}c_2e_n^2 - (c_2^2 - c_3)e_n^3 + O(e_n^4). \tag{4.5}$$

Again expand $f'(\frac{x^*_{n+1}+x_n}{2})$ by Tailor's series and use (4.5) to get

$$f'(\frac{x^*_{n+1} + x_n}{2}) = f'(\alpha)[1 + c_2e_n + (c_2^2 + \frac{3}{4}c_3)e_n^2 + (-2c_2^3 + \frac{7}{2}c_2c_3 + \frac{1}{2}c_4)e_n^3 + O(e_n^4)]. \tag{4.6}$$

From equations (4.2) and (4.4) respectively, we have

$$f'(x_n)^p = f'(\alpha)^p[1 + 2pc_2e_n + (3pc_3 + 2p(p-1)c_2^2)e_n^2 + O(e_n^3)], \tag{4.7}$$

$$f'(x^*_{n+1})^p = f'(\alpha)^p[1 + 2pc_2^2e_n^2 + O(e_n^3)]. \tag{4.8}$$

From (4.7) and (4.8), we get

$$sign(f'(x_n))\left(\frac{f'(x_n)^p + f'(x^*_{n+1})^p}{2}\right)^{1/p} = f'(\alpha)[1+c_2e_n+1/2(c_2^2+pc_2^2+3c_3)e_n^2+O(e_n^3)]. \tag{4.9}$$

Adding (4.9) and (4.6), we get

$$\begin{aligned} &sign(f'(x_n))\left(\frac{f'(x_n)^p + f'(x^*_{n+1})^p}{2}\right)^{1/p} + f'\left(\frac{x^*_{n+1} + x_n}{2}\right) \\ &\quad = 2f'(\alpha)\Big[1 + c_2e_n + \Big(\frac{3}{4}c_2^2 + \frac{p}{4}c_2^2 + \frac{9}{8}c_3\Big)e_n^2 + O(e_n^3)\Big]. \end{aligned} \tag{4.10}$$

Equations (4.1) and (4.10) are substituted in (3.7) to obtain

$$x_{n+1} = x_n - \frac{2f'(\alpha)\Big[e_n + c_2e_n^2 + c_3e_n^3 + O(e_n^4)\Big]}{2f'(\alpha)\Big[1 + c_2e_n + \Big(\frac{3}{4}c_2^2 + \frac{p}{4}c_2^2 + \frac{9}{8}c_3\Big)e_n^2 + O(e_n^3)\Big]}.$$

Further simplification yields

$$x_{n+1} = x_n - \Big[e_n - \Big(\frac{3}{4}c_2^2 + \frac{p}{4}c_2^2 + \frac{1}{8}c_3\Big)e_n^3 + O(e_n^4)\Big].$$

Since $e_n = x_n - \alpha$, we finally obtain

$$e_{n+1} = \Big(\frac{1}{4}(3+p)c_2^2 + \frac{1}{8}c_3\Big)e_n^3 + O(e_n^4),$$

which shows the method (3.7) has third order convergence.

Table 1: Number of Iterations

$f(x)$	x_0	NM	AN	TPMN							
				$p=1$	$p=-1$	$p=2$	$p=-2$	$p=1/2$	$p=0$	$p=3$	$p=-3$
$f_1(x)$	3	7	5	5	5	5	5	5	5	5	4
	4	8	5	5	5	5	5	5	5	6	5
$f_2(x)$	3.5	15	10	10	9	10	9	10	10	11	9
	4.5	26	18	18	16	18	15	17	17	19	15
$f_3(x)$	-3	15	10	10	9	10	9	10	9	11	9
	-2	9	6	6	6	6	6	6	6	7	6
$f_4(x)$	3	7	5	5	5	5	4	5	5	5	4
	3.5	8	7	6	5	6	5	6	5	6	4
$f_5(x)$	2	7	5	5	5	5	4	5	5	5	4
	4	7	5	5	5	5	5	5	5	5	4
$f_6(x)$	3.5	13	9	9	8	9	8	9	8	9	8
	4	20	14	13	12	14	12	13	13	14	11
$f_7(x)$	1.5	6	4	4	4	4	4	4	4	4	4
	2	7	NC	5	5	5	5	5	5	5	5

x_0 - Initial point and NC - Not convergent.

5. Numerical Examples and Discussion

In this section, we give numerical results on some test functions to demonstrate the order of convergence and efficiency of the new method. Also, we compare the results of the trapezoidal power means Newton methods $(TPMN)$ (for p = 1, -1, 2, -2, 1/2, 0, 3 and -3) with Newton's Method (NM) and Arithmetic Newton's method (AN). Numerical computations have been done using the MATLAB software rounding to 16 significant decimal digits. Depending on the precision (ϵ) of the computer, we use the following stopping criteria for the iterative process for our results: $|x_{n+1} - x_n| + |f(x_n)| < \epsilon$, where $\epsilon = 10^{-14}$.
The following test functions are used in the numerical results:

$$f_1(x) = x^3 + 4x^2 - 10, \quad \alpha = 1.365230013414097...$$
$$f_2(x) = (x-2)^{23} - 1, \quad \alpha = 3$$
$$f_3(x) = xe^{x^2} - sin^2x + 3cosx + 5, \quad \alpha = -1.207647827130919...$$
$$f_4(x) = ln(x-1), \quad \alpha = 2$$
$$f_5(x) = e^x + x - 20, \quad \alpha = 2.842438953784447...$$
$$f_6(x) = e^{x^2+7x-30}, \quad \alpha = 3$$
$$f_7(x) = x^2 sinx - cosx, \quad \alpha = 0.895206045384232...$$

Discussion: In this work, a class of Newton methods based on p-power means [12] and trapezoidal integration formula is proposed as a modification of Newton's method. By theoretical error analysis and numerical computations we have established third order convergence. The proposed new class of methods has the advantage of evaluating only the first order derivative and possesses third order convergence. It is found that the present methods perform better in terms number of iterations (particularly for p = -1, -2 and -3) and has higher rate of convergence when compared to Newton's method. The efficiency index of the current method is found to be $EI = 1.442$ which is greater than the efficiency index of Newton's method (1.414). It is observed that for p = 1, we get one of the method found in [9]. We can easily extend these methods to single equation with multiple roots and for system of non-linear equations with multiple variables, which will be taken-up for study separately.

Table 2: Computational Order of Convergence

$f(x)$	x_0	NM	AN	TPMN							
				$p=1$	$p=-1$	$p=2$	$p=-2$	$p=1/2$	$p=0$	$p=3$	$p=-3$
$f_1(x)$	3	2.00	2.93	2.93	2.97	2.91	3.00	2.94	2.95	2.89	3.18
	4	2.00	2.77	2.78	2.91	2.74	2.99	2.82	2.85	3.00	3.29
$f_2(x)$	3.5	2.00	2.82	2.93	2.93	2.83	3.02	2.97	2.97	2.99	3.17
	4.5	2.00	2.88	2.99	2.98	2.89	3.02	2.94	3.00	3.00	3.22
$f_3(x)$	-3	2.00	2.95	2.99	3.00	2.96	3.03	3.00	2.95	3.00	3.06
	-2	2.00	2.91	2.96	3.00	2.93	3.02	2.97	3.00	3.00	3.07
$f_4(x)$	3	2.00	2.97	3.00	3.00	3.00	2.70	2.98	2.98	2.97	2.52
	3.5	2.00	3.00	2.98	2.96	2.97	2.98	2.99	2.91	2.96	2.59
$f_5(x)$	2	2.00	3.03	3.01	3.00	3.03	2.91	3.01	3.00	3.03	2.87
	4	2.00	3.00	2.97	2.99	2.96	3.00	2.98	3.00	2.96	3.38
$f_6(x)$	3.5	2.00	2.96	3.00	3.00	2.96	3.01	2.99	2.91	2.93	3.08
	4	2.00	2.96	2.89	2.96	2.97	3.01	2.95	3.00	2.88	3.37
$f_7(x)$	1.5	2.00	2.37	2.46	2.52	2.43	2.54	2.47	2.49	2.40	2.91
	2	2.00	NC	3.17	3.13	3.16	3.04	3.17	3.17	3.14	3.36

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