New Newton's Method with Third-order Convergence for Solving Nonlinear Equations

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Abstract—For the last years, the variants of the Newton's method with cubic convergence have become popular iterative methods to find approximate solutions to the roots of non-linear equations. These methods both enjoy cubic convergence at simple roots and do not require the evaluation of second order derivatives. In this paper, we present a new Newton's method based on contra harmonic mean with cubically convergent. Numerical examples show that the new method can compete with the classical Newton's method.

Keywords—Third-order convergence, Non-linear equations, Root-finding, Iterative method.

I. Introduction

SOLVING non-linear equations is one of the most important problems in numerical analysis. In this paper, we consider iterative methods to find a simple root of a non-linear equation $f(x) = 0$, where $f : D \subset \mathbf{R} \rightarrow \mathbf{R}$ for an open interval D is a scalar function. The classical Newton method for a single non-linear equation is written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \quad (1)$$

This is an important and basic method [8], which converges quadratically. Recently, some modified Newton methods with cubic convergence have been developed in [1], [2], [3], [4], [5], [6] and [7]. Here, we will obtain a new modification of Newtons method. Analysis of convergence shows the new method is cubically convergent. Its practical utility is demonstrated by numerical examples.

Let α be a simple zero of a sufficiently differentiable function f and consider the numerical solution of the equation $f(x) = 0$. It is clear that

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt. \quad (2)$$

Suppose we interpolate f' on the interval $[x_n, x]$ by the constant $f'(x_n)$, then $(x - x_n)f'(x_n)$ provides an approximation for the indefinite integral in (2) and by taking $x = \alpha$ we obtain

$$0 \approx f(x_n) + (\alpha - x_n)f'(x_n).$$

Thus, a new approximation x_{n+1} to α is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

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On the other hand, if we approximate the indefinite integral in (2) by the trapezoidal rule and take $x = \alpha$, we obtain

$$0 \approx f(x_n) + \frac{1}{2}(\alpha - x_n)(f'(x_n) + f'(\alpha)),$$

and therefore, a new approximation x_{n+1} to α is given by

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_{n+1})}.$$

If the Newton's method is used on the right-hand side of the above equation to overcome the implicity problem, then

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(z_{n+1})}, \quad (3)$$

where

$$z_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

is obtained which is, for n = 0, 1, 2 , . . . , the trapezoidal Newton's method of Fernando et al. [1]. Let us rewrite equation (3) as

$$x_{n+1} = x_n - \frac{f(x_n)}{(f'(x_n) + f'(z_{n+1}))/2}, \quad n = 0, 1, ... \quad (4)$$

So, this variant of Newton's method can be viewed as obtained by using arithmetic mean of $f'(x_n)$ and $f'(z_{n+1})$ instead of $f'(x_n)$ in Newton's method defined by (1). Therefore, we call it arithmetic mean Newton's (AN) method.

In [3], the harmonic mean instead of the arithmetic mean is used to get a new formula

$$x_{n+1} = x_n - \frac{f(x_n)(f'(x_n) + f'(z_{n+1}))}{2f'(x_n)f'(z_{n+1})}, \quad n = 0, 1, ... \quad (5)$$

which is called *harmonic mean Newton's* (HN) *method* and used the midpoint to get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'((x_n + z_{n+1})/2)}, \quad n = 0, 1, ... \quad (6)$$

which is called *midpoint Newton's* (MN) *method*.

II. New iterative method and convergence analysis

If we use the contra harmonic mean instead of the arithmetic mean in (4), we get new Newton method

$$x_{n+1} = x_n - \frac{f(x_n)(f'(x_n) + f'(z_{n+1}))}{f'^2(x_n) + f'^2(z_{n+1})}, n = 0, 1, ... \quad (7)$$

where

$$z_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, ... \tag{8}$$

we call *contra harmonic Newton's* (CHN) *method.*

Theorem 2.1: *Let $\alpha \in D$ be a simple zero of a sufficiently differentiable function $f : D \subset \boldsymbol{R} \to \boldsymbol{R}$ for an open interval D. If x_0 is sufficiently close to α, then the methods defined by (7) converge cubically.*

Proof Let α be a simple zero of f. Since f is sufficiently differentiable, by expanding $f(x_n)$ and$f'(x_n)$ about α we get

$$f(x_n) = f'(\alpha)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + ...\right], \tag{9}$$

and

$$f'(x_n) = f'(\alpha)\left[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + ...\right], \tag{10}$$

where $c_k = (1/k!) f^{(k)}(\alpha)/f'(\alpha), k = 2, 3, ...$ and $e_n = x_n - \alpha$. Direct division gives us

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4),$$

and hence, for z_{n+1} given in (8) we have

$$z_{n+1} = \alpha + c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + O(e_n^4). \tag{11}$$

Again expanding $f'(z_{n+1})$ about α and using (11) we obtain

$$\begin{aligned} f'(z_{n+1}) &= f'(\alpha) + (z_{n+1} - \alpha)f''(\alpha) \\ &\quad + \tfrac{(z_{n+1}-\alpha)}{2!} f'''(\alpha) + ... \\ &= f'(\alpha) + [c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + O(e_n^4)]f''(\alpha) \\ &\quad + O(e_n^4) \\ &= f'(\alpha)[1 + 2c_2^2 e_n^2 + 4(c_2 c_3 - c_2^3)e_n^3 + O(e_n^4)]. \end{aligned} \tag{12}$$

By using (10) we obtain

$$f'^2(x_n) = f'^2(\alpha)$$

$$[1 + 4c_2 e_n + \left(4c_2^2 + 6c_3\right)e^2 + \left(12c_2 c_3 + 8c_4\right)e^3 + ...].$$

From (12), we get

$$f'^2(z_{n+1}) = f'^2(\alpha)\left[1 + 4c_2^2 e_n^2 + \left(8c_2 c_3 - 8c_2^3\right)e^3 + ...\right],$$

and

$$\begin{aligned} f'^2(x_n) + f'^2(z_{n+1}) = 2f'^2(\alpha)[1 + 2c_2 e_n + \left(4c_2^2 + 3c_3\right)e_n^2 \\ + \left(4c_4 + 10c_2 c_3 - 4c_2^3\right)e_n^3 + ...]. \end{aligned}$$

From (10) and (12) we get

$$\begin{aligned} f'(x_n) + f'(z_{n+1}) = 2f'(\alpha)[1 + c_2 e_n + \left(c_2^2 + \frac{3}{2}c_3\right)e_n^2 + \\ 2\left(c_2 c_3 - c_2^3 + c_4\right)e_n^3 + O(e_n^4)], \end{aligned}$$

and using (9) to get

$$f(x_n)\left(f'(x_n) + f'(z_{n+1})\right) =$$

$$2f'^2(\alpha)[e_n + 2c_2 e_n^2 + \left(2c_2^2 + \frac{5}{2}c_3\right)e_n^3 + O(e_n^4)].$$

Hence,

$$\frac{f(x_n)\left(f'(x_n) + f'(z_{n+1})\right)}{f'^2(x_n) + f'^2(z_{n+1})} = e_n - \left(2c_2^2 + \frac{1}{2}c_3\right)e_n^3 + O(e_n^4),$$

$$x_{n+1} = x_n - \frac{f(x_n)\left(f'(x_n) + f'(z_{n+1})\right)}{f'^2(x_n) + f'^2(z_{n+1})},$$

$$x_{n+1} = x_n - \left(e_n - \left(2c_2^2 + \frac{1}{2}c_3\right)e_n^3 + O(e_n^4)\right),$$

or subtracting α from both sides of this equation we get

$$e_{n+1} = \left(2c_2^2 + \frac{1}{2}c_3\right)e_n^3 + O(e_n^4),$$

which shows that contra harmonic Newton's method is of third order.

III. Numerical results and conclusions

In this section, we present the results of some numerical tests to compare the efficiencies of the new method (CHN). We employed (CN) method, (AN) method of Fernando et al.[1], and (HN) and (MN) methods in [3] . Numerical computations reported here have been carried out in a MTHEMATICA environment . The stopping criterion has been taken as $|x_{n+1} - x_n| < \varepsilon$, We used the fixed stopping criterion $\varepsilon = 10^{-14}$ and the following test functions have been used.

$f_1(x) = x^3 + 4x^2 - 10, \quad \alpha = 1.365230013414097,$
$f_2(x) = x^2 - e^x - 3x + 2, \quad \alpha = 0.2575302854398608,$
$f_3(x) = Sinx^2 - x^2 + 1, \quad \alpha = 1.404491648215341,$
$f_4(x) = Cosx - x, \quad \alpha = 0.7390851332151607,$
$f_5(x) = (x-1)^3 - 1, \quad \alpha = 2.$

In Table 1 and Table 2, we give the number of iterations (N) and total number of function evaluations (TNFE) required to satisfy the stopping criterion. As far as the numerical results are considered, for most of the cases HN and MN methods requires the least number of function evaluations.

All numerical results are in accordance with the theory and the basic advantage of the variants of Newton's method based on means or integration methods that they do not require the computation of second- or higher-order derivatives although they are of third order.

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TABLE I
ITERATION NUMBER (N)

$f(x)$	N CN	AN	HN	MN	CHN
$f_1, x_0 = 1$	6	4	4	4	5
$f_2, x_0 = 1$	5	4	4	4	4
$f_2, x_0 = 2$	6	5	5	4	5
$f_2, x_0 = 3$	7	5	5	5	5
$f_3, x_0 = 1$	7	5	4	5	5
$f_3, x_0 = 3$	7	5	4	4	5
$f_4, x_0 = 1$	5	3	4	4	4
$f_4, x_0 = 1.7$	5	4	4	4	4
$f_4, x_0 = -0.3$	6	4	5	5	5
$f_5, x_0 = 3$	7	5	5	5	5

TABLE II
THE TOTAL NUMBER OF FUNCTION EVALUATIONS (TNFE)

$f(x)$	TNEF CN	AN	HN	MN	CHN
$f_1, x_0 = 1$	12	12	12	12	15
$f_2, x_0 = 1$	10	12	12	12	12
$f_2, x_0 = 2$	12	15	15	15	15
$f_2, x_0 = 3$	14	15	15	15	15
$f_3, x_0 = 1$	14	15	12	15	15
$f_3, x_0 = 3$	14	15	12	12	15
$f_4, x_0 = 1$	10	9	12	12	12
$f_4, x_0 = 1.7$	10	12	12	12	12
$f_4, x_0 = -0.3$	12	12	15	15	15
$f_5, x_0 = 3$	14	15	15	15	15

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