A New Modification of Newton's Method

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Abstract

A new iterative modification of Newton's method for solving nonlinear scalar equations are proposed. Weerakoon and Fernando have been propose a variant of Newton's method in which they approximate the indefinite integral by a trapezoid instead of a rectangle. This modification has third-order convergence. We approximate the indefinite integral by a Simpson's formula instead of a trapezoid and obtain a new modification with third-order convergence. This new variant achieves less error than the Newton's method and the Weerakoon's and the Fernando's variant in special cases. It is showed that the convergence order of our method is three.

Keywords: Newton's method, nonlinear equations, iterative methods, order of convergence.

1 Introduction

It is well known that the iterative Newton's formula for numerical solving a nonlinear equation computes the function value the value of its first derivative. The Newton's method converges to the root quadratically.

Weerakoon and Fernando [1] have suggested an improvement to the iteration of Newton's method. They have approximated the indefinite integral by a trapezoid instead of a rectangle and the results is a new method with third-order convergence.

In this paper we propose a modification the Newton's iteration in which we approximate the integral by a Simpson formula. This new modification has third order convergence also.

2 Preliminaries

We use the following definitions [1].

Definition 1 *Let $\alpha \in R, x_n \in R, n = 0, 1, 2, \ldots$. Then the sequence $\{x_n\}$ is said to convergence to α if*

$$\lim_{n\to\infty} |x_n - \alpha| = 0.$$

If there exists a constant $c \geq 0$ an integer $n_0 \geq 0$ and $p \geq 0$ such that for all $n > n_0$ we have

$$|x_{n+1} - \alpha| \leq c|x_n - \alpha|^p,$$

then $\{x_n\}$ is said to convergence to α with convergence order at least p. If $p = 2$ or $p = 3$ the convergence is said to be quadratic or cubic, respectively.

The notation $e_n = x_n - \alpha$ is the error in the n^{th} iteration. The equation

$$e_{n+1} = ce_n^p + O(e_n^{p+1})$$

is called the error equation and the parameter p is called the order of this method.

Definition 2 *Let α be a root of the function $f(x)$ and suppose that x_{n-1}, x_n, x_{n+1} are three consecutive iterations closer to the root α. Then the computational order of convergence p can be computed using the formula*

$$p \approx \frac{\ln|(x_{n+1}-\alpha)/(x_n-\alpha)|}{\ln|(x_n-\alpha)/(x_{n-1}-\alpha)|}. \tag{1}$$

3 Iterative methods

3.1 Newton's method

The Newton's method for computing the α of the nonlinear equation $f(x) = 0$ uses the following iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2}$$

where x_n is the n-th approximation of α. It is well known this method is quadratically convergent.

The solution of the equation $M(x) = 0$ is denoted x_{n+1} where

$$M(x) = f(x_n) + f'(x_n)(x - x_n). \tag{3}$$

The equation (3) is the tangent's equation to the function $f(x)$ to point x_n.

If at the formula

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(\lambda)\, d\lambda \tag{4}$$

the integral is approximated by the rectangle, i.e.

$$\int_{x_n}^{x} f'(\lambda)\, d\lambda \approx f'(x_n)(x - x_n),$$

we obtain the linear equation (3).

3.2 The Weerakoon's and the Fernando's modification

Weerakoon and Fernando [1] have been proposed the following variant

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x^*_{n+1})}, \tag{5}$$

where $x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. The equation (5) is received from (4) when the integral is approximated by trapezoid. This method has third-order convergence.

3.3 A New Modification of Newton's method

We approximate the integral by Simpson's formula which is

$$\int_{x_n}^{x} f'(\lambda)\, d\lambda \approx \frac{x - x_n}{6}\left\{f'(x) + 4f'\left(\frac{x + x_n}{2}\right) + f'(x_n)\right\}.$$

We obtain the equation

$$\hat{M}_n(x) = f(x_n) + \frac{x - x_n}{6}\left\{f'(x) + 4f'\left(\frac{x + x_n}{2}\right) + f'(x_n)\right\}. \tag{6}$$

We find the solution x_{n+1} of the equation $\hat{M}_n(x_{n+1}) = 0$

$$f(x_n) + \frac{x_{n+1} - x_n}{6}\left\{f'(x_{n+1}) + 4f'\left(\frac{x_{n+1} + x_n}{2}\right) + f'(x_n)\right\} = 0$$

and obtain

$$x_{n+1} = x_n - \frac{6f(x_n)}{f'(x_{n+1}) + 4f'\left(\frac{x_{n+1}+x_n}{2}\right) + f'(x_n)} \tag{7}$$

In order to compute x_{n+1} it is used values of the derivative of the function at points x_{n+1} and $\frac{x_{n+1}+x_n}{2}$. Further on we replace x_{n+1} with x^*_{n+1} and use Newton's iterative step to compute x^*_{n+1}.

$$x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Define

$$\frac{x^*_{n+1}+x_n}{2} = \frac{1}{2}\left[x_n - \frac{f(x_n)}{f'(x_n)} + x_n\right] = x_n - \frac{f(x_n)}{2f'(x_n)}.$$

and obtain the formula

$$x_{n+1} = x_n - \frac{6f(x_n)}{f'(x^*_{n+1}) + 4f'(x^{**}_{n+1}) + f'(x_n)}, \tag{8}$$

where

$$x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad x^{**}_{n+1} = x_n - \frac{f(x_n)}{2f'(x_n)}.$$

Theorem 1 *Let f be a real function. Assume that $f(x)$ has first, second and third derivatives in the interval (a, b). If $f(x)$ has a simple root an $\alpha \in (a, b)$ and x_0 is sufficiently close to α, then the iterative method (8) satisfies the following error equation:*

$$e_{n+1} = (C_2)^2\, e_n^3 + O(e_n^4),$$

where $e_n = x_n - \alpha$ and $C_2 = \frac{1}{2}\frac{f^{(2)}(\alpha)}{f'(\alpha)}$.

Proof: Let α be a simple root of $f(x)$, i.e. $f(\alpha) = 0,\ f'(\alpha) \neq 0$. Assume $x_n = \alpha + e_n$,. We use the Taylor expansions

$$\begin{aligned} f(x_n) = f(\alpha + e_n) &= f(\alpha) + f'(\alpha)e_n + \frac{1}{2!}f^{(2)}(\alpha)e_n^2 + \frac{1}{3!}f^{(3)}(\alpha)e_n^3 + O(e_n^4) \\ &= f'(\alpha)\left[e_n + \frac{1}{2!}\frac{f^{(2)}(\alpha)}{f'(\alpha)}e_n^2 + \frac{1}{3!}\frac{f^{(3)}(\alpha)}{f'(\alpha)}e_n^3 + O(e_n^4)\right] \\ &= f'(\alpha)\left[e_n + C_2e_n^2 + C_3e_n^3 + O(e_n^4)\right], \end{aligned}$$

where $C_j = \frac{1}{j!}\frac{f^{(j)}(\alpha)}{f'(\alpha)},\quad j = 2, 3$. We have

$$\begin{aligned} f'(x_n) = f'(\alpha + e_n) &= f'(\alpha) + f^{(2)}(\alpha)e_n + \frac{1}{2!}f^{(3)}(\alpha)e_n^2 + O(e_n^3) \\ &= f'(\alpha)\left[1 + \frac{f^{(2)}(\alpha)}{f'(\alpha)}e_n + \frac{1}{2!}\frac{f^{(3)}(\alpha)}{f'(\alpha)}e_n^2 + O(e_n^3)\right] \\ &= f'(\alpha)\left[1 + 2C_2e_n + 3C_3e_n^2 + O(e_n^3)\right]. \end{aligned}$$

Compute

$$\begin{aligned} \frac{f(x_n)}{2f'(x_n)} &= \frac{1}{2}\left[e_n + C_2e_n^2 + C_3e_n^3 + O(e_n^4)\right]\left[1 + 2C_2e_n + 3C_3e_n^2 + O(e_n^3)\right]^{-1} \\ &= \frac{1}{2}\left[e_n + C_2e_n^2 + O(e_n^3)\right] \\ &\quad \times\left\{1 - \left[2C_2e_n + 3C_3e_n^2 + O(e_n^3)\right] + \left[2C_2e_n + 3C_3e_n^2 + O(e_n^3)\right]^2 - \cdots\right\} \\ &= \frac{1}{2}\left[e_n + C_2e_n^2 + O(e_n^3)\right]\left\{1 - \left[2C_2e_n + 3C_3e_n^2 + O(e_n^3)\right] + 4C_2^2e_n^2 + \cdots\right\} \\ &= \frac{1}{2}e_n - \frac{1}{2}C_2e_n^2 + O(e_n^3), \end{aligned}$$

Thus we obtain for x^*_{n+1} and x^{**}_{n+1}

$$\begin{aligned}
x^*_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
&= \alpha + e_n - \left[e_n - C_2 e_n^2 + O(e_n^3)\right] = \alpha + C_2 e_n^2 + O(e_n^3), \\
x^{**}_{n+1} &= x_n - \frac{f(x_n)}{2f'(x_n)} \\
&= \alpha + e_n - \left[\frac{1}{2}e_n - \frac{1}{2}C_2 e_n^2 + O(e_n^3)\right] = \alpha + \frac{1}{2}e_n + \frac{1}{2}C_2 e_n^2 + O(e_n^3).
\end{aligned}$$

Compute

$$\begin{aligned}
f'(x^*_{n+1}) &= f'(\alpha) + \left[C_2 e_n^2 + O(e_n^3)\right] f^{(2)}(\alpha) + O(e_n^4) \\
&= f'(\alpha)\left\{1 + \left[2C_2 e_n^2 + O(e_n^3)\right]\left[\frac{f^{(2)}(\alpha)}{2f'(\alpha)}\right]\right\} \\
&= f'(\alpha)\left\{1 + 2C_2^2 e_n^2 + O(e_n^3)\right\}, \\
f'(x^{**}_{n+1}) &= f'(\alpha) + \left[\frac{1}{2}e_n + \frac{1}{2}C_2 e_n^2 + O(e_n^3)\right] f^{(2)}(\alpha) \\
&\quad + \left[\frac{1}{2}e_n + \frac{1}{2}C_2 e_n^2 + O(e_n^3)\right]^2 \frac{f^{(3)}(\alpha)}{2!} + O(e_n^3) \\
&= f'(\alpha)\left\{1 + \left[e_n + C_2 e_n^2 + O(e_n^3)\right] C_2 + \left[\frac{1}{2}e_n + \frac{1}{2}C_2 e_n^2 + O(e_n^3)\right]^2 3C_3 + O(e_n^3)\right\} \\
&= f'(\alpha)\left\{1 + \left[e_n + C_2 e_n^2 + O(e_n^3)\right] C_2 + \left[\frac{1}{4}e_n^2 + O(e_n^3)\right] 3C_3 + O(e_n^3)\right\} \\
&= f'(\alpha)\left\{1 + C_2 e_n + \left[C_2^2 + \frac{3}{4}C_3\right] e_n^2 + O(e_n^3)\right\},
\end{aligned}$$

$$\begin{aligned}
f'(x_n) + 4f'(x^{**}_{n+1}) + f'(x^*_{n+1}) &= f'(\alpha)\left[1 + 2C_2 e_n + 3C_3 e_n^2 + O(e_n^3)\right] \\
&\quad + 4f'(\alpha)\left[1 + C_2 e_n + \left(C_2^2 + \frac{3}{4}C_3\right) e_n^2 + O(e_n^3)\right] \\
&\quad + f'(\alpha)\left[1 + 2C_2^2 e_n^2 + O(e_n^3)\right] \\
&= 6f'(\alpha)\left[1 + C_2 e_n + (C_3 + C_2^2) e_n^2 + O(e_n^3)\right].
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{6f(x_n)}{f'(x_n) + 4f'(x^{**}_{n+1}) + f'(x^*_{n+1})} &= \frac{e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4)}{1 + C_2 e_n + (C_3 + C_2^2)\, e_n^2 + O(e_n^3)} \\
&= \left[e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4)\right] \\
&\quad \times \{1 - \left[C_2 e_n + \left(C_3 + C_2^2\right) e_n^2 + O(e_n^3)\right] \\
&\quad\quad + \left[C_2 e_n + \left(C_3 + C_2^2\right) e_n^2 + O(e_n^3)\right]^2 - \cdots\} \\
&= \left[e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4)\right]\left\{1 - C_2 e_n - C_3 e_n^2 + O(e_n^3)\right\} \\
&= e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4) - C_2 e_n^2 - C_2^2 e_n^3 - C_3 e_n^3 + O(e_n^4) \\
&= e_n - C_2^2 e_n^3 + O(e_n^4).
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
e_{n+1} + \alpha &= e_n + \alpha - \left[e_n - C_2^2 e_n^3 + O(e_n^4)\right], \\
e_{n+1} &= C_2^2 e_n^3 + O(e_n^4).
\end{aligned}$$

The theorem is proved. $\square$

The new modification has third-order convergence. We use notations
MN - the Newton's method;
VT - the Weerakoon and the Fernando [1] variant;
VS - the proposed modification.
Table 1 shows iterative formulas of the methods and their error equations.

Table 1

MN	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$e_{n+1} = C_2 e_n^2 + O(e_n^3)$
VT	$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)+f'(x^*_{n+1})}$, $x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$e_{n+1} = \left(C_2^2 + \frac{1}{2}C_3\right) e_n^3 + O(e_n^4)$,
VS	$x_{n+1} = x_n - \frac{6f(x_n)}{f'(x_n)+4f'(x^{**}_{n+1})+f'(x^*_{n+1})}$, $x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $x^{**}_{n+1} = x_n - \frac{f(x_n)}{2f'(x_n)}$	$e_{n+1} = (C_2)^2 e_n^3 + O(e_n^4)$,

4 Numerical experiments

We were done numerical experiments for different functions and initial points. All programs were writen in MATLAB. We compare three iterative procedures for computing the root of nonlinear equations.

We use the following stopping criteria for computer programs

$$(i) \qquad |x_{n+1} - x_n| < \sqrt{\varepsilon},$$
$$(ii) \qquad |f(x_{n+1})| < \sqrt{\varepsilon},$$

where $\varepsilon = 2.22\ e - 16$ is a MATLAB constant.

We introduce notations: x_0 - a initial point; $iter$ - number of iterations for which (i) and (ii) are satisfied; $time_{500}$ - the execution time for 500 times execution; p - convergence order by (1).

Table 2

$f(x)$	x_0	$iter$			$time_{500}$		
		MN	VT	VS	MN	VT	VS
1) $x^3 + x^2 - 2$	-1.5	–	8	4	-	3.24	2.41
	1.5	5	3	3	1.92	1.96	1.92
	3	6	4	4	2.13	2.03	2.36
2) $\cos x - x$	2	3	3	2	1.59	1.64	1.64
	3	6	8	3	2.19	3.61	1.98
	1.7	4	3	2	1.75	1.86	1.86
	-1	7	3	3	2.03	1.69	1.69
	4	39	6	4	9.16	2.57	1.87
3) $(x-1)^3 - 1$	2.5	5	3	3	1.75	1.75	1.87
	4	7	4	4	2.31	2.03	2.19
	-0.5	15	15	6	3.42	4.78	2.91
	-1	9	7	4	2.64	2.69	2.20
	-2	10	8	6	3.14	3.24	3.14
4) $xe^{x^2} - \sin^2 x$ $+3\cos x + 5$	-3	16	13	12	6.48	6.76	8.11
	1.2	-	28	11	-	13.73	7.62
5) $e^{x^2+7x-30} - 1$	3.3	8	6	5	3.14	3.46	3.23
	3.5	11	8	7	4.17	4.28	4.28

Tables 2 and 3 show results from numerical experiments for different initial points. Methods VT and VS are convergent for every initial point.

Table 3

$f(x)$	$\|f(x_{n+1})\|$			p			α
	MN	VT	VS	MN	VT	VS	
1)	-	3.33e-15	9.37e-9	-	3.00	3.04	1.00
	0	4.76e-13	1.10e-13	-	-	2.99	
	6.44e-10	1.73e-11	1.50e-12	1.99	2.98	2.99	
2)	7.78e-12	1.11e-16	0	-	-	-	≈ 0.7390
	1.88e-15	8.04e-13	2.34e-10	2.00	2.89	3.08	
	4.44e-16	0	0	2.00	-	-	
	8.98e-9	0	1.36e-9	1.99	-	2.44	
	2.22e-16	1.18e-10	1.11e-16	1.96	2.98	1.96	
3)	3.46e-14	2.19e-11	5.23e-12	1.99	-	2.98	2.00
	1.33e-15	5.53e-9	6.71e-10	2.02	2.95	2.96	
	1.05e-10	1.08e-12	0	1.99	3.01	-	
	8.97e-9	1.01e-11	9.66e-9	1.99	3.02	3.05	
	1.33e-15	0	1.66e-9	2.04	-	3.04	
4)	5.29e-9	2.64e-9	1.26e-8	-	-	-	≈ -1.2862
	-	1.22e-9	6.05e-9	-	-	-	
5)	1.41e-10	0	5.23e-10	1.99	-	2.98	3.00
	3.28e-12	0	9.92e-13	1.99	-	2.99	

5 Conclusion

Our method and the Weerakoon's and the Fernando's variant do not need to compute the second or third derivatives of the function, while other third-order convergence methods require these computations. Our method makes less number of iterations than the Newton's method and uses different initial points when compared to Newton's method. The VS method has at least third-order convergence. Results from numerical experiments show that the convergence of the our method is independent from initial points. If $C_3 > 0$ then our modification achieves less error than the Weerakoon's and the Fernando's variant.

Unfortunately, we can not give mathematical statement for choice a initial point for our method. It is a problem of a future research.

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