Euler’s Constant Approximations

By

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Introduction

Euler’s constant (also known as Euler–Mascheroni constant) is approximately:

0.57721566490153286060651209008240243104215933593992

The following approximation for Euler’s constant is the starting point from which I point out better variants [1]:

$$\lim_{n\to\infty} \gamma = \sum_{k=1}^{n} 1/k - \ln(n) \qquad (1)$$

Equation 1 is simple to implement. The summation term also defines the *harmonic series* function, H_n. Equation 1 converges very slowly to Euler’s constant. This article discusses various approximations, including a few ones developed by the author, that converge to γ at a higher rate.

Table 1 lists the errors in evaluating Euler’s constant as a function of n using equation 1.

> ☞ Throughout this article I will be using the same sequence of values for n that appear in Table 1 for testing the various approximations. I also present the statistics of the approximation based on the range of 50 to 10,000 iterations. I chose the lower limit of 50 somewhat arbitrarily ensuring that all of the approximations can reach a reasonable level of yielding acceptable results.

> ☞ Throughout this article, I present the errors and percent errors *relative* to Euler’s constant value of 0.577215664901533 that I hard code in my Excel VBA code. Therefore a 0 error means that a calculated approximation matches the hard coded value of Euler’s constant and not the true value of that constant.

n	***Calculated γ***	***Error***
10	0.626383161	-0.049167496
20	0.602007384	-0.024791719
30	0.593789749	-0.016574084
40	0.589663585	-0.01244792
50	0.587182333	-0.009966668
60	0.585525851	-0.008310186
70	0.584341516	-0.007125851
80	0.583452644	-0.006236979
90	0.582760933	-0.005545268
100	0.582207332	-0.004991667
200	0.579713582	-0.002497917
300	0.578881406	-0.001665741
400	0.578465144	-0.001249479
500	0.578215332	-0.000999667
600	0.578048767	-0.000833102
700	0.577929781	-0.000714116
800	0.577840535	-0.00062487
900	0.577771118	-0.000555453
1000	0.577715582	-0.000499917
2000	0.577465644	-0.000249979
3000	0.577382322	-0.000166657
4000	0.57734066	-0.000124995
5000	0.577315662	-9.99967E-05
6000	0.577298996	-8.3331E-05
7000	0.577287092	-7.14269E-05
8000	0.577278164	-6.24987E-05
9000	0.577271219	-5.55545E-05
10000	0.577265664	-4.99992E-05

Table 1. Approximated values for the Euler’s constant using equation (1).

Slow Converging Approximations

This section presents approximations, based on equation 1, that converge faster than equation 1, but do not lead to spectacular results. As such, I will present these approximations and offer you the summary statistics for their performance.

The following infinite series [1] calculates an approximation for Euler’s constant:

$$\gamma = \sum_{k=1}^{\infty}[\frac{1}{k} - \ln\left(1 + \frac{1}{k}\right)] \quad (2)$$

Equation 2 incorporates the logarithm term into the summation in hope of reducing the error in calculating γ. Table 3 shows the summary statistics for the errors in equation 2 for 50 <= n <=10000:

	Error	*%Error*
Minimum	4.99958E-05	0.008661552
Maximum	0.009835959	1.704035406
Mean	0.002181204	0.377883658
Std Deviation	0.002999217	0.519600831

Table 3. The summary statistics for the errors in equation 2 for 50 ≤ n ≤10000.

The following series, which was developed by Nielsen [1] in 1897, calculates an approximation for Euler's constant using:

$$\gamma = 1 - \sum_{k=2}^{\infty}(-1)^k\ floor(\frac{log2(k)}{k+1})] \quad (3)$$

Where *floor*() is the floor function. Equation 3 stands out since it does not use the harmonic series in its calculations. Table 4 shows the summary statistics for the errors in equation 3 for 50 <= n <=10000:

	Error	*%Error*
Minimum	0.000588869	0.102018818
Maximum	0.034163835	5.918729732
Mean	0.010011106	1.734378707
Std Deviation	0.011352705	1.966804671

Table 4. The summary statistics for the errors in equation 3 for 50 ≤ n ≤10000.

A similar series, which was developed by Vacca [1] in 1910, calculates an approximation for Euler's constant using:

$$\gamma = \sum_{k=2}^{\infty}(-1)^k\ floor(\frac{log2(k)}{k})] \quad (4)$$

Table 5 shows the summary statistics for the errors in equation 4 for 50 <= n <=10000:

	Error	*%Error*
Minimum	0.000588931	0.102029646
Maximum	0.034718207	6.014772153
Mean	0.010137984	1.756359762

Std Deviation	0.011564218	2.003448468

Table 5. The summary statistics for the errors in equation 4 for 50 ≤ n ≤10000.

DeTemple [2] developed the following approximation for Euler's constant:

$$\gamma = H_n - \ln(n+1/2) \quad (5)$$

Equation 5 presents a slight variation on equation 1. Table 6 shows the summary statistics for the errors in equation 5 for 50 <= n <=10000:

	Error	***%Error***
Minimum	4.16587E-10	7.21717E-08
Maximum	1.63371E-05	0.002830337
Mean	2.25211E-06	0.000390167
Std Deviation	4.32722E-06	0.000749671

Table 6. The summary statistics for the errors in equation 5 for 50 ≤ n ≤10000.

Mortici [3] has derived several formulas to calculate Euler's constant. This section present the following three equations that are less accurate than the other Mortici equations that I present in the next section.

Mortici's first series is:

$$\gamma = H_{n-1} + 1/[(6 - 2\sqrt{6})n] - \ln(n + 1/\sqrt{6}) \quad (6)$$

Table 7 shows the summary statistics for the errors in equation 6 for 50 <= n <=10000:

	Error	***%Error***
Minimum	5.93969E-14	1.02902E-11
Maximum	1.79007E-07	3.10121E-05
Mean	1.87553E-08	3.24927E-06
Std Deviation	4.27679E-08	7.40935E-06

Table 7. The summary statistics for the errors in equation 6 for 50 ≤ n ≤10000.

Mortici's second series is:

$$\gamma = H_{n-1} + 1/[(6 + 2\sqrt{6})n] - \ln(n - 1/\sqrt{6}) \quad (7)$$

Table 8 shows the summary statistics for the errors in equation 7 for 50 <= n <=10000:

	Error	***%Error***
Minimum	8.88178E-15	1.53873E-12
Maximum	1.83895E-07	3.1859E-05
Mean	1.91745E-08	3.32189E-06
Std Deviation	4.38545E-08	7.5976E-06

Table 8. The summary statistics for the errors in equation 7 for 50 $\le n \le$10000.

Mortici's third series is:

$\gamma = H_n - \ln(n) + \ln((n - 1/12)/(n + 5/12))$ (8)

The second logarithm in equation 8 represents a correction term to equation 1. The algorithm calculates the value of first order rational polynomials. Table 9 shows the summary statistics for the errors in equation 8 for 50 <= n <=10000:

	Error	***%Error***
Minimum	6.28386E-14	1.08865E-11
Maximum	1.91916E-07	3.32485E-05
Mean	2.01062E-08	3.48331E-06
Std Deviation	4.58507E-08	7.94343E-06

Table 9. The summary statistics for the errors in equation 8 for 50 $\le n \le$10000.

The Mortici formulas in equations 6, 7, and 8 show an improvement in results over the previous equations in this section.

Chen[5] discusses several approximations by Negoi. The first approximation is:

$\gamma = H_n - \ln(n + ½ + 1/(24n))$ (9)

Equation 9 presents an enhancement on the logarithm appearing in equation 1. Table 10 shows the summary statistics for the errors in equation 9 for 50 <= n <=10000:

	Error	***%Error***
Minimum	5.79536E-14	1.00402E-11
Maximum	1.64367E-07	2.84759E-05
Mean	1.72226E-08	2.98374E-06
Std Deviation	3.92714E-08	6.80358E-06

Table 10. The summary statistics for the errors in equation 9 for 50 $\le n \le$10000.

The second Negoi [5] approximation is:

$\gamma = H_n - \ln(n + ½ + 1/(24n)) - (p + q)/2$ (10)

Where $p = 1/[48(n+1)^3]$ and $q = 1/(48n^3)$.

Table 11 shows the summary statistics for the errors in equation 10 for 50 <= n <=10000:

	Error	*%Error*
Minimum	7.88258E-14	1.36562E-11
Maximum	3.26228E-07	5.65175E-05
Mean	3.4228E-08	5.92985E-06
Std Deviation	7.79833E-08	1.35103E-05

Table 11. The summary statistics for the errors in equation 10 for 50 ≤ n ≤10000.

The third Negoi approximation which provides more accurate results appears in the next section.

Fast Converging Approximations

This section presents approximations that converge faster to Euler's constant. Many of the approximations in this section are better versions of ones presented in the last section. Therefore, the names of these approximation should be familiar by now.

Mortici's fourth equation to calculate Euler's constant is:

$\gamma = H_{n-1} + 1/(2n) - \ln(n^2 - 1/6)/2$ (11)

Table 12 shows the summary statistics for the errors in equation 11 for 50 <= n <=10000:

	Error	*%Error*
Minimum	1.22125E-15	2.11575E-13
Maximum	2.44424E-09	4.23453E-07
Mean	2.09611E-10	3.63142E-08
Std Deviation	5.48949E-10	9.51029E-08

Table 12. The summary statistics for the errors in equation 11 for 50 ≤ n ≤10000.

Mortici's fifth series that calculates Euler's constant is:

$\gamma = H_n - \ln(n) + \ln(p/q)$ (12)

Where,

$p = n^2 + 33/(140\ n) + 37/1680$ (12a)

and,

$q = n^2 + 103/(140\ n) + 61/336$ (12b)

Equation 12 presents a correction to equation 1 using quadratic rational polynomials. Table 13 shows the summary statistics for the errors in equation 12 for 50 <= n <=10000:

	Error	*%Error*
Minimum	7.77156E-16	1.34639E-13
Maximum	1.53277E-11	2.65546E-09
Mean	1.14316E-12	1.98048E-10
Std Deviation	3.31761E-12	5.74761E-10

Table 13. The summary statistics for the errors in equation 12 for 50 ≤ n ≤10000.

Batir and Chen [4] suggested two approximations to calculate Euler's constant. The first Batir-Chen approximation is:

$$\gamma = H_n - \ln(n + ½ + 1/(24\ n) - 1/(48\ n^2) + 23/(5760\ n^3) + 17/(3840\ n^4) + 10099/(2903040\ n^5) \qquad (13)$$

Table 14 shows the summary statistics for the errors in equation 13 for 50 <= n <=10000:

	Error	*%Error*
Minimum	1.11022E-15	1.92341E-13
Maximum	4.45644E-13	7.72057E-11
Mean	4.01716E-14	6.95954E-12
Std Deviation	9.2046E-14	1.59465E-11

Table 14. The summary statistics for the errors in equation 13 for 50 ≤ n ≤10000.

The second Batir-Chen approximation is:

$$\gamma = H_n - \ln(n + ½ + 1(24\ m) - 37/(5760\ m^3) + 10313/\ (2903040\ m^5) \qquad (14)$$

Where m = n + ½.

Table 15 shows the summary statistics for the errors in equation 14 for 50 <= n <=10000:

	Error	*%Error*
Minimum	6.66134E-16	1.15405E-13
Maximum	3.73711E-10	6.47437E-08
Mean	3.22365E-11	5.58482E-09
Std Deviation	8.40666E-11	1.45642E-08

Table 15. The summary statistics for the errors in equation 14 for 50 ≤ n ≤10000.

The third Negoi [5] approximation is:

$$\gamma = H_n - \ln(n + ½ + 1/(24n)) - p(n) \qquad (15)$$

Where p(n) =1/48/[n + 83/360 + 4909/64800/(n + 11976997/ 37112040)^3].

Table 16 shows the summary statistics for the errors in equation 15 for 50 <= n <=10000:

	Error	*%Error*
Minimum	2.22045E-16	3.84682E-14
Maximum	4.00791E-14	6.94351E-12
Mean	1.13567E-14	1.96749E-12
Std Deviation	1.44784E-14	2.50831E-12

Table 16. The summary statistics for the errors in equation 15 for 50 ≤ n ≤10000.

The Hurwitz [1] approximation is:

$$\gamma = H_n - \ln(n) - p(n) \tag{16}$$

Where $p(n) = 1/(2n) - 1/(12n^2) + 1/(120n^4)$.

Table 17 shows the summary statistics for the errors in equation 16 for 50 <= n <=10000:

	Error	*%Error*
Minimum	1.11022E-15	1.92341E-13
Maximum	2.55906E-13	4.43346E-11
Mean	2.76631E-14	4.7925E-12
Std Deviation	5.27728E-14	9.14266E-12

Table 17. The summary statistics for the errors in equation 16 for 50 ≤

The Author's Work

My attempts at approximating Euler's constant started with several polynomial models that fit the error in calculating Euler's constant with the number of iterations. As you will see soon, my last regression model pointed to a versatile analytical solution. After exploring this solution I was happy to come across the best approximation for Euler's constant, even though that relation was generally known already!

The first regression model, a fourth-order polynomial fit, contributes to calculating Euler's constant using the following approximation:

$$\gamma = H_n - \ln(n) - 10^{\mathrm{Poly}(n)} \tag{17}$$

Where Poly(n) is calculated using:

$$Poly(n) = -0.336890612822068 - 0.952200974992434 \log_{10}(n) - 2.40147901805659E\text{-}02 [\log_{10}(n)]^2 + 5.34486197957481E\text{-}03 [\log_{10}(n)]^3 - 4.42420215025482E\text{-}04 [\log_{10}(n)]^4 \quad (17a)$$

Table 18 shows the summary statistics for the errors in equation 17 for 50 <= n <=10000:

	Error	*%Error*
Minimum	2.38759E-10	4.1364E-08
Maximum	1.02886E-06	0.000178245
Mean	2.38267E-07	4.12787E-05
Std Deviation	3.73961E-07	6.47871E-05

Table 18. The summary statistics for the errors in equation 17 for 50 ≤ n ≤10000.

The second regression model, a fifth-order polynomial fit, contributes to calculating Euler's constant using the following approximation:

$$\gamma = H_n - \ln(n) - 10^{Poly(n)} \quad (18)$$

Where Poly(n) is calculated using:

$$Poly = -0.349390867440153 - 0.921506218334963 \log_{10}(n) - 5.22015860254543E\text{-}02 [\log_{10}(n)]^2 + 5.34486197957481E\text{-}03 [\log_{10}(n)]^3 - 2.95515486800227E\text{-}03 [\log_{10}(n)]^4 + 1.98213992961801E\text{-}04 [\log_{10}(n)]^5 \quad (18a)$$

Table 19 shows the summary statistics for the errors in equation 18 for 50 <= n <=10000:

	Error	*%Error*
Minimum	6.56377E-10	1.13714E-07
Maximum	3.74337E-07	6.48522E-05
Mean	6.53157E-08	1.13157E-05
Std Deviation	1.04699E-07	1.81386E-05

Table 19. The summary statistics for the errors in equation 18 for 50 ≤ n ≤10000.

The third regression model, a rational polynomial fit, contributes to calculating Euler's constant using the following approximation:

$\gamma = H_n - \ln(n) - 10^{Poly(n)}$ (19)

Where Poly(n) is calculated using:

Poly(n) = P(n)/Q(n)

Where,

$P(n) = -0.356484624590002 - 0.868207818526913 * \log_{10}(n)$ (19a)

$Q(n) = 1 - 9.50299272631852E\text{-}02 \log_{10}(n) +$

$3.94884084382336E\text{-}02 [\log_{10}(n)]^2 -$

$9.56562183896382E\text{-}03 [\log_{10}(n)]^3 +$

$1.25676353002308E\text{-}03 [\log_{10}(n)]^4 -$

$6.92420246695854E\text{-}05 [\log_{10}(n)]^5$ (19b)

Table 20 shows the summary statistics for the errors in equation 19 for 50 <= n <=10000:

	Error	***%Error***
Minimum	8.96294E-12	1.55279E-09
Maximum	1.19899E-07	2.07719E-05
Mean	1.57907E-08	2.73566E-06
Std Deviation	2.7957E-08	4.84342E-06

Table 20. The summary statistics for the errors in equation 19 for 50 ≤ n ≤10000.

The fourth regression model, a rational polynomial fit, contributes to calculating Euler's constant using the following approximation:

$\gamma = H_n - \ln(n) + Poly(n)$ (20)

Where,

$Poly(n) = 1.58197134978339E\text{-}14 - 0.5/n +$

$8.33333332716528E\text{-}02/n^2 -$

$8.33319446155676E\text{-}03/n^4 +$

$3.91413719514747E\text{-}03/n^6$ (20a)

Table 21 shows the summary statistics for the errors in equation 20 for 50 <= n <=10000:

	Error	*%Error*
Minimum	2.22045E-16	3.84682E-14
Maximum	2.39808E-14	4.15457E-12
Mean	1.34661E-14	2.33294E-12
Std Deviation	6.10631E-15	1.05789E-12

Table 21. The summary statistics for the errors in equation 20 for 50 ≤ n ≤10000.

Equation 20 can be rewritten as the following approximation by taking the reciprocal of the regression coefficients in equation 20a:

$$\gamma = H_n - \ln(n) + \text{Poly}(n) \tag{21}$$

Where,

$$\text{Poly}(n) = -1/(2\ n) + 1/(12\ n^2) - 1/(120\ n^4) + 1/\ (255.5\ n^6) \tag{21a}$$

Table 22 shows the summary statistics for the errors in equation 21 for 50 <= n <=10000:

	Error	*%Error*
Minimum	1.11022E-16	1.92341E-14
Maximum	3.9857E-14	6.90505E-12
Mean	1.1278E-14	1.95387E-12
Std Deviation	1.45986E-14	2.52913E-12

Table 22. The summary statistics for the errors in equation 21 for 50 ≤ n ≤10000.

Equation 21 encouraged me to use the approximation for Euler's constant that involves the Bernoulli numbers (see Havil [6] page 103 and Wikipedia's article on Harmonic number [8]):

$$\gamma = H_n - \ln(n) - \frac{1}{2n} + \sum_{r=1}^{\infty} \frac{B_{2r}}{2r\ n^{2r}} \tag{22}$$

Where B_{2r} is the Bernoulli number. Table 23 shows the sequence of the first few Bernoulli numbers is:

n	*B*
0	1
1	–1/2
2	1/6

n	B_n
4	–1/30
6	1/42
8	–1/30
10	5/66
12	–601/2730
14	7/6
16	–3617/510
18	43867/798
20	–174611/330

Table 23. The values for the first few Bernoulli numbers.

I used several terms from equation 22, as shown by the following equation:

$$\gamma = H_n - \ln(n) + \text{Poly}(n) \quad (23)$$

Where,

$$\text{Poly}(n) = -1/(2\,n) + 1/(12\,n^2) - 1/(120\,n^4) + 1/(252\,n^6) - 1/(240\,n^8) + 1/(132\,n^{10}) \quad (23a)$$

The above polynomial is a superset of Hurwitz's approximation that uses fewer terms to calculate Poly(n). Havil [6] also mentions an approximation that resembles equation 23.

Table 24 shows the summary statistics for the errors in equation 23 for 50 <= n <=10000:

	Error	*%Error*
Minimum	0	0
Maximum	3.9857E-14	6.90505E-12
Mean	1.10745E-14	1.9186E-12
Std Deviation	1.47136E-14	2.54906E-12

Table 24. The summary statistics for the errors in equation 23 for $50 \le n \le 10000$.

The minimum errors occur at n = 20 and 90 (using Excel 2013) matching the hardcoded value of Euler's constant in the VBA code. To show you how good equation 23 works, Table 25 lists the errors in Euler's constant for a low range of 2 to 20 iterations:

n	*Calculated Euler Constant*	*Error*
2	0.577218446	2.78067E-06
3	0.577215693	2.83493E-08
4	0.577215666	1.02043E-09
5	0.577215665	7.50449E-11
6	0.577215665	8.75888E-12
7	0.577215665	1.41243E-12
8	0.577215665	2.89102E-13
9	0.577215665	7.07212E-14
10	0.577215665	1.9762E-14
11	0.577215665	5.9952E-15
12	0.577215665	1.9984E-15
13	0.577215665	7.77156E-16
14	0.577215665	6.66134E-16
15	0.577215665	4.44089E-16
16	0.577215665	4.44089E-16
17	0.577215665	1.11022E-16
18	0.577215665	1.11022E-16
19	0.577215665	3.33067E-16
20	0.577215665	0

Table 25. The estimated values for Euler's constant and their errors for a low range of iterations.

☞ In researching reference for this article, I came across a web page by Wolfram-Math-World titled *Euler-Mascheroni Constant Approximations*. This web page contains a set of very clever short approximation for Euler's constant. These approximations, which range in accuracy, do not use summations and instead rely on simple empirical calculations. You can easily perform these calculations with your pocket calculator and get the value of Euler's constant accurate to a few decimals. I highly recommend visiting that the Wolfram-Math-World web page.

Conclusion

The best approximations for Euler's constant are ones based on equation 22 which uses a summation involving Bernoulli numbers. This article points out the advantages of such approximations, over others—something that several references that mention the approximation shy away from stressing its importance.

References

1. Wikipedia – Euler–Mascheroni constant.
2. D.W. DeTemple, *A geometric look at sequences that converge to Euler’s constant*. College Math. J., 3 (2006), 128–131.
3. Cristinel Mortici, *Fast convergences towards Euler-Mascheroni constant, Computational & Applied Mathematics*, Volume 29, N. 3, pp. 479-491, 2010.
4. Necdet Batin and Chao-Ping Chen, *Improving some sequences convergent to Euler-Mascheroni constant,* Proyecciones Journal of Mathematics Vol. 31, No 1, pp. 29-38, March 2012. Universidad Cat´olica del Norte Antofagasta – Chile.
5. Chao-Ping Chen, *Sharpness Of Negoi's Inequality For The Euler-Mascheroni Constant*, Bulletin of Mathematical Analysis and Applications ISSN: 1821-1291, URL: http://www.bmathaa.org, Volume 3 Issue 1(2011), Pages 134-141
6. Julian Havil, *Gamma: Exploring Euler’s Constant*, Princeton University Press, 2003.
7. Xavier Gourdon and Pascal Sebah, *The Euler constant : γ*, Numbers, Constants and Computation, April 14, 2004, http://numbers.computation.free.fr/Constants/constants.html
8. Wikipedia – Harmonic number.

