



To the Homepage of Factorial Algorithms

Approximation Formulas for the Factorial Function $n!$ Peter Luschny

The most beautiful approximation to the central binomial. $N = 8n + 2$

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{N\pi/2}} \left(2 - \frac{2}{N^2} + \frac{21}{N^4} - \frac{671}{N^6} + \frac{45081}{N^8} \right)$$

And to the Catalan numbers. $M = 4n + 3$

$$\frac{1}{n+1} \binom{2n}{n} \sim \frac{4^{n-2}}{M\sqrt{M\pi}} \left(128 + \frac{160}{M^2} + \frac{84}{M^4} + \frac{715}{M^6} - \frac{10180}{M^8} \right)$$

We give an overview of approximations for the factorial function, convergent or asymptotic, old or new, compare their efficiency and give hints for their application.

Although most formulas are variations of the asymptotic expansion of James Stirling (1692–1770) we will reach a conclusion different from those given in most places. We will recommend Stieltjes' formula whenever a numerical approximation of the factorial function is required. The implementation for two of these formulas is given in pseudo-code below.

Some abbreviations:

```
kern0(n) = sqrt(2Pi/n) * (n/e)^n = kern2(n)/sqrt(n)
kern1(n) = sqrt(2Pi*n) * (n/e)^n = kern2(n)*sqrt(n)
kern2(n) = sqrt(2Pi) * (n/e)^n = sqrt(2Pi) * n^n * exp(-n)
```

kern0 - approximations

```
stieltjes0(n) : N=n+1; kern0(N)
stieltjes1(n) : N=n+1; kern0(N) * exp((1/12)/N)
stieltjes2(n) : N=n+1; kern0(N) * exp((1/12)/(N+(1/30)/N))
stieltjes3(n) : N=n+1; kern0(N) * exp((1/12)/(N+(1/30)/(N+(53/210)/N)))
stieltjes4(n) : N=n+1; kern0(N) * exp((1/12)/(N+(1/30)/(N+(53/210)/(N+(195/371)/N))))

henrici0(n) : N=n+1; kern0(N)
henrici1(n) : N=n+1; kern0(N) * exp(1/(12*N+1/N))
henrici2(n) : N=n+1; kern0(N) * exp(5/2*1/(30*N+1/N))
henrici3(n) : N=n+1; kern0(N) * exp((315*N-53/N)/(3780*N^2-510-53/N^2))

stirser0(n) : N=n+1; kern0(N)
stirser1(n) : N=n+1; kern0(N) * exp(1/(12*N))
stirser2(n) : N=n+1; kern0(N) * exp(1/(12*N) * (1-1/(30*N^2)))
stirser3(n) : N=n+1; kern0(N) * exp(1/(12*N) * (1-1/(30*N^2) * (1-2/(7*N^2))))
stirser4(n) : N=n+1; kern0(N) * exp(1/(12*N) * (1-1/(30*N^2) * (1-2/(7*N^2) * (1-3/(4*N^2)))))
```

kern1 - approximations

```
ramanujan0(n) : kern1(n)
ramanujan1(n) : N=2*n; kern1(n) * (1+1/N)^(1/6)
ramanujan2(n) : N=2*n; kern1(n) * (1+1/N * (1+1/(2*N)))^(1/6)
ramanujan3(n) : N=2*n; kern1(n) * (1+1/N * (1+1/(2*N) * (1+1/(15*N))))^(1/6)
ramanujan4(n) : N=2*n; kern1(n) * (1+1/N * (1+1/(2*N) * (1+1/(15*N) * (1-11/(4*N)))))^(1/6)
ramanujan5(n) : N=2*n; kern1(n) * (1+1/N * (1+1/(2*N) * (1+1/(15*N) * (1-1/(4*N) * (11-79/(7*N)))))^(1/6)

_____ L0(n) : kern1(n)
_____ L1(n) : N=6*n; kern1(n) * (1+1/N)^(1/2)
_____ L2(n) : N=6*n; kern1(n) * (1+1/N * (1+1/(2*N)))^(1/2)
_____ L3(n) : N=6*n; kern1(n) * (1+1/N * (1+1/(2*N) * (1-31/(15*N))))^(1/2)
_____ L4(n) : N=6*n; kern1(n) * (1+1/N * (1+1/(2*N) * (1-1/(15*N) * (31+139/(4*N)))))^(1/2)
_____ L5(n) : N=6*n; kern1(n) * (1+1/N * (1+1/(2*N) * (1-1/(15*N) * (31+1/(4*N) * (139-9871/(7*N)))))^(1/2)
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```

stirling0(n) :      kern1(n)
stirling1(n) : N=12*n; kern1(n) * (1+1/N)
stirling2(n) : N=12*n; kern1(n) * (1+1/N * (1+1/(2*N)))
stirling3(n) : N=12*n; kern1(n) * (1+1/N * (1+1/(2*N) * (1-139/(15*N))))
stirling4(n) : N=12*n; kern1(n) * (1+1/N * (1+1/(2*N) * (1-1/(15*N) * (139+571/(4*N))))))

nemes0(n) :      kern1(n)
nemes1(n) : N=12*n^2; kern1(n) * (1+1/N)^n
nemes2(n) : N=12*n^2; kern1(n) * (1+1/N * (1+1/(10*N)))^n
nemes3(n) : N=12*n^2; kern1(n) * (1+1/N * (1+1/(10*N) * (1+239/(21*N))))^n
nemes4(n) : N=12*n^2; kern1(n) * (1+1/N * (1+1/(10*N) * (1+1/(21*N) * (239-46409/(20*N))))))^n

BuricElezovic0(n) :      kern1(n)
BuricElezovic1(n) :      kern1(n) * exp(1/(12*n))
BuricElezovic2(n) : N=1/(360*n^2); kern1(n) * exp(1/(12*n)) * (1-N)^(1/n)
BuricElezovic3(n) : N=1/(360*n^2); kern1(n) * exp(1/(12*n)) * (1-N * (1-(1447/14)*N))^(1/n)
BuricElezovic4(n) : N=1/(360*n^2); kern1(n) * exp(1/(12*n)) * (1-N * (1-(1447/14)*N * (1-(1170727/4341)*N)))

nemesCF0(n) : kern1(n)
nemesCF1(n) : kern1(n) * (n/(n-1/12*1/n))^n
nemesCF2(n) : kern1(n) * (n/(n-1/12*1/(n+3/40*1/n)))^n
nemesCF3(n) : kern1(n) * (n/(n-1/12*1/(n+3/40*1/(n+2369/22680*1/n))))^n
nemesCF4(n) : kern1(n) * (n/(n-1/12*1/(n+3/40*1/(n+2369/22680*1/(n+7828759/10745784*1/n))))))^n

FengWang0(n) : kern1(n)
FengWang1(n) : kern1(n) * ((n+1)/(n-1))^(1/24)
FengWang2(n) : kern1(n) * ((n+1)/(n-1))^( (1/24) * (1-(11/30)/n^2) )
FengWang3(n) : kern1(n) * ((n+1)/(n-1))^( (1/24) * (1-(11/30)/n^2 * (1+(43/231)/n^2) ) )
FengWang4(n) : kern1(n) * ((n+1)/(n-1))^( (1/24) * (1-(11/30)/n^2 * (1+(43/231)/n^2 * (1+(1019/1290)/n^2) ) ) )

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kern2 - approximations

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demoivre0(n) : N=n+1/2; kern2(N)
demoivre1(n) : N=n+1/2; kern2(N) * (1-1/(24*N))
demoivre2(n) : N=n+1/2; kern2(N) * (1-1/(24*N)+1/(1152*N^2))
demoivre3(n) : N=n+1/2; kern2(N) * (1-1/(24*N)+1/(1152*N^2)+(1003/414720)/N^3)
demoivre4(n) : N=n+1/2; kern2(N) * (1-1/(24*N)+1/(1152*N^2)+(1003/414720)/N^3-(4027/39813120)/N^4)
demoivre5(n) : N=n+1/2; kern2(N) * (1-1/(24*N)+1/(1152*N^2)+(1003/414720)/N^3-(4027/39813120)/N^4
              - (5128423/6688604160)/N^5)

maclser0(n) :      kern2(n+1/2)
maclser1(n) : N=2*n+1; kern2(n+1/2) * exp(-1/(12*N))
maclser2(n) : N=2*n+1; kern2(n+1/2) * exp(-1/(12*N) * (1-7/(30*N^2)))
maclser3(n) : N=2*n+1; kern2(n+1/2) * exp(-1/(12*N) * (1-7/(30*N^2) * (1-62/(49*N^2))))

wdsmith0(n) : N=n+1/2; kern2(N)
wdsmith1(n) : N=n+1/2; kern2(N) * exp((-1/24)/N)
wdsmith2(n) : N=n+1/2; kern2(N) * exp((-1/24)/(N+(7/120)/N))
wdsmith3(n) : N=n+1/2; kern2(N) * exp((-1/24)/(N+(7/120)/(N+(1517/5880)/N)))
wdsmith4(n) : N=n+1/2; kern2(N) * exp((-1/24)/(N+(7/120)/(N+(1517/5880)/(N+(164715/297332)/N))))

cantrell0(n) : N=n+1/2; kern2(N)
cantrell1(n) : N=n+1/2; kern2(N) / (1+1/(24*N-1/2))
cantrell2(n) : N=n+1/2; kern2(N) / (1+1/(24*N-1/2+1/((1440/2021)*N)))
cantrell3(n) : N=n+1/2; kern2(N) / (1+1/(24*N-1/2+1/((1440/2021)*N+1/((686186088/125896643)*N))))
cantrell4(n) : N=n+1/2; kern2(N) / (1+1/(24*N-1/2+1/((1440/2021)*N+1/((686186088/125896643)*N
              +1/((1521596612992267104/4596084813365743279)*N))))))

lanczos0(n) : N=n+1/2; kern2(N)
lanczos1(n) : N=n+1/2; kern2(N) * (1-(1/24) * (1/(n+1)))
lanczos2(n) : N=n+1/2; kern2(N) * (1-(1/24) * (1/(n+1)) - (23/1152) * (1/((n+1)*(n+2))))
lanczos3(n) : N=n+1/2; kern2(N) * (1-(1/24) * (1/(n+1)) - (23/1152) * (1/((n+1)*(n+2)))
              - (11237/414720) * (1/((n+1)*(n+2)*(n+3))))

wehmeier0(n) : kern2(n) * sqrt(n)
wehmeier1(n) : kern2(n) * sqrt(n+1/6)
wehmeier2(n) : kern2(n) * sqrt(n+1/6+(1/72)/n)
wehmeier3(n) : kern2(n) * sqrt(n+1/6+(1/72)/n - (31/6480)/n^2)
wehmeier4(n) : kern2(n) * sqrt(n+1/6+(1/72)/n - (31/6480)/n^2 - (139/155520)/n^3)

gosper0(n) : kern2(n) * sqrt(n)
gosper1(n) : kern2(n) * sqrt(n+1/6)
gosper2(n) : kern2(n) * sqrt(n+1/6) * (1+(1/144)/n^2)
gosper3(n) : kern2(n) * sqrt(n+1/6) * (1+(1/144)/n^2 - (23/6480)/n^3)

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gospers4(n) : kern2(n)*sqrt(n+1/6)*(1+(1/144)/n^2-(23/6480)/n^3+(5/41472)/n^4)
gospers5(n) : kern2(n)*sqrt(n+1/6)*(1+(1/144)/n^2-(23/6480)/n^3+(5/41472)/n^4+(4939/6531840)/n^5)

nemesG1(n) : N=n+1/4; kern2(n)*sqrt(n+1/6)
nemesG2(n) : N=n+1/4; kern2(n)*sqrt(n+1/6)*(1+(1/144)/N^2)
nemesG3(n) : N=n+1/4; kern2(n)*sqrt(n+1/6)*(1+(1/144)/N^2-(1/12960)/N^3)
nemesG4(n) : N=n+1/4; kern2(n)*sqrt(n+1/6)*(1+(1/144)/N^2-(1/12960)/N^3-(257/207360)/N^4)
nemesG5(n) : N=n+1/4; kern2(n)*sqrt(n+1/6)*(1+(1/144)/N^2-(1/12960)/N^3-(257/207360)/N^4-(53/2612)
nemesG6(n) : N=n+1/4; kern2(n)*sqrt(n+1/6)*(1+(1/144)/N^2-(1/12960)/N^3-(257/207360)/N^4-(53/2612
+ (5741173/9405849600)/N^6)

gospersmith0(n): N=n+1/2; NN=N^2; sqrt(2*Pi)*exp(-N)*(NN)^(N/2)
gospersmith1(n): N=n+1/2; NN=N^2; sqrt(2*Pi)*exp(-N)*(NN-(1/12))^(N/2)
gospersmith2(n): N=n+1/2; NN=N^2; sqrt(2*Pi)*exp(-N)*(NN+(1/12)*(-1+(1/(10*NN))))^(N/2)
gospersmith3(n): N=n+1/2; NN=N^2; sqrt(2*Pi)*exp(-N)*(NN+(1/12)*(-1+(1/(10*NN)))*(1-(185/(756*NN))))

luschny0(n): N=n+1/2; NN=24*N^2; kern2(N)
luschny1(n): N=n+1/2; NN=24*N^2; kern2(N)*(1-1/NN)^N
luschny2(n): N=n+1/2; NN=24*N^2; kern2(N)*(1-1/NN*(1-19/(10*NN)))^N
luschny3(n): N=n+1/2; NN=24*N^2; kern2(N)*(1-1/NN*(1-1/(10*NN)*(19-2561/(21*NN))))^N
luschny4(n): N=n+1/2; NN=24*N^2; kern2(N)*(1-1/NN*(1-1/(10*NN)*(19-1/(21*NN)*(2561-874831/(20*NN))))

luschnyCF0(n): N=n+1/2; kern2(N)
luschnyCF1(n): N=n+1/2; kern2(N)*(N/(N+1/24*1/N))^N
luschnyCF2(n): N=n+1/2; kern2(N)*(N/(N+1/24*1/(N+3/80*1/N)))^N
luschnyCF3(n): N=n+1/2; kern2(N)*(N/(N+1/24*1/(N+3/80*1/(N+18029/45360*1/N))))^N
luschnyCF4(n): N=n+1/2; kern2(N)*(N/(N+1/24*1/(N+3/80*1/(N+18029/45360*1/(N+6272051/14869008*1/N))))

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miscellaneous approximations

```

maclstir0(n) : ( 2*maclser0(n)+ stirser0(n))/3
maclstir1(n) : ( 32*maclser1(n)+ 31*stirser1(n))/63
maclstir2(n) : ( 64*maclser2(n)+ 63*stirser2(n))/127
maclstir3(n) : (128*maclser3(n)+127*stirser3(n))/255

windschitl(n): N=n+1; sqrt(2*Pi/N)*((N/e)*sqrt(N*sinh(1/N)))^N
wdsmith*(n) : N=n+1/2; sqrt(2*Pi)*((N/e)*sqrt(2*N*tanh(1/(2*N))))^N

nemes*(n) : N=n; sqrt(2*Pi*N)*((N+1/(12*N-1/(10*N)))/e)^N
nemesgam(n) : N=n+1; sqrt(2*Pi/N)*((N+1/(12*N-1/(10*N)))/e)^N
luschny*(n) : N=n+1/2; sqrt(2*Pi)*((N-1/(24*N+19/(10*N)))/e)^N

```

A popular formula (but not recommended)

```

Digits := 20;
lanczos7 := proc(z) local x, t, p;
p := [0.9999999999980993, 676.5203681218851, -1259.1392167224028,
771.32342877765313, -176.61502916214059, 12.507343278686905,
-0.13857109526572012, 9.9843695780195716e-6, 1.5056327351493116e-7];

x := p[1]+p[2]/(z+1)+p[3]/(z+2)+p[4]/(z+3)+p[5]/(z+4)+
p[6]/(z+5)+p[7]/(z+6)+p[8]/(z+7)+p[9]/(z+8);

t := z+7.5; sqrt(2*Pi)*t^(z+0.5)*exp(-t)*x end;

```

This formula, due to Lanczos, is popular presumably because it was included in the 'Numerical Recipes in C' book in the early 1990. Why it is not recommended (at least not for the real domain) can be seen from this little test computing 100!, 1000! and 10000!. It shows the number of exact decimal digits.

| n | 100 | 1000 | 10000 |
|----------------------|------|------|-------|
| exact decimal digits | 13.2 | 12.8 | 12.7 |

A simple new formula (recommended)

```

Digits := 50;
luschnyCF4 := proc(n) local c, N, p, logF; N := n + 1/2;
c := array(0..4, [1/24, 3/80, 18029/45360, 6272051/14869008]);
p := N^2/(N+c[0]/(N+c[1]/(N+c[2]/(N+c[3]/N))));
logF := ln(2*Pi)/2 + N*(ln(p)-1);
return exp(logF) end;

```

The same test applied to this formula shows the new formula is far more than a match for the

Lanczos formula. The formula is due to the present writer.

| | | | |
|-----------------------------|-------------|-------------|--------------|
| n | 100 | 1000 | 10000 |
| exact decimal digits | 21.5 | 30.5 | 39.5 |

$$n! = \sqrt{2\pi} \left(\frac{p}{e}\right)^N, \quad N = n + 1/2,$$

$$p = \frac{N^2}{N + \frac{c_0}{N + \frac{c_1}{N + \frac{c_2}{N + \frac{c_3}{N}}}}}$$

A numerical example

To give an idea how this formula works numerically we show the successive approximations to 100!. At the left hand margin the number of exact decimal digits are displayed.

| | |
|--------|--|
| -8.81 | $e^{\left(\frac{1}{2} \ln(2\pi) + \frac{201}{2} \ln\left(\frac{24361803}{242407}\right) - \frac{201}{2}\right)}$ |
| 13.22 | $e^{\left(\frac{1}{2} \ln(2\pi) + \frac{201}{2} \ln\left(\frac{487237869}{4848158}\right) - \frac{201}{2}\right)}$ |
| -17.60 | $e^{\left(\frac{1}{2} \ln(2\pi) + \frac{201}{2} \ln\left(\frac{11161775900427210}{111062904935789}\right) - \frac{201}{2}\right)}$ |
| 21.48 | $e^{\left(\frac{1}{2} \ln(2\pi) + \frac{201}{2} \ln\left(\frac{8131073198218254231}{80906534739251638}\right) - \frac{201}{2}\right)}$ |
| -25.39 | $e^{\left(\frac{1}{2} \ln(2\pi) + \frac{201}{2} \ln\left(\frac{456388040617595736096084309567}{4541193267193529742287221337}\right) - \frac{201}{2}\right)}$ |

How well do these formulas perform?

We now look at the general case. The following table displays the number of exact decimal digits (edd) of all the formulas above for some values of n. *Exact decimal digits* are defined as $-\log_{10}(\text{abs}(1 - \text{approximation}/\text{trueValue}))$. \log_{10} denotes the logarithm to the base 10.

$$\text{edd}(n) = -\log_{10}(\text{abs}(1 - \text{approximation}(n) / n!)).$$

In the following table the i-th entry (i=0,1,2,... from the left hand side to the right hand side) in a line starting with 'name' is the edd of formula name(i)(n). The '-' sign indicates that the approximation is smaller and the '+' sign (not displayed) indicates that the approximation is larger than the true value.

For example we see from the table that formula nemes1 gives the same number of exact decimal digits as the formula ramanujan2.

| Formula | n! | exact decimal digits, edd(n) | | | | | | | | | |
|-----------|--------|------------------------------|------|------|-------|-------|-------|-------|-------|-------|--|
| Stirling | 00100! | -3.1 | -6.5 | 8.6 | 11.7 | -13.1 | 16.2 | 17.2 | -20.4 | -21.1 | |
| De Moivre | 00100! | 3.4 | -7.1 | -8.6 | 12.0 | 13.1 | -16.4 | -17.2 | 20.5 | -22.8 | |
| NemesG | 00100! | | -6.2 | 10.1 | 10.9 | -14.9 | -15.2 | 19.4 | 19.2 | -23.0 | |
| Gosper | 00100! | -3.1 | -6.2 | 8.5 | -11.9 | 13.1 | 17.5 | 17.2 | 21.2 | 21.1 | |
| Wehmeier | 00100! | -3.1 | -6.2 | 8.6 | 11.4 | -13.1 | -15.9 | 17.2 | -20.0 | -21.1 | |

| | | | | | | | | | | |
|---------------|--------|------|-------|-------|-------|-------|-------|-------|-------|-------|
| Ramanujan | 00100! | -3.1 | -5.7 | -9.2 | 11.0 | -13.3 | -15.4 | 17.3 | 19.5 | -21.1 |
| Nemes | 00100! | -3.1 | | -9.2 | | -13.2 | | 17.3 | | -21.1 |
| Luschny | 00100! | 3.4 | | -8.5 | | 13.1 | | -17.2 | | 21.1 |
| Henrici | 00100! | -3.1 | | -8.4 | | -13.2 | | 17.2 | | |
| GosperSonin | 00100! | 3.4 | | -8.4 | | 13.0 | | -17.2 | | |
| MaclSeries | 00100! | 3.4 | | -8.6 | | 13.1 | | -17.2 | | |
| StirSeries | 00100! | -3.1 | | 8.6 | | -13.1 | | 17.3 | | -21.1 |
| BuricElezovic | 100! | -3.1 | | 8.6 | | -13.1 | | 17.2 | | -21.1 |
| FengWang | 00100! | -3.1 | | 7.5 | | 12.2 | | 16.3 | | 20.8 |
| NemesCF | 00100! | 3.1 | | 8.2 | | 13.2 | | 17.3 | | 21.5 |
| W.D.Smith | 00100! | 3.4 | | -8.6 | | 13.2 | | -17.5 | | 21.5 |
| Cantrell | 00100! | 3.4 | | -8.6 | | 13.2 | | -17.5 | | 21.5 |
| Stieltjes | 00100! | -3.1 | | 8.6 | | -13.2 | | 17.5 | | -21.5 |
| LuschnyCF | 00100! | 3.4 | | -8.8 | | 13.2 | | -17.6 | | 21.5 |
| Lanczos7 | 00100! | | | | | -13.2 | | | | |
| Windschitl | 00100! | | | | | -13.2 | | | | |
| W.D.Smith* | 00100! | | | | | 13.2 | | | | |
| Nemes* | 00100! | | | | | -13.2 | | | | |
| NemesGamma | 00100! | | | | | -13.2 | | | | |
| Luschny* | 00100! | | | | | 13.2 | | | | |
| Stirling | 01000! | -4.1 | -8.5 | 11.6 | 15.6 | -18.1 | 22.2 | 24.2 | -28.3 | -30.1 |
| De Moivre | 01000! | 4.4 | -9.1 | -11.6 | 16.0 | 18.1 | -22.4 | -24.3 | 28.6 | -29.1 |
| NemesG | 01000! | | -8.2 | 13.1 | 14.9 | -19.7 | -21.2 | 26.9 | 27.2 | -33.5 |
| Gosper | 01000! | -4.1 | -8.2 | 11.4 | -15.9 | 18.1 | 23.1 | 24.2 | 29.7 | 30.1 |
| Wehmeier | 01000! | -4.1 | -8.2 | 11.6 | 15.4 | -18.1 | -21.9 | 24.2 | -28.0 | -30.1 |
| Ramanujan | 01000! | -4.1 | -7.7 | -12.2 | 15.0 | -18.3 | -21.4 | 24.3 | 27.5 | -30.1 |
| Nemes | 01000! | -4.1 | | -12.2 | | -18.2 | | 24.3 | | -30.1 |
| Luschny | 01000! | 4.4 | | -11.5 | | 18.1 | | -24.2 | | 30.1 |
| Henrici | 01000! | -4.1 | | -11.4 | | -18.2 | | 24.2 | | |
| GosperSonin | 01000! | 4.4 | | -11.4 | | 18.0 | | -24.2 | | |
| MaclSeries | 01000! | 4.4 | | -11.6 | | 18.1 | | -24.2 | | |
| StirSeries | 01000! | -4.1 | | 11.6 | | -18.1 | | 24.2 | | -30.1 |
| BuricElezovic | 1000! | -4.1 | | 11.6 | | -18.1 | | 24.2 | | -30.1 |
| FengWang | 01000! | -4.1 | | 10.5 | | 17.2 | | 23.3 | | 29.8 |
| NemesCF | 01000! | 4.1 | | 11.2 | | 18.2 | | 24.3 | | 30.5 |
| W.D.Smith | 01000! | 4.4 | | -11.6 | | 18.2 | | -24.5 | | 30.5 |
| Cantrell | 01000! | 4.4 | | -11.6 | | 18.2 | | -24.5 | | 30.5 |
| Stieltjes | 01000! | -4.1 | | 11.6 | | -18.2 | | 24.4 | | -30.4 |
| LuschnyCF | 01000! | 4.4 | | -11.8 | | 18.2 | | -24.6 | | 30.5 |
| Lanczos7 | 01000! | | | -12.8 | | | | | | |
| Windschitl | 01000! | | | | | -18.2 | | | | |
| W.D.Smith* | 01000! | | | | | 18.2 | | | | |
| Nemes* | 01000! | | | | | -18.2 | | | | |
| NemesGamma | 01000! | | | | | -18.2 | | | | |
| Luschny* | 01000! | | | | | 18.2 | | | | |
| Stirling | 10000! | -5.1 | -10.5 | 14.6 | 19.6 | -23.1 | 28.2 | 31.2 | -36.3 | -39.1 |
| De Moivre | 10000! | 5.4 | -11.1 | -14.6 | 20.0 | 23.1 | -28.5 | -31.2 | 36.6 | -37.1 |
| NemesG | 10000! | | -10.2 | 16.1 | 18.9 | -24.7 | -27.2 | 33.7 | 35.2 | -42.1 |
| Gosper | 10000! | -5.1 | -10.2 | 14.4 | -19.9 | 23.1 | 29.1 | 31.2 | 37.6 | 39.1 |
| Wehmeier | 10000! | -5.1 | -10.2 | 14.6 | 19.3 | -23.1 | -27.9 | 31.2 | -36.0 | -39.1 |
| Ramanujan | 10000! | -5.1 | -9.7 | -15.2 | 19.0 | -23.3 | -27.4 | 31.3 | 35.5 | -39.1 |
| Nemes | 10000! | -5.1 | | -15.2 | | -23.2 | | 31.3 | | -39.1 |
| Luschny | 10000! | 5.4 | | -14.5 | | 23.1 | | -31.2 | | 39.1 |
| Henrici | 10000! | -5.1 | | -14.4 | | -23.2 | | 31.2 | | |
| GosperSonin | 10000! | 5.4 | | -14.4 | | 23.0 | | -31.2 | | |
| MaclSeries | 10000! | 5.4 | | -14.6 | | 23.1 | | -31.2 | | |
| StirSeries | 10000! | -5.1 | | 14.6 | | -23.1 | | 31.2 | | -39.1 |
| BuricElezovic | 10000! | -5.1 | | 14.6 | | -23.1 | | 31.2 | | -39.1 |
| FengWang | 10000! | -5.1 | | 13.5 | | 22.2 | | 30.3 | | 38.8 |
| NemesCF | 10000! | 5.1 | | 14.2 | | 23.2 | | 31.3 | | 39.5 |
| W.D.Smith | 10000! | 5.4 | | -14.6 | | 23.2 | | -31.5 | | 39.5 |
| Cantrell | 10000! | 5.4 | | -14.6 | | 23.2 | | -31.5 | | 39.5 |
| Stieltjes | 10000! | -5.1 | | 14.6 | | -23.2 | | 31.4 | | -39.4 |

| | | | | | | |
|------------|--------|-------|-------|------|-------|------|
| LuschnyCF | 10000! | 5.4 | -14.8 | 23.2 | -31.6 | 39.5 |
| Lanczos7 | 10000! | -12.7 | | | | |
| Windschitl | 10000! | -23.2 | | | | |
| W.D.Smith* | 10000! | 23.2 | | | | |
| Nemes* | 10000! | -23.2 | | | | |
| NemesGamma | 10000! | -23.2 | | | | |
| Luschny* | 10000! | 23.2 | | | | |

Exact digits and efficiency

Note that the above table is misleading in one respect: It shows only the exact decimal digits of a formula; however it does not to give evidence for efficiency. Although the asymptotic *even* Nemes formulas nemesG2, nemesG4, etc., the *odd* De Moivre formulas demoivre1, demoivre3 and the odd Gosper formulas gosper5, gosper7 score highest in the respective columns they are *not* the most efficient ones. The most efficient ones are the continued fraction formulas (distinguished by color (blue and purple) in the table).

To illustrate the last remark we give a 'nemesG' formula in a form which has a comparable amount of computational cost as the simple new formula recommended above.

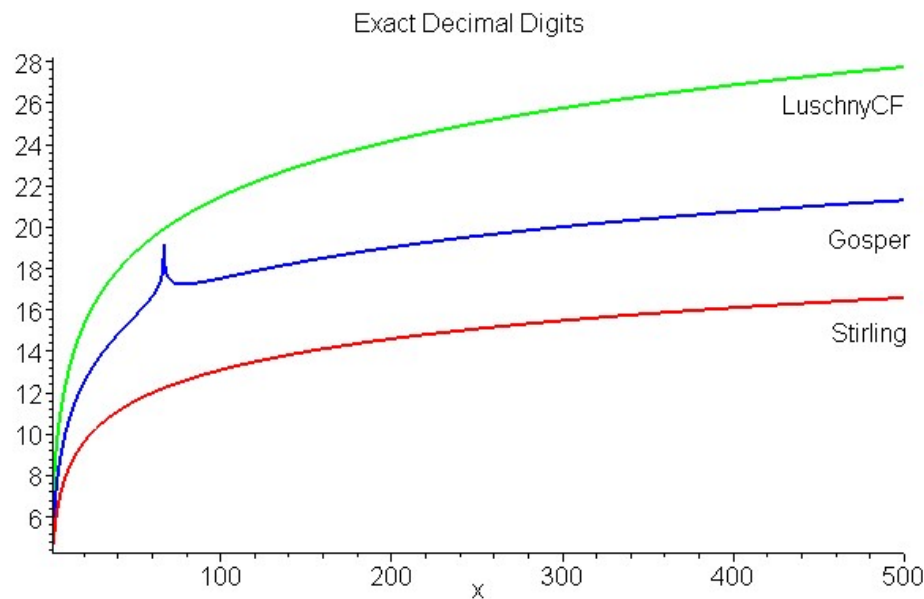
```
Digits:= 30;
nemesG5 := proc(x) local X, G, S, n; X := x + 1/4;
G := array(0..4, [1/144, -1/12960, -257/207360, -53/2612736]);
S := 1 + G[0]/X^2 + G[1]/X^3 + G[2]/X^4 + G[3]/X^5;
return x*x*exp(-x)*sqrt(2*Pi*(x+1/6))*S end;
```

The test applied to this formula shows that the new formula 'LuschnyCF' is far more efficient than the 'NemesG' formula.

| Formula | n | 100 | 1000 | 10000 |
|-----------------|-----|------|------|-------|
| NemesG5 | edd | 14.9 | 19.7 | 24.7 |
| LuschnyCF4 | edd | 21.5 | 30.5 | 39.5 |
| StirlingSeries4 | edd | 21.1 | 30.1 | 39.1 |

With regard to the ratio (exact decimal digits) / (computational cost) the popular 'Lanczos7' is perhaps the most inefficient formula shown on this page.

The plot below compares on a continuous scale the exact decimal digits of the Stirling and the Gosper formula with the Luschny-CF formula. The peak in the blue edd-function is due to the fact that the Gosper5 formula is exact at $x = 67.0033148435486248\dots$



Which approximation to choose?

The formulas given are mostly asymptotic formulas and this means they only give good approximations if x is large. However, for small values of x there is a simple trick to overcome this shortcoming at no great cost. Shift x in the direction $\rightarrow +\infty$ before evaluating and divide by the shifted amount afterwards. This is, of course, nothing else but a clever application of the functional

equation of $x!$.

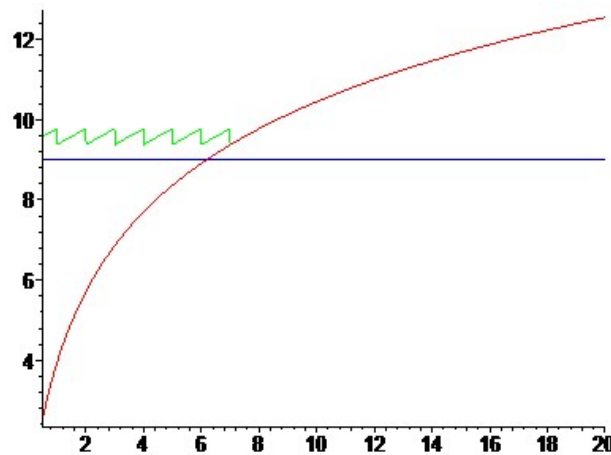
I am quite fond of the continued fraction approximation of T. J. Stieltjes because of its simplicity and the small constants involved. *Note especially that this is a convergent approximation!* So I built the above-mentioned trick in the following simple function based on the stieltjes3 formula and recommend its use if only moderate precision is needed.

```
Stieltjes3Factorial(x)
y = x+1; p = 1;
while y < 7 do p = p*y; y = y+1; od;
r = exp(y*log(y)-y + 1/(12*y+2/(5*y+53/(42*y)))));
if x < 7 then r = r/p fi;
return r*sqrt(2*Pi / y) end;
```

This approximation guarantees 9 valid decimal digits. And you can expect about $7/2+3*\log(x)$ valid significant decimal digits for $x \geq 10$. For programming a [pocket calculator](#) the following simplification of the fourth line is even better suited:

```
r = exp(y*(log(y)-1)+1/(2*(6*y+1/(5*(y+1/(4*y))))));
```

In the following plot the blue line represents the guaranteed 9 decimal digits, the green zigzag line shows the effect of the shift/divide-trick, and the red curve is the number of exact decimal digits of the 'pure' stieltjes3 approximation.



How to display approximation values?

It is good practice to display only the valid digits of an approximation and not to leave the user more or less helplessly looking at long strings of digits figuring out how much of these digits he might assume as valid.

However, how this is done depends much on the implementation language. I give an example using pseudo-Maple-style, which in turn uses a formatting scheme which is very similar to the one used in the computer language 'C'.

```
PrintFactorial(x)
# Number of valid significant decimal digits for StieltjesFactorial.
l = floor(7/2+3*log(x));
# Put formatting specifications in the format string.
format = sprintf("%d! = %s%de\n",x,"%1.",l-1);
# Add some internal guarding digits and evaluate.
F = evalf(Stieltjes3Factorial(x),l+6);
printf(format, F) end;
```

Using this special print function to display the approximations to $(2^i)!$

```
Sequence( PrintFactorial(2^i), i = 0..12);
```

gives the following output, which shows only valid digits (the last digit might be rounded).

```
1! = 1.00e+00
2! = 2.0000e+00
4! = 2.400000e+01
8! = 4.03200000e+04
16! = 2.0922789888e+13
32! = 2.631308369337e+35
64! = 1.26886932185884e+89
```

```

128! = 3.85620482362580422e+215
256! = 8.5781777534284265412e+506
512! = 3.477289793132605363283e+1166
1024! = 5.41852879605885728307692e+2639
2048! = 1.6726919319100117051699525e+5894
4096! = 3.642736389457041931565827470e+13019

```

A higher precision approximation

... and now we are going for the Big Pizza. The next approximation guarantees 16 valid decimal digits. And you can expect about $5/2 + (13/2) \cdot \log(x)$ valid significant decimal digits for $x \geq 10$ (see the next figure). I just give the pseudo-code because things are very similar to the things explained in the last two sections.

However, there is a small difference. This time we make explicit that we are using the logarithmic version of the continued fraction formula. This gives an additional layer of flexibility.

```

StieltjesLnFactorial(z)
a0 = 1/12; a1 = 1/30; a2 = 53/210; a3 = 195/371;
a4 = 22999/22737; a5 = 29944523/19733142;
a6 = 109535241009/48264275462;
Z = z+1; (1/2)*ln(2*Pi)+(Z-1/2)*ln(Z)-Z +
a0/(Z+a1/(Z+a2/(Z+a3/(Z+a4/(Z+a5/(Z+a6/Z)))))) end;

StieltjesFactorial(x)
y = x; p = 1;
while y < 8 do p = p*y; y = y+1; od;
r = exp(StieltjesLnFactorial(y));
if x < 8 then r = (r*x)/(p*y) fi;
r end;

```

```

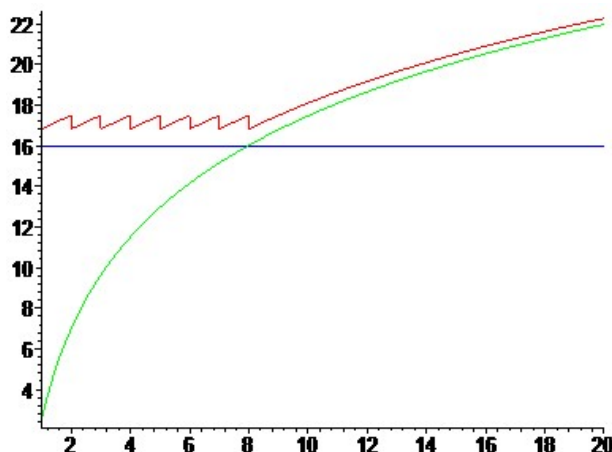
PrintFactorial(x)
l = floor((5+13*log(x))/2);
format = sprintf("%d! = %s%de\n",x,"%1.",l-1);
F = evalf(StieltjesFactorial(x),l+6);
printf(format, F) end;

```

```

1! = 1.00e+00
2! = 2.000000e+00
4! = 2.4000000000e+01
8! = 4.032000000000000e+04
16! = 2.092278988800000000e+13
32! = 2.631308369336935301672180e+35
64! = 1.2688693218588416410343338934e+89
128! = 3.856204823625804217356770659234637e+215
256! = 8.5781777534284265411908227168123262516e+506
512! = 3.477289793132605363283045917545604711992251e+1166
1024! = 5.4185287960588572830769219446838547380015539636e+2639
2048! = 1.672691931910011705169952468793676234018507002356737e+5894
4096! = 3.6427363894570419315658274703116469205712449235098554300e+13019

```



The 100 digits challenge: Using Stieltjes' CF for unlimited precision

Approximations like the Stirling or the De Moivre or the Nemes approximations are asymptotic in their character. They do not converge to the true value. On the other hand the continued fraction developments of Stieltjes or Cantrell or Luschny do converge. The asymptotic developments are fine if you work with fixed length arithmetic. For instance if you have to work with standard formats of computer languages like 'float' or 'double' it is easy to find a formula which is appropriate for these types in the table above and that's it. (Well, a careful implementation taking the rounding error into account is still needed, of course.) However, increasingly often applications demand higher precision and assuming that your computing environment supports unlimited arithmetic we can implement also these continued fraction algorithms without pain.

Just to keep things simple and concrete let us pose the 100 digits question. Suppose I want to approximate $100!$ (which has 158 decimal digits) to 100 decimal digits. How can I proceed? We stay with our favorite convergent approximation, the Stieltjes continued fractions. First we have to compute all the constants involved. This is far from being trivial. Stieltjes himself groaned about it: "Le calcul est très pénible ... la loi de ces nombres paraît extrêmement compliqué". Using the algorithms of Rutishauser and Akiyama-Tanigawa we can compute nowadays the coefficients much more easily than Stieltjes could:

```
StieltjesCF := proc(len)
local s, m, n, k, a, b, c, AkiyamaTanigawa;

# Computes Bernoulli(2*p+2)/((2*p+1)*(2*p+2))
# using the Akiyama-Tanigawa algorithm

AkiyamaTanigawa := proc(l) local a,t,n,m,j,k;
n := 2*l+1; t := array(1..n); a := array(1..1);
t[1] := 1; k := 1;
for m from 2 to n do
    t[m] := 1/m;
    for j from m-1 by -1 to 1 do
        t[j] := j*(t[j]-t[j+1]);
    od;
    if type(m,odd) then
        a[k] := (-1)^(k+1)*t[1]/((2*k-1)*(2*k));
        k := k+1;
    fi;
od;
a end;

# Computes the Stieltjes continued fraction for the
# Gamma function using Rutishauser's QD-algorithm.

s := AkiyamaTanigawa(len);
m := array(1..len,1..len);
for n to len do m[n,1] := 0 od;
for n to len-1 do m[n,2] := s[n+1]/s[n] od;
for k from 3 to len do
    for n to len-k+1 do
        a := m[n+1,k-2]; b := m[n+1,k-1]; c := m[n,k-1];
        m[n,k] := `if`(type(k,odd), a+b-c, a*b/c);
    od;
od;
m[1,1] := s[1]; [seq(m[1,k],k=1..len)] end;

# Example call
StieltjesCF(6);
[1/12, 1/30, 53/210, 195/371, 22999/22737]
```

Having found a way to compute the constants it is easy now to finish the job and compute Stieltjes approximations for $n!$ to any order we wish.

```
Stieltjes := proc(n, ord) local N, c, q, i;
N := n + 1; q := N;
c := StieltjesCF(ord);
for i from ord by -1 to 2 do
    q := N + c[i] / q od;
sqrt(2*Pi/N)*(N/exp(1))^N*exp(1/(12*q))
end;
```

In our example (and using Maple) we can now call 'Stieltjes(n, 32); evalf(%, 100)' for $n = 100$. For comparison the first value given below is $\text{evalf}(n!, 100)$ and the second value is the approximation. ('evalf(x, 100)' here means that we want to compute the floating point value of x to 100 digits.)

```
0.9332621544394415268169923885626670049071596826438162146859296389521759999322991560894146397615651
0.9332621544394415268169923885626670049071596826438162146859296389521759999322991560894146397615651
```

We see that the first 97 decimal digits are exact; so we learned that it is always a good idea to compute with some extra digits (say $100+5$ in our case) to compensate for the rounding error.

The computation of the Stirling-coefficients

Stirling's approximation is the most famous approximation to the factorial function — for historic reasons. Here we have found no reasons to use this formula. Nevertheless we show how to compute the coefficients of the Stirling formula.

```
h := proc(k) option remember; local j;
`if` (k=0,1, (h(k-1)/k+add((h(k-j)*h(j))/(j+1),j=1..k-1))/(1+1/(k+1))) end:

coeffStirling := proc(n) option remember;
h(2*n)*2^n*pochhammer(1/2,n) end:
```

The computation of the 'demoivre'-coefficients

This is amazingly simple, provided you already have the Bernoulli numbers. (If not, use the Akiyama-Tanigawa algorithm given above.) Here is a Maple implementation:

```
G := proc(n) option remember; local j, R;
R := seq(2*j, j = 1 .. iquo(n+1,2));
`if` (n=0,1,add(bernoulli(j,1/2)*G(n-j+1)/(n*j),j=R)) end:
```

The relationship between these coefficients and the Bernoulli numbers are due to [De Moivre](#), 1730, which of course also explains the name of the formula. 281 years old but still better than those "ultimate extremely accurate formulas" of Mortici.

The computation of the Lanczos-coefficients

C. Lanczos, A precision approximation of the gamma function, J. SIAM Numer. Anal., Ser. B, 1 (1964), 86-96.

Here we look at formula (21) on page 90. A Maple implementation:

```
Lanczos := proc(n) exp(1+LambertW((x^2-1)/exp(1)));
coefTayl(taylor(%,x=0,2*n+2),x=0,2*n+1);
simplify(-%*(2*n+1)*pochhammer(1/2,n)/sqrt(2),exp) end:
```

The computation of the 'nemesG'-coefficients

[Now some fiction, also a puzzle: *The mysterious J-function.*] One day, while studying -- no, not Mortici but Y. L. Luke's treatise -- Gery saw the beautiful expansion given at the top of this page (in fact the formula on this page is by a hairbreadth better). "Wow" he thought, "I'd like to have also such a nice formula for the factorial." So what was important was to find a way to mimic the *even* expansion from Luke's book, which is due to J. L. Fields (A note on the asymptotic expansion of a ratio of gamma functions. Proc. Edinburgh Math. Soc., 15:43-45, 1966.) This means he had to look for a sequence of even polynomials. Well, after some hard thinking Gery came up with these polynomials:

```
G := proc(n,x) local j;
add(x^(2*j)*2^j*6^(j-n)*GAMMA(1/2+j)/(GAMMA(n-j+1)*GAMMA(1/2+j-n)),j=0..n)-
add(G(j,x)*(-4)^(j-n)*(GAMMA(n)/(GAMMA(n-j+1)*GAMMA(j))),j=1..n-1);
sort(%) end:

for i from 0 to 5 do G(i,x)*24^i od;

1
24 x^2 - 2
1728 x^4 + 96 x^2 - 6
207360 x^6 + 17280 x^4 + 432 x^2 - 20
34836480 x^8 + 3317760 x^6 + 134784 x^4 + 2112 x^2 - 74
7524679680 x^10 + 766402560 x^8 + 36080640 x^6 + 960768 x^4 + 10896 x^2 - 300
```

At this moment an email from James S. arrived announcing a totally mysterious function. James

wrote: "Dear Gery, please look at this convolved function. Can you make any sense out of it?"

```
J := proc(k) option remember; local j;
`if` (k=0,1, (J(k-1)/k-add((J(k-j)*J(j))/(j+1),j=1..k-1))/(1+1/(k+1))) end;
```

Gery again thought hard but this time he could not make any sense with this function. However, his just discovered polynomials still were on his desktop and he decided to play just for fun the umbral game. This means to replace x^i by a function call $x(i)$. Well, so he evaluated his polynomials with $J(i)$ in lieu of x^i . And here is what he saw:

```
seq(subs({seq(x^i=J(i), i=0..degree(G(k,x),x))}, G(k,x)),k=0..7);
```

```
1, 0, 1/144, -1/12960, -257/207360, -53/2612736, 5741173/9405849600, 37529/18811699200, ...
```

Conclusions and Recommendations

The Stirling formula is a useful formula when we have to do some Big-O (complexity-) analysis. But if we are only interested in the numerical evaluation of the factorial function then there is no reason to use the Stirling approximation or some of its many asymptotic variants. Because there exist convergent approximations which are

- computational as simple as the Stirling formula,
- computational more efficient then the Stirling formula,
- and can equally well be used in the context where only a simple estimate is required or in a context where an arbitrary precision computation is required.

In our judgement there are three outstanding formulas which we declare the winners of our little contest.

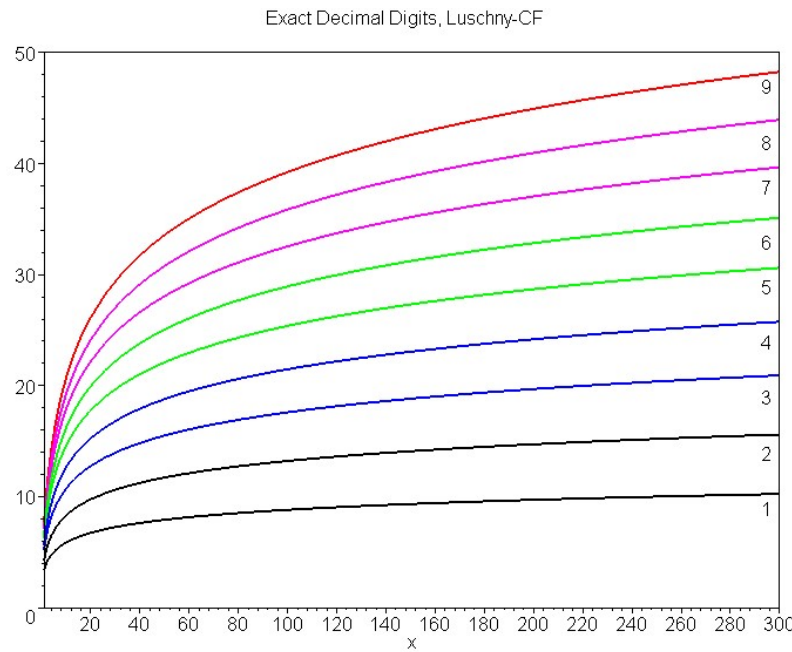
- The continued fraction formula of Thomas J. Stieltjes.
- The continued fraction 'half-shift' formula of Warren D. Smith.
- The continued fraction formula of Peter Luschny (luschnyCF).

Clearly the continued fraction of W. D. Smith is a variant of Stieltjes' formula. And the coefficients of this formula are slightly larger than those of the Stieltjes formula. Another runner-up is Cantrell's formula, which is numerically on par with Smith's formula but has even larger rational constants (which does not really matter if we are using floating point arithmetic with fixed precision). Note that we looked only at the real range $x > 1$. If considered throughout the complex half-plane $\text{Re}(z) > 1$ there might be some special strength or weakness in the formulas which we have not noticed.

Summa summarum we recommend Stieltjes's formula or Luschny's formula whenever a numerical approximation of the factorial function is required.

S * H * O * W * D * O * W * N

| | Stieltjes Luschny | | Stieltjes Luschny | | Stieltjes Luschny | |
|---|-------------------|--------|-------------------|--------|-------------------|--------|
| n | 100 | | 1000 | | 10000 | |
| 1 | 8.57 | -8.81 | 11.56 | -11.81 | 14.56 | -14.81 |
| 2 | -13.18 | 13.22 | -18.16 | 18.21 | -23.15 | 23.21 |
| 3 | 17.46 | -17.60 | 24.44 | -24.58 | 31.43 | -31.58 |
| 4 | -21.47 | 21.48 | -30.43 | 30.46 | -39.43 | 39.46 |
| 5 | 25.30 | -25.39 | 36.25 | -36.37 | 47.25 | -47.36 |
| 6 | -28.95 | 28.94 | -41.90 | 41.92 | -54.89 | 54.91 |
| 7 | 32.48 | -32.55 | 47.42 | -47.52 | 62.41 | -62.51 |
| 8 | -35.88 | 35.86 | -52.82 | 52.83 | -69.81 | 69.83 |
| 9 | 39.19 | -39.25 | 58.12 | -58.21 | 77.11 | -77.20 |



Miscellaneous notes

****** The reference for the Ramanujan formula is: S. Raghavan and S. S. Rangachari (eds.) "S. Ramanujan: The lost notebook and other unpublished papers", Springer, 1988, p. 339. Ramanujan's statement is as follows:

$$\Gamma(x+1) = \sqrt{\pi} (x/e)^x (8x^3 + 4x^2 + x + \theta(x)/30)^{1/6}$$

where $\theta(x) \rightarrow 1$ as $x \rightarrow \infty$ and $(3/10) < \theta(x) < 1$.

****** Paul Abbott commented Ramanujan's statement on the Usenet (sci.math.num-analysis): "For

$$\sqrt{\pi} (n/e)^n (n(4n(2n+1)+1) + (1/30)(1 - (11/(8n))(1 - (79/(154n))) * (1 + (3539/(4740n))(1 - (9511/(7078n))(1 + 90459/(152176n))))))^{1/6}$$

one finds that

$$\theta(x) = 1 - 11/(8x) + 79/(112x^2) + 3539/(6720x^3) - 9511/(13440x^4) - 30153/(71680x^5)$$

Clearly $\theta(x) \rightarrow 1$ for $x \rightarrow \infty$. Also $(3/10) < \theta(x) < 1$ for $x > \sim 1.3878$. So this form does agree with Ramanujan's statement."

****** Michael Hirschhorn [The Mathematical Gazette, Vol. 90, July 2006, 286-292] proved the following sharper bounds for Ramanujan's $\theta(x)$, valid for $x > 6053/3987$ (or $x > 1.52$).

$$1 - 11/(8x) + 5/(8x^2) < \theta(x) < 1 - 11/(8x) + 11/(8x^2)$$

****** The reference for Stieltjes' continued fraction formula is: M. Abramowitz, I. A. Stegun (eds.), Handbook of Mathematical Functions, p. 258, [6.1.48]. Note also: B. W. Char: "On Stieltjes' continued fraction for the gamma function", Mathematics of Computation 34, 150 (1980) 547-551.

****** There are many Lanczos formulas for approximating $n!$. The form we use here is his equation (21) given in "A precision approximation of the gamma function", J. SIAM Numer. Anal., Ser. B, Vol. 1 (1964), pp. 86-96. Lanczos also devised other series (not included here) which might be more useful in practical computation. So he might not be fairly represented here! (Thanks to D. W. Cantrell for this remark.)

****** The formulas of Warren D. Smith were taken from *The Gamma function revisited*, a typeset dated 29 Mar 2006.

****** The formulas cantrell and wdsmith have an almost identical numerical performance.

****** In our setup the approximation of Robert H. Windschitl is only a curiosity. Warren D. Smith's formula $\text{wdsmith}^*(n)$ provides a similar approximation based on $\Gamma(n+1/2)$. The two formulas wdsmith^* and windschitl bracket the true value. (Note that our interpretation of the Windschitl formula differs from the interpretation used by W. D. Smith. However, our $(\Gamma(n-))$ interpretation seems to be more useful.)

| z | Windschitl | z! | W.D.Smith* |
|------------|-------------------|-----------------|-------------------|
| 0.0 | 0.999658 | 1.0 | 1.00429 |
| 0.5 | 0.88617 | 0.88623 | 0.88652 |
| 1.0 | 0.999984 | 1.0 | 1.000057 |
| 1.5 | 1.329333 | 1.329340 | 1.329361 |
| 2.0 | 1.9999954 | 2.0 | 2.0000107 |

****** I do not understand why the Windschitl formula is called 'a version suitable for calculators' on Wikipedia and other web-sides. The appearance of sinh (or tanh in the case of wdsmith*) make them look like stars of the 'Pimp My Formula' TV-show. Well, it is time to un-pimp these formulas and to introduce the cute nemesGamma(n) and luschny*(n) formulas, which are *much* simpler and computational less expensive but equal powerful. (For the definitions see the paragraph *Miscellaneous Approximations* above.)

And there is an added value. Comparing windschitl(n) with nemesGamma(n) we find the following approximation:

$$\sqrt{x \sinh(1/x)} = 1 + 1/(12 x^2 - 1/10) + O(x^{-6})$$

In fact $|\sqrt{x \sinh(1/x)} - (1 + 1/(12 x^2 - 1/10))| < 1 / (24192 x^6)$.

Comparing wdsmith*(n) with luschny*(n) shows the approximation:

$$\sqrt{x \tanh(1/x)} = 1 - 1/(6 x^2 + 19/10) + O(x^{-6})$$

In fact $|\sqrt{x \tanh(1/x)} - (1 - 1/(6 x^2 + 19/10))| < 1 / (54 x^6)$.

These two approximations are noteworthy. It does not make sense to replace the simple right hand sides by the functions on the left hand side in the above approximations to the factorial function as this will *not* decrease the approximation error. It will only produce pimped formulas ;-)

****** Some numerical methods for computing x! (the tricks of the trade) can be found in the *Handbook of Mathematical Functions* (6.7).

****** Let us define the Noerlund scheme as

$$\text{noerlund}(z, h, m) \\ \ln(\sqrt{2\pi}) + (z+h-1/2) \ln(z) - z + \sum_{j=1}^{m-1} ((-1)^{j+1} \text{bernoulli}(j+1, h) / ((j+1)^j z^j))$$

The Noerlund scheme [*Vorlesungen über Differenzenrechnung*, Berlin 1924, p.111] gives us a common framework to classify approximation formulas to x!. There are three basic types, which correspond to our more general abbreviations kern#(x) used in the table above.

$$\begin{aligned} \text{KERN0}(x) &= \exp(\text{noerlund}(x+1, 0, 0)) = \sqrt{2\pi/Y} (Y/e)^Y \text{ where } Y=x+1 \\ \text{KERN1}(x) &= \exp(\text{noerlund}(x, 1, 0)) = \sqrt{2\pi*Y} (Y/e)^Y \text{ where } Y=x \\ \text{KERN2}(x) &= \exp(\text{noerlund}(x+1/2, 1/2, 0)) = \sqrt{2\pi} (Y/e)^Y \text{ where } Y=x+1/2 \end{aligned}$$

Note that KERN1 has a problem when x=0. Therefore KERN0 is to be preferred to KERN1. KERN2 has the most simple form and might be called the Hermite-Sonin variant.

****** Asymptotic approximation of the Bernoulli Numbers.

Combining the nemes* formula and the well known connection of the **Bernoulli Numbers** with the Riemann Zeta Function we find the following remarkable formula, valid for even n >= 4.

$$|B(n)| \sim 2 \sqrt{2 \pi n} ((n + 1/(12 n - 1/(10 n)))/(2 \pi e))^n$$

For example this approximation gives

$$|B(100)| \sim 0.28382249570691.. 10^{79}$$

compared with

$$|B(100)| = 0.28382249570693.. 10^{79}.$$

We can write this formula also as an approximation to Zeta(1-n), valid for even n >= 4.

$$|Zeta(1-n)| \sim 2 \sqrt{2 \pi / n} ((n / (2 \pi e))((120 n^2 + 9)/(120 n^2 - 1)))^n$$

Asymptotic formulas for the central binomial coefficient

and the Catalan numbers.

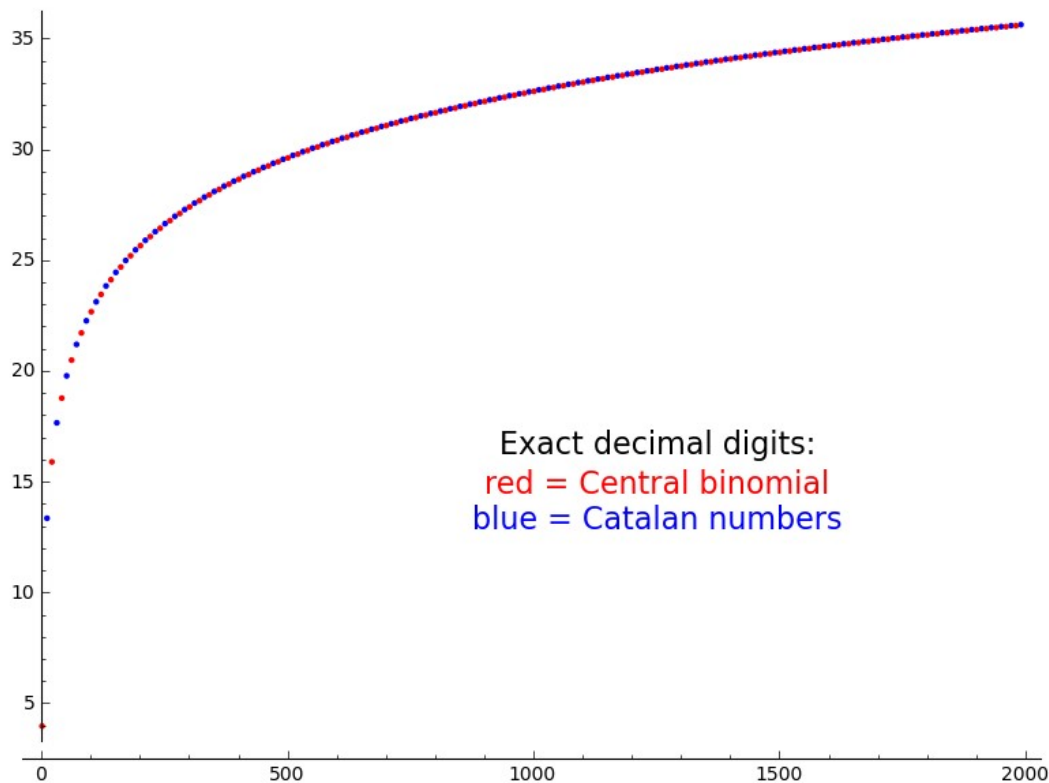
The two asymptotic formulas for the central binomial coefficient and the Catalan numbers at the top of this page have a more precise (but less elegant) form which is:

$$N = 8n + 2 \text{ and } M = 4n + 3.$$

$$\binom{2n}{n} = \frac{4^n}{\sqrt{(n+1/4)\pi}} \left(1 - \frac{1}{N^2} + \frac{21}{2N^4} - \frac{671}{2N^6} + \frac{180323}{8N^8} - O\left(\frac{1}{N^{10}}\right) \right)$$

$$\frac{1}{n+1} \binom{2n}{n} = \frac{4^n 8}{M\sqrt{M\pi}} \left(1 + \frac{5}{4M^2} + \frac{21}{32M^4} + \frac{715}{128M^6} - \frac{162877}{2048M^8} + O\left(\frac{1}{M^{10}}\right) \right)$$

More on this can be found in: Peter Luschny, "Divide, swing and conquer the factorial and the lcm{1,2,...,n}", preprint, April 2008. More coefficients, hints to the literature and to the interesting connection of coefficients with the Euler numbers can be found in the Online-Encyclopedia of Integer Sequences [A220002](#) and [A220422](#).



Asymptotic formulas for the Euler and the Bernoulli numbers.

can be found here: [Euler](#) and [Bernoulli](#).

Update-history.

=> Update June 13, 2013:

Lei Feng and Weiping Wang published six new formulas in their paper *Two families of approximations for the gamma function*. We added the most efficient one to the above data. It is a kern1 formula which performs under average. Feng and Wang cite parts of our tables but curiously they claim "these six formulae perform better than Nemes' another result, which, to our knowledge, is numerically more accurate than all the other formulae presented in the literature ... as well as the other formulae in Luschny's web page." Apparently they totally misunderstood the structure of our tables and how we have set up the successive approximations of a formula.

=> Update Nov 30, 2012:

Prof. Neven Elezović kindly pointed out an error in the asymptotic formula of the Catalan numbers on the top of this page: 10177 has to be replaced by 10180. Now corrected. Thanks!

=> Update Jun 3, 2011: A formula of T. Buric and N. Elezović added.

Formula (3.3) in "New asymptotic expansions of the gamma function and improvements of Stirling's type formulas."

```
1000!      0.402387260077093773543702433923003*10^2568
Buric-Elezović: 0.402387260077093773543702433922664*10^2568
StirlingSeries: 0.402387260077093773543702433922668*10^2568
```

There is no noticeable difference (in terms of significant decimal digits) compared to the Stirling series. The Stirling series (see the formulas for stirser above) has smaller coefficients and is simpler.

=> Update Mar 24, 2011: More annoying things from Cristinel Mortici!

Cristinel Mortici writes in: "A substantially improvement of the Stirling formula", to appear in: Applied Mathematics Letters. Received date: 30 August 2009.

"We introduce the new approximation formula

$$n! \sim \sqrt{2\pi n} (n/e + 1/(12e n))^n = \mu_n \quad (1.1)$$

which has great superiority over all the previous formulas. ... but our new formula (1.1) has the advantage of simplicity. ... Unlike most formulas which are variations of the asymptotic expansion of the Stirling formula, our formula (1.1) has an original construction mode. ... Our new approximation formula $n! \sim \mu_n$ gives surprisingly good results. It is much better than the Gosper's approximation $n! \sim \gamma_n$. "

Fact is: Formula 1.1 is on this page since Nov 12, 2006. "The new approximation formula" and its "original construction mode" are due to Gergő Nemes. It is described here as the nemes1 formula. But much more is true: Nemes went ahead and gave a full expansion to an asymptotic approximation of the factorial function. And Nemes published his results also in other places: [First version](#), [second version](#), [third version](#).

=> Update Feb 5, 2011: New approximation added. And read about Cristinel Mortici!

Gergő Nemes has published a clever and efficient new formula! [\[Nemes\]](#) I include the formula under the name NemesG in the tables above. The new formula scores 12 (!) times as the best result in the tables! Well done, Gergő. This was the good news.

Now some other news. The Comptes Rendus de l'Académie des Sciences, Paris, Ser. I 348 (2010) 137–140, presented by Jean-Pierre Kahanep, begins: "The purpose of this Note is to construct a new type of Stirling series, which extends the Gosper's formula for big factorials." It ends: "Finally, we demonstrate the superiority of our new series..."

OK, so I had to include it here, n'est-ce pas? Well, not really. It is already here, for many years. It is the Wehmeier formula. So perhaps there is a new idea in deriving this formula? The author, Mortici, says the basic tool is a lemma which "...was used by Mortici.." several times. Looking at the lemma I see that it is exactly the way Wehmeier derived his formula. So Mortici's result is neither 'new' nor does the above table indicate any 'superiority' of the formula. On the other hand Wehmeier never published his result (Mortici publishes every month or so a 'new sharp' approximation to the 'big factorials'). Wehmeier just participated in a discussion on the Usenet group de.sci.mathematik. It is nice to know that once upon a time, before the cranks took over, a discussion on dsm could be on the level of the Comptes Rendus de l'Académie des Sciences, Paris. [\[Mortici\]](#) [\[dsm1\]](#) [\[dsm2\]](#)

=> A000207 at OEIS.

Recently (Feb 2007) Rainer Rosenthal drew my attention to the number of planar 2-trees, sequence A000207 at OEIS. The sequence can be computed as (Maple):

```
A000108 := n -> (2*n)! / (n!*(n+1)!);
A000207 := n -> A000108(n)/(2*n+4)+ A000108(floor(n/2))* (3-modp(n,2))/4 +
`if` (modp(n,3)=1,A000108(floor((n-1)/3))/3,0);
```

I noticed that the quotient $A000207(n+1) / A000207(n)$ has a remarkable regular asymptotic expansion:

```
QuotAsymptotic := n -> 4 - (10/n)* (1-(3/n)* ((1-(3/n)* ((1-(3/n)* ((1-(3/n)
*((1-(3/n)* ((1-(3/n)* ((1-(3/n)))))))))))).
```

=> Update Jan 11, 2007: The computation of Stieltjes' continued fraction.

After calculating his famous continued fraction for the Gamma function Stieltjes remarked: "Le calcul est très pénible ... la loi de ces nombres paraît extrêmement compliqué". I looked at the state of the art and wrote a summary. See the link list below.

=> Update Jan 9, 2007: Two new continued fraction approximations added.

My latest additions are the formulas luschnyCT and nemesCT. A first comparison shows that the luschnyCT formula performs slightly better than Stieltjes' formula (or the W.D.Smith or Cantrell formula).

=> Update Jan 4, 2007: Three simple but powerful formulas added.

The formulas nemes*, nemesGamma and luschny* are new. They are much simpler than the windschitl or wdschitl* formulas but equal powerful.

=> Update Dec 28, 2006: The Luschny formulas added.

The new Luschny formulas added following Sonin's 'half-shift' idea and similar to the Nemes approach.

=> Update Dec 14, 2006: The Gosper-Smith formulas added.

Gosper's $z! \sim \sqrt{(2z+1/3)\pi} (z/e)^z$ recasted by W. D. Smith following Sonin's 'half-shift' idea.

=> Update Dec 13, 2006: Warren D. Smith's formulas added.

The continued fraction formulas of W. D. Smith are based on the Hermite-Sonin discussion: *Sur les polynômes d'Bernoulli, Extrait d'une correspondance entre M. Sonin et M. Hermite*. J. für Math. 116 (1896), 133-156.

=> Update Nov 18, 2006: The Continued Fraction Formula corrected

Gergö Nemes spotted an error in stieltjes3. Thanks, Gergö! With this correction, the author hopes that the formulas are now error-free.

=> Update Nov 12, 2006: The Nemes Formula announced

Gergö Nemes, a student from Hungary, sent me his formula. Thank you Gergö for sharing your findings with us!

$$\text{Nemes}(x) = \left(x + \frac{1}{12} \frac{1}{x} + \frac{1}{1440} \frac{1}{x^3} + \frac{239}{362880} \frac{1}{x^5} \right)^x e^{(-x)} \sqrt{2} \sqrt{\pi x}$$

=> Update Nov 9, 2006: The Wehmeier Formula rediscovered

$$L(x) = \sqrt{\pi} \left(\frac{x}{e} \right)^x \sqrt{2x + \frac{1}{3} + \frac{1}{36} \frac{1 - \frac{1}{90} \frac{31 + \frac{1}{24} \frac{139 - \frac{9871}{42} \frac{1}{x}}{x}}{x}}$$

In expanded normal form:

$$L(x) = \sqrt{2} \sqrt{\pi x} x^x (e)^{(-x)} \sqrt{1 + \frac{1}{6} \frac{1}{x} + \frac{1}{72} \frac{1}{x^2} - \frac{31}{6480} \frac{1}{x^3} - \frac{139}{155520} \frac{1}{x^4} + \frac{9871}{6531840} \frac{1}{x^5}}$$

First I thought that I had discovered a new family of formulas. However, a closer examination showed that I had rediscovered the Wehmeier formula while I was contemplating the Ramanujan formula. Nevertheless, I left the L-formula in the table because I think this alternative way to look at the Wehmeier formula throw some light on the structure of the Ramanujan formula.

=> Update Aug, 2006: S. Ramanujan's Formula

$$R(x) = \sqrt{\pi} \left(\frac{x}{e}\right)^x \left(x(1+4x(1+2x)) + \frac{1}{30} - \frac{11}{240} \frac{1 - \frac{79}{154} \frac{1}{x}}{x} \right)^{\left(\frac{1}{6}\right)}$$

In expanded normal form:

$$R(x) = \sqrt{2} \sqrt{\pi x} x^x (e)^{(-x)} \left(1 + \frac{1}{2} \frac{1}{x} + \frac{1}{8} \frac{1}{x^2} + \frac{1}{240} \frac{1}{x^3} - \frac{11}{1920} \frac{1}{x^4} + \frac{79}{26880} \frac{1}{x^5} \right)^{\left(\frac{1}{6}\right)}$$

Coefficients on OEIS

Formula Numerator Denominator

| | | |
|-----------|---------|---------|
| Stirling | A001163 | A001164 |
| De Moivre | A182935 | A144618 |
| Nemes | A181855 | A181856 |
| NemesG | A182912 | A182913 |
| Lanczos | A090674 | A090675 |

Formula Numerator Denominator

| | | |
|-----------|---------|---------|
| Stieltjes | A005146 | A005147 |
| LuschnyCF | A182914 | A182915 |
| Gosper | A182919 | A182920 |
| Wehmeier | A182916 | A182917 |

Links

- * David W. Cantrell's announcement "A new convergent expansion for the gamma function", OEIS A119422
- * Knud Thomsen's formula for small positive values
- * Stefan Wehmeier's announcement of his approximation.
- * G. R. Pugh's thesis "An analysis of the Lanczos gamma approximation"
- * Hugo Pfoertner's Fortran implementation of Cantrell's expansion.
- * 'Calculators and the Gamma Function' about the origin of the Windschitl-formula.
- * The computation of Stieltjes' continued fraction.

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Web. Accessed June 3, 2012. Available <http://www.luschny.de/math/factorial/approx/SimpleCases.html>.

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